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Enriques' Classification of Surfaces in Char. p, II

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Introduction and Preliminary Reductions

The purpose of this paper is to carry further the extension of Enriques' classification of surfaces from the case of a char. 0 groundfield to the case of a char. p groundfield. The first part of this extension was made in the paper [10] of one of the present authors. The main results of that paper are as follows[1] : let X be a non-singular complete algebraic surface without exceptional curves over a field k of any characteristic. We may divide such X's into 4 classes :

a) $\exists$ a curve C on X with $(K_X \cdot C)<0$

b) $\forall$ curve C on X, $(K_X \cdot C)=0$, or equivalently, for any $l \neq$ char. p, the fundamental class $[K_X] \in H^2_{et}(X, \boldsymbol{Q}_l)$ is zero.

c) $(K_X \cdot C) \geq 0$ for all curves C and $(K_X^2)=0$ but $(K_X \cdot H)>0$ for all ample divisors H.

d) $(K_X \cdot C) \geq 0$ for all C and $(K_X^2)>0$, hence $(K_X \cdot H)>0$ for all ample H.

(Other cases are excluded by using the following well-known consequences of Hodge's Index Theorem : (1) $(K_X \cdot H)=0$ for some ample H, $(K_X^2) \geq 0$ implies $(K_X \cdot C)=0$ all C and (2) $(K_X \cdot C) \geq 0$ all curves C implies $(K_X^2) \geq 0$). Then in [10], it is proven that

(a) holds $\Leftrightarrow$ X is ruled, in which case $|nK_X|=\phi$, all n.

(b) holds $\Leftrightarrow$ either i) $2K_X \equiv 0$
or ii) $\exists\ \pi : X \to D$, D a curve, almost all fibres of π non-singular elliptic and hence $nK_X=\pi^*(\mathfrak{A})$, $\mathfrak{A}$ divisor on D of degree 0, $n \geq 1$ an integer.

(c) holds $\Leftrightarrow$ $\exists\ \pi : X \to D$ almost all fibres either non-singular elliptic or rational with one cusp, hence $nK_X=\pi^*(\mathfrak{A})$ where deg $(\mathfrak{A})>0$, $n \geq 1$.

(d) holds $\Leftrightarrow$ $|nK_X|$ is base-point free and defines a birational map from X to $\boldsymbol{P}^N$, for $n \gg 0$. Moreover, in this case $|2K_X| \neq \phi$.

Our *first goal* in this paper is to prove the following result, well known in char. 0 :

Theorem 1. *In cases* (b) *and* (c), *either* $|4K_X| \neq \phi$ *or* $|6K_X| \neq \phi$. *Therefore, in case* (b), *either* $4K_X \equiv 0$ *or* $6K_X \equiv 0$, *and in case* (c), *either* $4K_X$ *or* $6K_X$ *is represented by a*

1) The notation used is summarized below in "list of notations".

positive divisor.

In particular, this shows that the 4 cases above correspond to the classification of surfaces by Kodaira-dimension κ, i.e.,

$$\kappa = \text{tr. deg.}_k \bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{O}(nK_X)) - 1.$$

Then we see that :

In case (a), $\kappa = -1$
In case (b), $\kappa = 0$
In case (c), $\kappa = 1$
In case (d), $\kappa = 2$.

Thereafter, our *next goal* in this and a subsequent 3rd paper is the further analysis of all surfaces in case (b). It turns out that these can be divided into 4 types *by their Betti numbers*. This division into 4 types is based on a rather mysterious calculation that appears again and again in all work on the classification of surfaces. This calculation is as follows :

Assume $(K_X^2)=0$. Then by the Riemann-Roch theorem on X,

$$\text{(1)} \qquad \begin{aligned} 12(\dim H^0(\mathcal{O}_X) - \dim H^1(\mathcal{O}_X) + \dim H^2(\mathcal{O}_X)) &= c_{2,X} \\ &= B_0 - B_1 + B_2 - B_3 + B_4 \end{aligned}$$

hence substituting $1 = B_0 = B_4 = \dim H^0(\mathcal{O}_X)$, we find

$$\text{(2)} \qquad 10 + 12\, p_g = 8 \dim H^1(\mathcal{O}_X) + 2(2 \dim H^1(\mathcal{O}_X) - B_1) + B_2.$$

Write $\Delta = 2 \dim H^1(\mathcal{O}_X) - B_1$. This is a "non-classical" term because when $\text{char}(k)=0$, then $\Delta=0$. In fact, we know that for *almost all* primes l :

$$\begin{aligned} (\boldsymbol{Z}/l\boldsymbol{Z})^{B_1} &\approx H^1_{ét}(X, \boldsymbol{Z}/l\boldsymbol{Z}) \\ &\cong \{x \in \text{Pic}(X) \mid \ lx = 0\} \\ &\cong \{x \in \text{Pic}^0(X) \mid \ lx = 0\} \\ &\approx (\boldsymbol{Z}/l\boldsymbol{Z})^{2q} \end{aligned}$$

hence in any characteristic $B_1 = 2q$. On the other hand,

$$H^1(\mathcal{O}_X) \cong [\text{tangent space to Pic}(X) \text{ at } 0].$$

Thus if $\text{char}(k)=0$, $\text{Pic}(X)$, like any group scheme, is reduced, hence

$$\dim H^1(\mathcal{O}_X) = q$$

and $\Delta=0$. In general, we conclude that

$$\dim H^1(\mathcal{O}_X) \geq q$$

hence $\Delta \geq 0$, Δ even. We can say a bit more : if β_i are the Bockstein operators from $H^1(\mathcal{O}_X)$ to $H^2(\mathcal{O}_X)$, we know ([9], Lecture 27) that

$$\text{tangent space to } \text{Pic}^0_{\text{red}} \cong \bigcap_{i=1}^{\infty} \ker(\beta_i)$$

hence

$$\dim(\overbrace{\text{tang.sp.to Pic}^0}^{T_P})-\dim(\overbrace{\text{tang.sp.to Pic}^0_{\text{red}}}^{T_{P,\text{red}}}) = \dim H^1(\mathcal{O}_X)-\dim\bigcap_{i=1}^{\infty}\ker\beta_i$$
$$\leq \dim\bigcup_{i=1}^{\infty}\operatorname{Im}\beta_i$$
$$\leq p_g.$$

Thus

$$\varDelta = 2(\dim T_P-\dim T_{P,\text{red}}) \leq 2p_g.$$

Although it is not used in what follows, it is interesting at this point to consider what happens for arbitrary *analytic* surfaces over $\boldsymbol{C}$. The equation (2) is perfectly valid and in this case Kodaira ([4], p. 755, Th. 3) has shown that $\varDelta=0$ or 1 according to the parity of B_1.

Now assume that the surface X has $\kappa=0$, i.e. : $(K_X\cdot C)=0$ for all curves C. Then not only is $(K_X^2)=0$ but either $p_g=0$ or $p_g=1$ and $K_X\equiv0$. It is then easy to list *all* solutions to equation (2) :

Table of Possible Invariants for Surfaces with $\kappa=0$

B_2	B_1	c_2	$\chi(\mathcal{O}_X)$	$\dim H^1(\mathcal{O}_X)$	p_g	$\varDelta$
22	0	24	2	0	1	0
14	2	12	1	1	1	0
10	0	12	1	0	0	0
				1	1	2
6	4	0	0	2	1	0
2	2	0	0	1	0	0
				2	1	2

invariants under deformation (B_2, B_1, c_2, $\chi(\mathcal{O}_X)$)

invariants which are in general only upper semi-continuous under deformation ($\dim H^1(\mathcal{O}_X)$, p_g, $\varDelta$)

Concerning these categories of surfaces, we shall prove in this paper the following results :

Theorem 5. *The surfaces with $\kappa=0$, $B_2=22$, known as K3-surfaces, have the following properties:*

i) *for all divisors D on X, $(D\cdot C)=0$ for all curves C implies $D\equiv0$, hence*

$$\operatorname{Pic}^0(X) = (0).$$

ii) *X has no connected étale coverings, i.e.,*

$$\pi_{1,\text{alg}}(X) = (e).$$

No surfaces with $\kappa=0$, $B_2=14$ exist.

Surfaces with $\kappa=0$, $B_2=10$, $p_g=1$ cannot exist if $\operatorname{char}(k)\neq2$.

Theorem 6. *All surfaces with* $\kappa=0$, $B_2=6$ *are abelian varieties.*

Moreover, the following is easy to see from the above table and the results of [15] :

Proposition. *If* X *is a surface with* $\kappa=0$, $B_2=2$, *then* $B_1=2$, *hence* $\mathrm{Alb}(X)$ *is an elliptic curve and the fibres of the canonical map*

$$\pi : X \to \mathrm{Alb}(X)$$

are either almost all non-singular elliptic curves, or almost all rational curves with ordinary cusps. The latter is only possible if $\mathrm{char}(k)=2$ *or* 3.

We call surfaces of this type *hyperelliptic* or *quasi-hyperelliptic* surfaces, depending on which type of fibre π has. In this paper, we shall also analyze hyperelliptic surfaces. However, the analysis of the case of quasi-hyperelliptic surfaces and the case of surfaces with $\kappa=0$, $B_2=10$, which we propose to call *Enriques surfaces* (regardless of whether $K\equiv 0$ or $K\not\equiv 0$!), we postpone to a 3^rd^ part of the paper. Since Enriques surfaces in $\mathrm{char}(k)\neq 2$ are fairly easily seen to have the same behaviour as in char. 0, Part III of this paper will deal largely with the curious pathology of char. 2 and 3.

Finally, for use in § 2, we note that the analysis leading to the Table does not use completely the assumption $\kappa=0$: in fact, it really only uses $(K_X^2)=0$, $p_g\leq 1$. Thus the analysis also shows :

Corollary. *If* X *is a non-singular complete surface with* $(K_X^2)=p_g=0$, *then* X *belongs to one of the 2 following types*:

i) $B_1=\dim H^1(\mathcal{O}_X)=0$, *hence* $\mathrm{Pic}^0(X)=(0)$; $\chi(\mathcal{O}_X)=1$; $B_2=10$

ii) $B_1=2$, $\dim H^1(\mathcal{O}_X)=1$, *hence* $\mathrm{Pic}^0(X)$ *is a reduced elliptic curve* ; $\chi(\mathcal{O}_X)=0$; $B_2=2$.

List of Notations

X usually a non-singular projective surface
Alb X = Albanese variety of X
Pic X = Picard scheme of X
$\mathrm{Pic}^0\, X$ = connected component of $0\in \mathrm{Pic}(X)$
$q=\dim \mathrm{Pic}\, X=\dim \mathrm{Alb}\, X$, the "irregularity" of X
K_X = the canonical divisor class on X
B_i = i^{th} Betti number of X
$h^{p,q}=\dim H^q(X,\Omega^p)$
$p_g=h^{0,2}=h^{2,0}$, the geometric genus of X
$\omega_X=\Omega_X^2$, the sheaf of 2–forms, if X is smooth
= the dualizing sheaf of Grothendieck for general Cohen-Macauley surfaces.

1. K_X of Elliptic or Quasi-elliptic Surfaces

An elliptic or quasi-elliptic surface is a fibration $f: X \to B$ of a surface X over a non-singular curve B, with $f_*\mathcal{O}_X = \mathcal{O}_B$, with almost all fibres elliptic or rational with a cusp (by a result of Tate [15], the latter situation can occur only if char$(k)=2$ or 3). Note that since the function field $k(X)$ is separable over $k(B)$, almost all fibres are generically smooth. Also every fibre of f is a curve of canonical type[1]. At finitely many points $b_1, \cdots, b_r \in B$ the fibre $f^{-1}(b_\lambda)$ is multiple, i.e.,

$$f^{-1}(b_\lambda) = m_\lambda P_\lambda$$

with $m_\lambda \geqq 2$ and P_λ indecomposable of canonical type. We have

$$R^1 f_* \mathcal{O}_X = L \oplus T$$

where L is an invertible sheaf and T is supported precisely at the points $b \in B$ at which

$$\dim H^0(f^{-1}(b), \mathcal{O}_{f^{-1}(b)}) \geqq 2.$$

To see this, note that by E. G. A. III 7. 8, the sheaf $R^1 f_* \mathcal{O}_X$ is locally free at b if and only if $\mathcal{O}_X$ is cohomologically flat at b in dimension 0.

This suggests

Definition. The fibres of f over supp T are called *wild fibres*.

Noting that if C is indecomposable of canonical type then $\dim H^0(C, \mathcal{O}_C) = 1$ (see Mumford [10], p. 332), we get

Proposition 3. *Every wild fibre is a multiple fibre.*

In the following, we consider only relatively minimal fibrations $f: X \to B$, i.e., no exceptional curve of the first kind is a component of a fibre.

Theorem 2. *Let $f: X \to B$ be a relatively minimal elliptic or quasi-elliptic fibration and let $R^1 f_* \mathcal{O}_X = L \oplus T$. Then*

$$\omega_X = f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}(\sum a_\lambda P_\lambda)$$

where

(i) *$m_\lambda P_\lambda$ are the multiple fibres*

(ii) $0 \leqq a_\lambda < m_\lambda$

(iii) *$a_\lambda = m_\lambda - 1$ if $m_\lambda P_\lambda$ is not wild*

(iv) $\deg (L^{-1} \otimes \omega_B) = 2p(B) - 2 + \chi(\mathcal{O}_X) + \text{length } T$

where $p(B)$ is the genus of B.

1) In the notation of [10], a curve $D = \sum n_i E_i$ is said to be of canonical type if $(K \cdot E_i) = (D \cdot E_i) = 0$ for all i.

Note that in the case char$(k)=0$ or in the complex analytic case there are no wild fibres, so that $a_i=m_i-1$; see Kodaira [4], p. 772, Th. 12.

Proof. For any non-multiple fibre $f^{-1}(y)$ we have

$$\mathcal{O}_{f^{-1}(y)}\otimes\omega_X\cong\omega_{f^{-1}(y)}\cong\mathcal{O}_{f^{-1}(y)},$$

hence if $y_1,\cdots,y_r$ are distinct general points of B the cohomology sequence of

$$0\longrightarrow\omega_X\longrightarrow\omega_X\otimes\mathcal{O}(\sum_{i=1}^r f^{-1}(y_i))\longrightarrow\bigoplus_{i=1}^r\mathcal{O}_{f^{-1}(y_i)}\longrightarrow 0$$

yields

$$\dim\left|\omega_X\otimes\mathcal{O}(\sum_{i=1}^r f^{-1}(y_i))\right|\geqq 0$$

for large enough r. If D is a divisor in the linear system above, we have

$$(D\cdot f^{-1}(y))=0$$

hence we can write

$$K_X\equiv(\text{sum of fibres})+\Delta$$

where $\Delta\geqq 0$ is contained in a union of fibres and does not contain fibres of f. Let Δ_0 be a connected component of Δ and let $C=f^{-1}(y)$ be the fibre containing Δ_0. Then Δ_0 is a rational submultiple of C, i.e., we have

$$C=mP,\qquad\Delta_0=aP$$

where P is indecomposable of canonical type and $0\leqq a<m$. This follows from

Lemma. *Let $D=\sum n_iC_i$ be an effective divisor on a surface X with each C_i irreducible. Assume that*

$$(C_i\cdot D)\leqq 0,\qquad \textit{all } i$$

and that D is connected.

Then every divisor $Z=\sum m_iC_i$ satisfies $Z^2\leqq 0$ and equality holds if and only if $D^2=0$ and $Z=\lambda D$, $\lambda\in\boldsymbol{Q}$.

Proof. Write $x_i=m_i/n_i$. We have

$$\begin{aligned}Z^2&=\sum x_ix_jn_in_j(C_i\cdot C_j)\\&\leqq\sum x_i^2n_i^2(C_i\cdot C_i)+\sum_{i\neq j}\frac{1}{2}(x_i^2+x_j^2)n_in_j(C_i\cdot C_j)\\&=\sum x_i^2n_i(C_i\cdot D)\leqq 0.\end{aligned}$$

If equality holds everywhere, we have either $x_i=x_j$ or $(C_i\cdot C_j)=0$ for all i,j ; since D is connected, x_i is constant, i.e., $m_i=\lambda n_i$, $\lambda\in\boldsymbol{Q}$. q. e. d.

Going back to the proof that $\Delta_0=aP$, if Δ_ν are the connected components of Δ, we have

$$0=K_X^2=\sum\Delta_\nu^2;$$

since each $\Delta_\nu^2\leqq 0$ by the previous lemma, we must have $\Delta_\nu^2=0$ and now the equality case of the lemma proves that Δ_ν is a rational multiple of the fibre containing it.

We have proved that

$$\omega_X = f^*\mathcal{O}_B(\mathfrak{A}) \otimes \mathcal{O}(\sum a_\lambda P_\lambda)$$

for some divisor $\mathfrak{A} \in \operatorname{div}(B)$ and integers a_λ with $0 \leqq a_\lambda < m_\lambda$. We deduce that

$$f_*(\omega_X) = \mathcal{O}_B(\mathfrak{A}).$$

Now the duality theorem for a map says that

$$\begin{aligned} f_*\omega_X &= \operatorname{Hom}(R^1 f_*\mathcal{O}_X, \omega_B) \\ &= L^{-1} \otimes \omega_B \end{aligned}$$

because the dual of the torsion sheaf is 0 ; this can be found in Deligne-Rapoport [2], pp. 19–20, formula (2. 2. 3). Hence

$$\omega_X = f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}(\sum a_\lambda P_\lambda).$$

The spectral sequence of the map f yields

$$\begin{aligned} \chi(\mathcal{O}_X) &= \chi(\mathcal{O}_B) - \chi(R^1 f_*\mathcal{O}_X) \\ &= \chi(\mathcal{O}_B) - \chi(L) - \text{length } T \\ &= -\deg L - \text{length } T, \end{aligned}$$

by the Riemann-Roch theorem on the curve B, and since $\deg(\omega_B) = 2p(B) - 2$ we obtain (iv) of Theorem 2.

It remains to prove (iii), and this follows from

Proposition 4. *Let m_λ, P_λ, a_λ be as in Theorem 2 and let*

$$\nu_\lambda = \operatorname{order}(\mathcal{O}_{P_\lambda} \otimes \mathcal{J}_{P_\lambda}^{-1})$$

where $\mathcal{J}_{P_\lambda}$ is the sheaf of ideals of P_λ, be the order of the normal sheaf of P_λ in X. Then we have

i) *ν_λ divides m_λ and $a_\lambda + 1$,*

ii) $\dim H^0(P_\lambda, \mathcal{O}_{(\nu_\lambda+1)P_\lambda}) \geqq 2$, $\dim H^0(P_\lambda, \mathcal{O}_{\nu_\lambda P_\lambda}) = 1$,

iii) $\dim H^0(P_\lambda, \mathcal{O}_{rP_\lambda})$ *is non-decreasing with r.*

In particular, if $a_\lambda < m_\lambda - 1$ then $\nu_\lambda < m_\lambda$ and this is equivalent to the multiple fibre $m_\lambda P_\lambda$ being wild.

Proof. Let us write $m, P, a, \nu, \mathcal{J}$ for $m_\lambda, P_\lambda, a_\lambda, \nu_\lambda, \mathcal{J}_{P_\lambda}$. If $r \geqq s \geqq 1$, the restriction map $\mathcal{O}_{rP} \to \mathcal{O}_{sP}$ is surjective, hence $\dim H^1(P, \mathcal{O}_{rP})$ is non-decreasing with r. Since $\chi(\mathcal{O}_{rP}) = 0$, this proves that $\dim H^0(P, \mathcal{O}_{rP})$ is non-decreasing too.

We have an isomorphism

$$\mathcal{O}_P \otimes \mathcal{J}^\nu \cong \mathcal{O}_P$$

and via this isomorphism we get an exact sequence

$$0 \to \mathcal{O}_P \to \mathcal{O}_{(\nu+1)P} \xrightarrow{\text{res}} \mathcal{O}_{\nu P} \to 0$$

where res is the restriction. Since constants in $H^0(P, \mathcal{O}_{(\nu+1)P})$ are mapped into constants in $H^0(P, \mathcal{O}_{\nu P})$, the cohomology sequence shows that $\dim H^0(P, \mathcal{O}_{(\nu+1)P}) \geqq 2$. Finally, ν divides both m and $a+1$, because $\mathcal{O}_P \otimes \mathcal{J}^{-m} \cong \mathcal{O}_P$ (trivial) and

$$\mathcal{O}_P \otimes \mathcal{J}^{-a-1} \cong \omega_P \cong \mathcal{O}_P$$

(Mumford [10], p. 333). q. e. d.

It is shown in Raynaud [13], Prop. 6. 3. 5, that m_λ/ν_λ is a power of the characteristic p of k. In particular the multiplicity of a wild fibre is divisible by p, and wild fibres do not occur in char. 0.

Corollary. *If* $\dim H^1(X, \mathcal{O}_X) \leqq 1$ *we have either*

$$a_\lambda + 1 = m_\lambda \quad \text{or} \quad \nu_\lambda + a_\lambda + 1 = m_\lambda.$$

Proof. Since $\chi(\mathcal{O}_{(\nu+1)P}) = 0$ and $\dim H^0(P, \mathcal{O}_{(\nu+1)P}) \geqq 2$, using duality we find that

$$\dim H^0(P, \omega_{(\nu+1)P}) \geqq 2.$$

Now the cohomology sequence of

$$0 \longrightarrow \omega_X \longrightarrow \mathcal{J}^{-\nu-1} \otimes \omega_X \longrightarrow \omega_{(\nu+1)P} \longrightarrow 0$$

yields

$$\dim H^0(X, \mathcal{J}^{-\nu-1} \otimes \omega_X) > \dim H^0(X, \omega_X),$$

since we have $\dim H^1(X, \omega_X) = \dim H^1(X, \mathcal{O}_X) \leqq 1$ by hypothesis. This increase in dimension is possible only if $\nu + a + 1 \geqq m$, or $1 + (a+1)/\nu \geq m/\nu$. Therefore $(a+1)/\nu = m/\nu$ or $m/\nu - 1$. q. e. d.

We conclude this section with a remark on hyperelliptic or quasi-hyperelliptic surfaces.

Proposition 5. *Let* $f: X \to E$, $E = \mathrm{Alb}(X)$ *be an hyperelliptic surface. Then every fibre of* f *is smooth.*

Moreover if $f: X \to E$ *is quasi-hyperelliptic then every fibre of* f *is a rational curve with a cusp, i.e., there are no reducible fibres.*

Proof. Since $p(E) = 1$, $\chi(\mathcal{O}_X) = 0$ and $K_X \sim 0$ ($\sim$ is numerical equivalence), Theorem 2 gives

$$(\text{length } T) f^{-1}(y) + \sum a_\lambda P_\lambda \sim 0$$

therefore there are no multiple fibres. Also since the Picard number is $\rho \leqq B_2 = 2$, there are no reducible fibres. In the elliptic case the smoothness of f follows by considering the differential $f^*(\omega)$, where $\omega \in \Gamma(\Omega_E^1)$. $f^*(\omega)$ will only be zero at the points where f is not smooth and since these are finite in number,

$$c_{2,X} = [\text{number of zeroes of } f^*(\omega) \text{ counted with multiplicity}].$$

But $c_{2,X} = 0$, so $f^*\omega$ has no zeroes, so f is smooth. In any elliptic or quasi-elliptic surface, every irreducible fibre is either a) non-singular elliptic, b) rational with a node, or c) rational with a cusp. In the quasi-elliptic case, the generic fibre is of type (c) and since such a curve cannot specialize to type (a) or type (b), every irreducible fibre is rational with a cusp.

q. e. d.

2. Proof of Theorem 1

We shall prove here that if $f: X \to B$ is elliptic or quasi-elliptic, $(K_X \cdot C) \geqq 0$ for all curves C and $K_X^2 = 0$, then :

$$(*) \qquad |4K_X| \neq \phi \quad \text{or} \quad |6K_X| \neq \phi.$$

In proving this result we may assume $p_g = 0$ and use Table 1 as a list of numerical invariants. Theorem 2 implies

$$p_g = \dim H^0(B, L^{-1} \otimes \omega_B)$$

and since $\chi(\mathcal{O}_X) \geqq 0$, the Riemann-Roch theorem on B shows that $p_g = 0$ implies $p(B) = 0$ or 1 and if $p(B) = 1$ we must also have $T = (0)$. So if $p(B) = 1$ there are no wild fibres and $a_\lambda = m_\lambda - 1$ in Theorem 2. If there is a multiple fibre, it is easily seen that $|2K_X| \neq \phi$. If there are no multiple fibres at all, then

$$\omega_X = f^*(L^{-1} \otimes \omega_B)$$

and $\deg(L^{-1} \otimes \omega_B) = 0$, thus $K_X \sim 0$ and X is hyperelliptic or quasi-hyperelliptic.

Theorem 3. *If X is hyperelliptic or quasi-hyperelliptic, then there is a second structure $f: X \to \boldsymbol{P}^1$ of X as an elliptic surface over $\boldsymbol{P}^1$.*

Proof. By the results in [10], it is sufficient to show the existence of a curve C of canonical type, transversal to the Albanese fibration, $\varphi: X \to E$ with $E = \mathrm{Alb}(X)$. Let F_t be the fibre $\varphi^{-1}(t)$ of φ over $t \in E$. There exists a divisor D on X such that

$$(D^2) = 0, \qquad (D \cdot F_0) > 0,$$

for example some linear combination of an ample divisor and F_0 ; let

$$D_t = D + F_t - F_0.$$

There is a point $t \in E$ such that $|D_t| \neq \phi$. If not, use $\chi(\mathcal{O}(D_t)) = 0$ and the Riemann-Roch theorem to prove

$$\dim H^0(X, \mathcal{O}(D_t)) = \dim H^1(X, \mathcal{O}(D_t)) = 0$$

for all t. The cohomology sequence of

$$0 \to \mathcal{O}(D_t) \to \mathcal{O}(D + F_t) \xrightarrow{r_{F_0}} \mathcal{O}_{F_0} \otimes \mathcal{O}(D) \to 0$$

then gives an isomorphism

$$r_{F_0}: H^0(X, \mathcal{O}(D + F_t)) \simeq H^0(F_0, \mathcal{O}_{F_0} \otimes \mathcal{O}(D))$$

where r_{F_0} is the restriction. Since $(D \cdot F_0) > 0$, there is a non-trivial section $\sigma \in \Gamma(\mathcal{O}_{F_0} \otimes \mathcal{O}(D))$, and let $s_t = r_{F_0}^{-1}(\sigma)$. Clearly $X = \text{closure} \bigcup_{t \neq 0} \mathrm{div}(s_t)$ and $\mathrm{div}(s_t) \cap F_0$ has support in $\mathrm{div}(\sigma)$, for all $t \neq 0$. It follows that as $t \to 0$ we must have $\mathrm{div}(s_t) \to F_0 + C \equiv D + F_0$, and $C \in |D|$, proving our assertion.

We have found a curve $C > 0$ with $(C^2) = 0$ and $(C \cdot F_0) > 0$, and we claim that C is of canonical type. In fact, since $K_X \sim 0$ and $(C^2) = 0$, our assertion will follow from the fact that X has no irreducible curve Γ with $(\Gamma^2) = -2$. Such a curve Γ

cannot be transversal to the Albanese fibering because Γ is rational, and cannot be a component of a fibre, since every fibre is irreducible by Proposition 5.

q. e. d.

In view of Theorem 3, we have only to examine the case in which $p(B)=0$. Since B is rational, the canonical bundle formula becomes

$$K_X \equiv rf^{-1}(y)+\sum_{\lambda} a_\lambda P_\lambda$$

where

$$r = -2+\chi(\mathcal{O}_X)+\text{length } T.$$

If H is an ample divisor on X, since $(K_X \cdot H) \geqq 0$ we have

$$r+\sum_{\lambda} \frac{a_\lambda}{m_\lambda} \geqq 0.$$

Moreover

$$\dim |nK_X| = nr+\sum_{\lambda} \left[\frac{na_\lambda}{m_\lambda}\right].$$

It is now easy to see, using $\chi(\mathcal{O}_X) \geqq 0$ and Proposition 4, Corollary that we can have only the following cases:

(A) length $T=0$, so $a_\lambda = m_\lambda - 1$, $\nu_\lambda = m_\lambda$.

If $\chi(\mathcal{O}_X)=0$, then there are at least 3 multiple fibres and we can have:

a) there are 4 or more multiple fibres, i. e., $m_\lambda \geq 2$, $1 \leq \lambda \leq 4$, and then $|2K_X| \neq \phi$.

b) there are 3 multiple fibres with all multiplicities $m_\lambda \geq 3$. Then $|3K_X| \neq \phi$.

c) there are 3 multiple fibres with $m_1=2$, m_2, $m_3 \geq 4$. Then $|4K_X| \neq \phi$.

d) there are 3 multiple fibres with $m_1=2$, $m_2=3$, $m_3 \geq 6$. Then $|6K_X| \neq \phi$.

If $\chi(\mathcal{O}_X)=1$, then there are at least 2 multiple fibres, m_1, $m_2 \geq 2$, and $|2K_X| \neq \phi$.

If $\chi(\mathcal{O}_X) \geq 2$, then $|K_X| \neq \phi$.

(B) length $T=1$. If $\chi(\mathcal{O}_X)=0$, then $|K_X| = \phi$, so $\dim H^1(\mathcal{O}_X)=1$ and Prop. 4, Cor. applies. So if $f^{-1}(P_1)$ is the wild fibre, then we have $a_1=m_1-1$ or $a_1=m_1-1-\nu_1$ where $\nu_1 | \text{g. c. d.} (m_1, a_1+1)$, while $a_\lambda = m_\lambda - 1$, $\nu_\lambda = m_\lambda$ for $\lambda \geq 2$. Moreover there are at least 2 multiple fibres and we can have:

a′) there are 2 or more multiple fibres with $a_\lambda = m_\lambda - 1$ and then $|2K_X| \neq \phi$.

b′) the wild fibre satisfies $m_1=3$, $a_1=1$, $\nu_1=1$ (hence char.$=3$) and the tame fibre satisfies $m_2 \geq 3$. Then $|3K_X| \neq \phi$.

c_1') the wild fibre satisfies $m_1=4$, $a_1=1$, $\nu_1=2$ (hence char.$=2$) and the tame fibre satisfies $m_2 \geq 4$. Then $|4K_X| \neq \phi$.

c_2') the wild fibre satisfies $m_1=\mu_1\nu_1$, where $\mu_1 \geq 4$ (any positive char.). In this case, $a_1/m_1 \geq 1/2$ and $|2K_X| \neq \phi$.

d_1') the wild fibre satisfies $m_1=2\nu_1$, $a_1=\nu_1-1$, $\nu_1 \geq 3$ (hence char.$=2$) and the tame fibre satisfies $m_2 \geq 3$. then $|3K_X| \neq \phi$.

d_2') the wild fibre satisfies $m_1=3\nu_1$, $a_1=2\nu_1-1$, $\nu_2 \geq 2$ (hence char.$=3$). In this case $|2K_X| \neq \phi$.

If $\chi(\mathcal{O}_X)\geq 1$, then $|K_X| \neq \phi$.
(C) length $T\geq 2$, then also $|K_X| \neq \phi$.

If we specialize to the case $\kappa=0$, then we easily get the following list of possible multiple fibres for elliptic or quasi-elliptic surfaces $f\colon X\to \boldsymbol{P}^1$ with K_X a torsion divisor :

		length T	$\chi(\mathcal{O}_X)$	a_λ/m_λ (* = wild fibre)	order K_X	char.
	i)	0	0	(1/2, 1/2, 1/2, 1/2)	2	
	ii)	0	0	(2/3, 2/3, 2/3)	3	
tame	iii)	0	0	(1/2, 3/4, 3/4)	4	
cases	iv)	0	0	(1/2, 2/3, 5/6)	6	
	v)	0	1	(1/2, 1/2)	2	
	vi)	0	2	none	1	
	vii)	1	0	(0/2*, 1/2, 1/2)	2	2
	viii)	1	0	(1/2*, 1/2)	2	2
	ix)	1	0	(1/3*, 2/3)	3	3
	x)	1	0	(1/4*, 3/4)	4	2
wild	xi)	1	0	(2/4*, 1/2)	2	2
cases	xii)	1	0	(2/6*, 2/3)	3	2
	xiii)	1	0	(3/6*, 1/2)	2	3
	xiv)	1	1	(0/2*)	1	2
	xv)	2	0	one or two wild fibres $0/p^r$	1	p

Note that each of the wild cases may be thought of as coming from the confluence of 2 tame fibres in one of the tame cases.

3. Analysis of Hyperelliptic Surfaces

In this section, we study more closely surfaces X such that :

a) $\kappa = 0$

b) the Albanese mapping is $\pi : X \to E$, E elliptic

c) almost all fibres C_x of π are non-singular.

By the Table of the Introduction, it follows also that

d) $B_2 = 2$, $c_2 = 0$, $\chi(\mathcal{O}_X) = 0$.

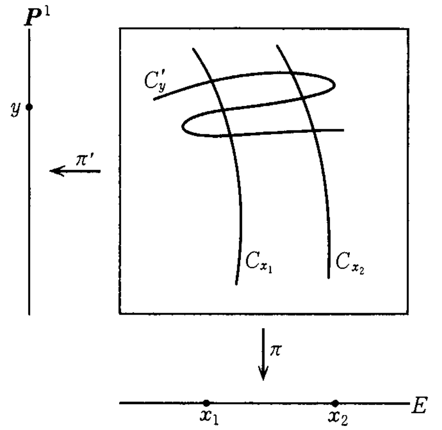
Moreover, by Proposition 5 it follows that

c′) *all* fibres C_x are non-singular elliptic.

By Theorem 3, § 2, we see :

e) There is a second elliptic pencil $\pi' : X \to \boldsymbol{P}^1$ on X.

We want to compare π and π' and see the effect of 2 simultaneous elliptic fibrations! Let C'_y be the fibres of π'. Then all the C'_y are finite coverings of E :



Hence all the C'_y are either non-singular elliptic or multiples of non-singular elliptic curves, and

$$p_y = \text{res } \pi : C'_y \to E$$

is an isogeny. Let $S = \{y \in \boldsymbol{P}^1 | C'_y \text{ multiple}\}$. p_y defines a pull-back on Pic^0:

$$\text{Pic}^0(C'_y) \overset{p_y^*}{\longleftarrow} \text{Pic}^0(E).$$

Choosing a base point $x_0 \in E$, we can identify $\text{Pic}^0(E)$ with E by associating the sheaf $\mathcal{O}_E(x - x_0)$ with the point x. As usual, this makes E into an algebraic group with identity x_0. Now we cannot choose base points on each C'_y varying nicely with y unless $\pi' : X \to \boldsymbol{P}^1$ has a section. However, we can instead note that $\text{Pic}^0(C'_y)$ *acts* canonically on C'_y by translations: i.e., the sheaf L of degree 0 maps $u \in C'_y$ to the unique point v such that $L(u) \cong \mathcal{O}_{C'_y}(v)$. Then via the maps p_y^*, we find that E is acting by translations simultaneously on *all* the curves C'_y. If we stick to the non-multiple curves, it follows easily that this is an algebraic action of E:

$$\sigma_0 : E \times \pi'^{-1}(\boldsymbol{P}^1 - S) \to \pi'^{-1}(\boldsymbol{P}^1 - S).$$

But since X is a minimal model, any automorphism of the Zariski-open set $\pi'^{-1}(\boldsymbol{P}^1 - S)$ extends to an automorphism of X so we actually get an action:

$$\sigma : E \times X \to X.$$

To relate this action to π, say $x \in E$, $u \in C'_y$. Then x takes u to v where

$$\pi^*(\mathcal{O}_E(x - x_0)) \otimes \mathcal{O}_{C'_y}(u) \cong \mathcal{O}_{C'_y}(v).$$

Let $n = (C'_y \cdot C_x) = (\text{degree of res } \pi : C'_y \to E)$. Then taking $\text{Norm}_{C'_y/E}$ of the 2 sides of the above isomorphism:

$$\mathcal{O}_E(nx - nx_0 + \pi u) \cong \mathcal{O}_E(\pi v),$$

hence we get a commutative diagram

$$(*)\qquad \begin{tikzcd} X \arrow[r, "\text{action of } x"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ E \arrow[r, "\text{translation}"', "\text{by } n(x-x_0)"'] & E \end{tikzcd} \quad ; \quad \begin{tikzcd} E\times X \arrow[r, "\sigma"] \arrow[d, "1\times\pi"'] & X \arrow[d, "\pi"] \\ E\times E \arrow[r] & E \\ x,y \arrow[r] & nx+y \end{tikzcd}$$

We can now use this action of E to describe the whole surface X as follows : let $E_0 = C_{x_0}$ be the fibre over x_0, and let $A_n = \text{Ker}\ (n_E : E \to E)$ considered as a subgroup scheme of E. Then by $(*)$ the action of A_n on X preserves the fibres of π, hence A_n acts on E_0, and give this action the name α :

$$\alpha : A_n \to \text{Aut}(E_0) = \text{group scheme of automorphisms of } E_0.$$

Then by restriction of the action σ of E, we get a morphism :

$$\tau : E \times E_0 \to X$$

which by $(*)$ fits into a diagram :

$$\begin{tikzcd} E\times E_0 \arrow[rr, "\tau"] \arrow[dr, "n_E \cdot p_1"'] & & X \arrow[dl, "\pi"] \\ & E & \end{tikzcd}$$

Note that

$$\begin{aligned} \tau(x,y) = \tau(x',y') &\Leftrightarrow \sigma(x-x',y) = y' \\ &\Leftrightarrow x-x' \in A_n \text{ and } \alpha(x-x')(y) = y' \end{aligned}$$

hence it follows that $X \cong$ quotient $(E \times E_0/A_n)$, via the action

$$x(u, v) = (u+x,\ \alpha(x)(v)), \quad x \in A_n,\ u \in E,\ v \in E_0.$$

If we replace E by $E_1 = E/\text{Ker}\ \alpha$, this proves :

Theorem 4. *Every hyperelliptic surface X is of the form :*

$$X = E_1 \times E_0/A, \qquad E_1,\ E_0 \textit{ elliptic curves}$$

where A is a finite subgroupscheme of E_1, and A acts by

$$k(u, v) = (u+k,\ \alpha(k)(v))$$

for some injective homomorphism

$$\alpha : A \to \text{Aut}(E_0).$$

Moreover, the 2 elliptic fibrations on X are given by :

$$\begin{tikzcd} E_1 \times E_0/A \arrow[d] \\ E_1/A \\ \text{(elliptic)} \end{tikzcd} \qquad \begin{tikzcd} E_1 \times E_0/A \arrow[d] \\ E_0/\alpha(A) \\ \wr\| \\ \boldsymbol{P}^1 \end{tikzcd}$$

This theorem can easily be used a) to classify such X's and b) to compute the order of K_X in $\mathrm{Pic}\,(X)$. We use the fact that choosing a base point $0 \in E_0$, $\mathrm{Aut}(E_0)$ becomes a semi-direct product:

$$\mathrm{Aut}\,(E_0) = E_0 \cdot \mathrm{Aut}\,(E_0, 0)$$

normal subgroup of translations — finite, discrete group of autos, fixing 0

Note that $\alpha(A) \not\subset E_0$, or else $E_0/\alpha(A)$ would be elliptic instead of rational as required. Moreover, from the tables in Lang [5], Appendix 1, we find:

$$\begin{aligned}
\mathrm{Aut}\,(E_0, 0) &= \{1_E, -1_E\} \cong \boldsymbol{Z}/2\boldsymbol{Z} && \text{if } j(E_0) \neq 0, 12^3 \\
&\cong \boldsymbol{Z}/4\boldsymbol{Z} && \text{if } j(E_0) = 12^3, \text{ char} \neq 2, 3 \\
&\cong \boldsymbol{Z}/6\boldsymbol{Z} && \text{if } j(E_0) = 0, \text{ char} \neq 2, 3
\end{aligned}$$

$\cong$ semi-direct product $\boldsymbol{Z}/4\boldsymbol{Z} \cdot \boldsymbol{Z}/3\boldsymbol{Z}$, $\boldsymbol{Z}/3\boldsymbol{Z}$ normal, $i \in \boldsymbol{Z}/4\boldsymbol{Z}$ acting by mult. by $(-1)^{2i}$

if $j(E_0) = 0$, char $= 3$

$\cong$ semi-direct product (Quat. gp. of order 8) $\cdot\, \boldsymbol{Z}/3\boldsymbol{Z}$, Quat. gp. normal, $\boldsymbol{Z}/3\boldsymbol{Z}$ permuting cyclically $i, j, k \in$ Quat. gp.

if $j(E_0) = 0$, char $= 2$

The important point here is that since A is commutative, so is $\alpha(A)$ and now even in the last 2 nasty cases, the maximal *abelian* subgroups are still $\boldsymbol{Z}/4\boldsymbol{Z}$ and $\boldsymbol{Z}/6\boldsymbol{Z}$, which in all cases are cyclic.

Let $k \in A$ be such that

$$\mathrm{Im}\,\alpha(k) \in \mathrm{Aut}\,(E_0)/E_0$$

generates

$$\mathrm{Im}\,\alpha(A) \subset \mathrm{Aut}\,(E_0)/E_0.$$

Then $\alpha(k) \notin E_0$, hence it has some fixed point. Replacing 0 by this fixed point, it follows that $\alpha(A)$ itself is a direct product:

$$\alpha(A) = A_0 \cdot \boldsymbol{Z}/n\boldsymbol{Z}$$

finite gp. scheme of translations $A_0 \subset E_0$ — cyclic gp. generated by k, $n = 2, 3, 4$ or 6

Since A_0 and k must commute, $A_0 \subset$ (fix pt. set F of k). Again referring to Lang to check the fix point sets, we find:

a) $n = 2$, (so $k = -1_E$), then $F = \mathrm{Ker}\ 2_{E_0}$

b) $n = 3$, then $\sharp F = 3$ so $F \cong \boldsymbol{Z}/3\boldsymbol{Z}$ if char $\neq 3$
$F \cong \alpha_3$ if char $= 3$ (because E_0 is supersingular !)
c) $n = 4$, then $\sharp F = 2$ so $F \cong \boldsymbol{Z}/2\boldsymbol{Z}$ if char $\neq 2$
$F \cong \alpha_2$ if char $= 2$ (because E_0 is supersingular !)
d) $n = 6$, then $F = (e)$

We can now mechanically compile a list of all possible $\alpha(K)$' s, hence all possible X' s :

a1) $E_1 \times E_0/(\boldsymbol{Z}/2\boldsymbol{Z})$;	action $(x, y) \mapsto (x+a, -y)$
a2) $E_1 \times E_0/(\boldsymbol{Z}/2\boldsymbol{Z})^2$;	action $(x, y) \mapsto (x+a, -y)$, $(x+b, y+c)$ (here char $\neq 2$)
a3) $E_1 \times E_0/(\boldsymbol{Z}/2\boldsymbol{Z}) \cdot \mu_2$;	action $(x, y) \mapsto (x+a, -y)$, μ_2 acts by transl. on both factors.
b1) $E_1 \times E_0/(\boldsymbol{Z}/3\boldsymbol{Z})$;	action $(x, y) \mapsto (x+a, \omega y)$ where $j(E_0) = 0$, ω : $E_0 \to E_0$ an automorphism of order 3
b2) $E_1 \times E_0/(\boldsymbol{Z}/3\boldsymbol{Z})^2$;	action $(x, y) \mapsto (x+a, \omega y)$, $(x+b, y+c)$, E_0, ω as before and $\omega c = c$, order $c = 3$ (here char $\neq 3$)
c1) $E_1 \times E_0/(\boldsymbol{Z}/4\boldsymbol{Z})$;	action $(x, y) \mapsto (x+a, iy)$, where $j(E_0) = 12^3$, $i : E_0 \to E_0$ an automorphism of order 4
c2) $E_1 \times E_0/(\boldsymbol{Z}/2\boldsymbol{Z}) \cdot (\boldsymbol{Z}/4\boldsymbol{Z})$;	action $(x, y) \mapsto (x+a, iy)$, $(x+b, y+c)$, E_0, i as before and $ic = c$, order $c = 2$ (here char $\neq 2$)
d) $E_1 \times E_0/\boldsymbol{Z}/6\boldsymbol{Z}$;	action $(x, y) \mapsto (x+a, -\omega y)$, E_0, ω as in b.

The list obtained here coincides with the classical list in characteristic 0 (see Bagnera and DeFranchis [1], Enriques and Severi [3], pp. 283–392, Šafarevič [14], p. 181). Note here that the requirements $A_0 \subset E_0$ *and* $A \subset E_1$ eliminate the possibilities $n=2$, $A_0 = \operatorname{Ker} 2_{E_0}$ and $n=3$ or 4, $A_0 = \alpha_3$ or α_2. A striking feature of this list are the missing cases. From a moduli point of view, even in case a1), one may ask what happens if we start with such an X in characteristic 0 and specialize to characteristic 2 in such a way that the point a goes to $0 \in E_1$. One would hope for instance that the moduli spaces of these X's were proper over $\boldsymbol{Z}[j(E_0), j(E_1)]$ but this is not true. The answer seems to be that the X's become quasi-hyperelliptic ! This is an interesting point to investigate.

The order of K_X is easily obtained, since if ω is the 2-form on $E_1 \times E_0$ with no zeros or poles, then

$$\text{order of } K_X = \text{least } n \text{ such that } A \text{ acts trivially on } \omega^{\otimes n}$$

and we find

$$\begin{aligned} \text{order of } K_X &= 2, 3, 4, 6 \text{ in cases a), b), c), d)} \\ &\quad\ \text{and } \operatorname{char}(k) \neq 2, 3 \\ &= 1, 3, 1, 3 \text{ in cases a), b), c), d)} \\ &\quad\ \text{and } \operatorname{char}(k) = 2 \\ &= 2, 1, 4, 2 \text{ in cases a), b), c), d)} \\ &\quad\ \text{and } \operatorname{char}(k) = 3 \end{aligned}$$

It is interesting to check exactly which wild multiple fibres (in the sense of § 1) occur here for $\pi' : X \to \boldsymbol{P}^1$. One can check that we get the following cases in the list of § 2 :

case	char. $\neq$ 2,3	char. 3	char. 2
a	(i)	(i)	(xv)-one or two fibres 0/2
b	(ii)	(xv)-one fibre 0/3	(ii)
c	(iii)	(iii)	(xv)-one fibre 0/4
d	(iv)	(xiii)	(xii)

4. Proof of Theorem 5

First of all, let X be a $K3$-surface, i. e., $K_X \equiv 0$, $B_2 = 22$, $B_1 = 0$, $\chi(\mathcal{O}_X) = 2$, $H^1(\mathcal{O}_X) = (0)$ (cf. Table in Introduction). Then

i) if $\pi : Y \to X$ were a connected étale covering of degree d, one would have $K_Y \equiv \pi^* K_X \equiv 0$, hence Y would be a surface in the Table too. But

$$c_{2,Y} = \pi^{-1}(c_{2,X})$$

hence

$$\deg c_{2,Y} = 24d > 24$$

and there are no such surfaces in the table.

ii) Since $H^-(\mathcal{O}_X)$ is isomorphic to the tangent space to $\mathrm{Pic}(X)$, it follows that Pic^0_X is a finite discrete group. Let $L = \mathcal{O}_X(D)$ represent a point of Pic^0_X. Then $(D^2) = (D \cdot K_X) = 0$, so $\chi(L) = \chi(\mathcal{O}_X) = 2$, Therefore $H^0(L) \neq (0)$ or $H^2(L) \neq (0)$. But by Serre duality $H^2(L)$ is dual to $H^0(L^{-1})$. Thus L or L^{-1} is represented by an effective divisor E, but since it is in Pic^0, $E = 0$. So finally $L \cong \mathcal{O}_X$ and $\mathrm{Pic}^0_X = (0)$.

Secondly, let X be a surface with $K_X \equiv 0$, $B_2 = 14$, $B_1 = 2$, $\chi(\mathcal{O}_X) = 1$, dim $H^1(\mathcal{O}_X) = 1$. Since $B_1 > 0$, X has a positive dimensional Picard variety. This means that X does indeed support invertible sheaves $L = \mathcal{O}_X(D)$ such that D is numerically equivalent to zero but $D \not\equiv 0$. Then $\chi(L) = \chi(\mathcal{O}_X) = 1$, so $H^0(L) \neq (0)$ or $H^2(L) \neq (0)$. As above, Serre duality shows that $H^2(L) \neq (0) \Rightarrow H^0(L^{-1}) \neq (0)$, so L or L^{-1} is represented by an effective divisor E. E numerically equivalent to 0 implies $E = 0$, so $L \cong \mathcal{O}_X$ contrary to our assumption.

Alternatively, we could argue that because $B_1 > 0$, X has connected cyclic étale coverings $\pi : Y \to X$ of every order d prime to the characteristic. As in (i) above, $c_{2,Y} = 12d$ and if $d > 2$, no such Y appears in our table.

Arguments of the above type, using μ_p or α_p-coverings of X (cf. Mumford [11]) do not quite seem to be strong enough to prove that if X is a $K3$-surface, then $H^0(X, \Omega^1_X) = (0)$. It remains a very intriguing open question[1)] whether or not

1) (added in proof) Rudakov and Šafarevič have just settled this. They show that Ω^1_X has no sections when X is a $K3$-surface. Moreover, P. Deligne has used their result to prove that all $K3$-surfaces lift to char. 0.

$H^0(X, \Omega^1_X)$ is (0) for every $K3$-surface of char. p.

Thirdly, let X be a surface with $K_X \equiv 0$, $B_2=10$, $B_1=0$, $\chi(\mathcal{O}_X)=1$, $\dim H^1(\mathcal{O}_X)=1$. Let $\{a_{ij}\} \in Z^1(\mathcal{O}_X)$ be a non-trivial cocycle and consider the $\boldsymbol{G}_a$-bundle

$$\pi : W \to X$$

defined locally as $\boldsymbol{A}^1 \times U_i$, coordinate z_i on $\boldsymbol{A}^1$, and glued by

$$z_i = z_j + a_{ij}.$$

If ω is a non-zero 2-form on X with no zeroes or poles,

$$\eta = dz_i \wedge \omega$$

is a non-zero 3-form on W with no zeroes or poles, i.e., $K_W \equiv 0$. Now since $H^1(\mathcal{O}_X)$ is 1-dimensional, there is a constant $\lambda \in k$ such that $\{a_{ij}^p\}$, $\{\lambda a_{ij}\}$ are cohomologous :

$$a_{ij}^p = \lambda a_{ij} + b_i - b_j$$

Consider the global function f on W defined locally by

$$f = z_i^p - \lambda z_i - b_i.$$

Let Y be the 2-dimensional scheme $f=0$. If $\lambda \neq 0$, Y is étale over X, hence non-singular. If $\lambda=0$, still $b_i \notin \mathcal{O}_X^p$ (or else $a_{ij}=b_i^{1/p}-b_j^{1/p}$ is cohomologous to zero), so Y is a reduced Gorenstein surface. Since $K_W \equiv 0$ and Y has trivial normal sheaf in W, in both cases $\omega_Y \cong \mathcal{O}_Y$. Thus

$$\chi(\mathcal{O}_Y) \leq \dim H^0(\mathcal{O}_Y) + \dim H^2(\mathcal{O}_Y) = \dim H^0(\mathcal{O}_Y) + \dim H^0(\omega_Y) = 2.$$

On the other hand,

$$\operatorname{res} \pi : Y \to X$$

is finite and flat and $(\operatorname{res} \pi)_* \mathcal{O}_Y$ is filtered by the subsheaves :

$$\mathcal{O}_X \subset [\mathcal{O}_X \oplus \mathcal{O}_X \cdot z_i] \subset [\mathcal{O}_X \oplus \mathcal{O}_X \cdot z_i \oplus \mathcal{O}_X \cdot z_i^2] \subset \cdots \subset (\operatorname{res} \pi)_* \mathcal{O}_Y.$$

The quotients here are all isomorphic to $\mathcal{O}_X$, thus

$$\chi(\mathcal{O}_Y) = p \cdot \chi(\mathcal{O}_X) = p.$$

Thus $p \leq 2$ as asserted.

5. Analysis of the Case Leading to Abelian Surfaces

In this section, we prove Theorem 6, that a surface X with $K_X \equiv 0$ and $B_2=6$ is an abelian surface. As we see from the table in § 1, the surface X also has the properties :

a) $\dim H^1(\mathcal{O}_X) = 2$, $\dim H^2(\mathcal{O}_X) = 1$, $\chi(\mathcal{O}_X) = 0$,

b) $c_{2,X} = 0$, $B_1 = 4$, $q = 2$.

In particular, $\mathrm{Pic}^0 X$ is reduced and 2-dimensional and its dual Alb X is 2-dimensional. Let

$$\phi : X \to \mathrm{Alb}\, X$$

be the Albanese mapping. First of all, we can see that ϕ is surjective as follows : if not, since $\phi(X)$ generates Alb X, $\phi(X)$ is a curve of genus $g \geq 2$. Consider the diagram :

$$\begin{tikzcd} X'=X\times_{\mathrm{Alb}}\mathrm{Alb} \arrow[r, "\psi"] \arrow[d] & \mathrm{Alb}\,X \arrow[d, "n"] \\ X \arrow[r, "\phi"'] & \mathrm{Alb}\,X \end{tikzcd}$$

where n denotes multiplication by n and $p \nmid n$. Then $\psi(X')$ is an étale covering of $\phi(X)$ of degree n^{2q}. Also $\psi(X')$ is connected because $\psi(X')=n^{-1}(\phi(X))$ and $\phi(X)$ is an ample curve[1] on Alb X. Therefore, $\psi(X')$ has genus $g'>2$. Therefore, Alb X' can be mapped onto $\mathrm{Jac}(\psi(X'))$ which is an abelian variety of dimension>2 : i. e., $q(X')>2$. But X' is an étale cover of X. So $K_{X'}\equiv 0$ and looking in the Table, we see that no such surface X' exists.

Therefore, ϕ is surjective, and hence of finite degree. If ϕ were separable, e.g., if char.$=0$, then we could quickly finish up as follows :

Let ω be the translation-invariant 2-form on Alb X. Then ω has no zeroes or poles and because ϕ is separable, $\phi^*\omega\neq 0$. But $\phi^*\omega$ has zeroes at all points where ϕ is not étale, and $K_X=(\phi^*\omega)$. Since $K_X\equiv 0$, $\phi^*\omega$ has no zeroes, hence ϕ is everywhere étale. But then by the Theorem of § 18 [12], X itself is an abelian surface. Unfortunately if ϕ is inseparable, this argument breaks down. However, when we are in characteristic p, we can use another trick and reduce the Theorem to the case where the ground field k is finite! In fact X lies in a smooth and proper algebraic family of surfaces defined over a finite field and all members of this family have the same invariants (e.g., because by the table in § 1, these surfaces are also characterized by saying $K_X\sim 0$ in étale cohomology and $q=2$). Therefore, if we prove that the surfaces in this family over closed points of the base are abelian, it follows that all are abelian (cf. Theorem 6. 14, [8]).

Now assume the ground field k is finite. We follow a line of argument similar to that in Tate [16]. Consider the infinite sequence of surfaces :

$$\begin{tikzcd} X_l=X\times_{\mathrm{Alb}}\mathrm{Alb} \arrow[r, "\phi_l"] \arrow[d] & \mathrm{Alb}\,X \arrow[d] \\ X \arrow[r, "\phi"] & \mathrm{Alb}\,X \end{tikzcd}$$

for all $l\geq 2$ with $p \nmid l$. Note that $\deg\phi_l=\deg\phi$ for all : call this degree d. Note that $X_l\to X$ is étale and hence X_l is a surface of the same type as X (in fact, $K_{X_l}\equiv 0$ and $q(X_l)\geq 2$, hence by table I, $q(X_l)=2$). We can deduce quickly that ϕ and hence ϕ_l are all finite morphisms : in fact, if not, let $E\subset X$ be a curve such that $\phi(E)$ is a point $e\in\mathrm{Alb}\,X$. Then considering the Stein factorization $X\to Y\to\mathrm{Alb}$ of ϕ, we see that E can be blown down in a birational map $X\to Y$, hence $(E^2)<0$. Now

1) A suitable multiple of an ample curve C on any surface Y is a hyperplane section of Y for some projective embedding and all hyperplane sections of varieties of dimension>1 are connected.

for each l, $l^{-1}(e)$ consists of l^4 points $e_i \in \mathrm{Alb}\, X$, and $\phi^{-1}(e_i)$ contains a curve E_i that is contracted by ϕ_l. These curves are disjoint since $\phi_l(E_i)=e_i \neq e_j=\phi_l(E_j)$. Thus X_l has l^4 disjoint curves E_i with $(E_i^2)<0$; thus $B_2(X_l) \geq l^4$. But for all surfaces of the same type as X, $B_2=6$. This is a contradiction if $l>1$, hence ϕ is finite.

Next, fix L_0, an ample sheaf on Alb X. It follows that $L_l=\phi_l^*(L_0)$ is ample on X_l, with Hilbert polynomial

$$\chi(L_l^{\otimes n}) = d \cdot \chi(L_0^{\otimes n})$$

independent of l. By the Main Theorem of Matsusaka-Mumford [7], there is also a number N independent of l such that $L_l^{\otimes N}$ is very ample for all l. Therefore the infinite set of k-varieties X_l can all be embedded in a fixed $\boldsymbol{P}^M$ with fixed degree. Since there are only finitely many k-varieties of this degree (as k is finite), it follows that all the pairs $(X_l, L_l^{\otimes N})$ are isomorphic to finitely many of them !

Now consider the facts—

a) for any variety X and ample sheaf L, the group of automorphisms f of X such that f^*L is numerically equivalent to L is an algebraic group ; esp. it has only finitely many components (Matsusaka [6]),

b) The group A_l of translations by points of order l acts on X_l since by definition, it is the fibre product $X \times_{\mathrm{Alb}} (\mathrm{Alb}, l)$; moreover each $g \in A_l$ carries L_l into a sheaf algebraically equivalent to L_l.

Let $(X_l, L_l^{\otimes N})$ be isomorphic to infinitely many other $(X_{l'}, L_{l'}^{\otimes N})$'s. Then $A_{l'}$ acts on X_l. Let $G_l \subset \mathrm{Aut}(X_l)$ be the group of automorphisms f such that f^*L_l is numerically equivalent to L_l. Then $A_{l'} \subset G_l$ which implies that the order of G_l is infinite, hence G_l^0 (the connected component) is positive dimensional. But if G_l^0 contains a non-trivial *linear* subgroup, then when this acts on X_l, it would follow that X_l was a ruled surface : since $K_{X_l} \equiv 0$, this is absurd. Therefore G_l^0 is an abelian variety. On the other hand, $A_{l'} \cong (\boldsymbol{Z}/l'\boldsymbol{Z})^4$, and subgroups of fixed bounded index in $A_{l'}$ are inside G_l^0. Therefore dim $G_l^0 \geq 2$. It follows that X_l consists in only one orbit under G_l^0, hence X_l is a coset space G_l^0/H, hence X_l itself is an abelian variety. Finally X itself is now caught in the middle between 2 abelian varieties :

$$X_l \xrightarrow{\text{étale}} X \longrightarrow \mathrm{Alb}\, X.$$

With a suitable origin, $X_l \to \mathrm{Alb}\, X$ is then a homomorphism, hence if K is its kernel, we find :

$$X_l \times_{\mathrm{Alb}\, X} X_l = \{(x, x+k) \mid x \in X_l, k \in K\}.$$

But $X_l \times_X X_l \subset X_l \times_{\mathrm{Alb}\, X} X_l$ and $X_l \times_X X_l$ is (i) étale over X_l, and (ii) the graph of an equivalence relation on X_l. (i) implies that

$$X_l \times_X X_{l'} = \{(x, x+k) \mid x \in X_l, k \in K'\}$$

for some subset $K' \subset K$, and (ii) implies that K' is a subgroup. It follows that $X \cong X_l/K'$, hence X is also an abelian variety.

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