MATH. SCAND. 39 (1976), 19–55

THE PROJECTIVITY OF THE MODULI SPACE OF STABLE CURVES I: PRELIMINARIES ON "det" AND "Div"

FINN KNUDSEN and DAVID MUMFORD

Introduction.

This paper is the first in a sequence of three. In the last paper Mumford will prove that the coarse moduli space of "stable" curves is a projective variety. The proof is a direct application of the very powerful Grothendieck relative Riemann-Roch Theorem.

The notion of a stable curve was introduced by Deligne and Mumford [1]. A stable curve is a reduced, connected curve with at most ordinary double points such that every non-singular rational component meets the other components in at least 3 points.

In this first paper we deal with some essential preliminary constructions which may also have other applications.

In the first paragraph we give the details of a construction whose existence was asserted by Grothendieck and described in the unpublished expose of Ferrand in SGA "Theorie des Intersections —". The construction is to assign to every perfect complex $\mathscr{F}^\cdot$ an invertible sheaf det $\mathscr{F}^\cdot$ in such a way that det becomes a functor from the category of perfect complexes and isomorphisms (in the derived categorical sense) to the category of invertible sheaves and isomorphisms. Roughly det $\mathscr{F}^\cdot$ is the alternating tensor product of the top exterior products of a locally free resolution of $\mathscr{F}^\cdot$. However in making this precise a certain very nasty problem of sign arises. The authors' first solution to these sign problems was described by Grothendieck in a letter as very alambicated* and he suggested to use the "Koszul rule of signs" which we follow in this paper.

The second paragraph deals with a generalization of Chow's construction assigning a "chow form" to every subvariety of P^n. We functorialize this and analyse the invertible sheaves involved, following some ideas in an unpublished letter of Grothendieck to Mumford (1962) and in

*) This apparently means similar to an alchemical apparatus.

Received April 10, 1975.

[3, p. 109]. Finally we must mention that we have several overlaps with J. Fogarty "Truncated Hilbert functors" [4]. He analyses the relation between Div and Chow in the case $\mathscr{F}^{\cdot}$ is an 0-dimensional perfect complex, i.e. a coherent sheaf of finite Tor. dimension. In his notation Div and Chow correspond to ∇ and ω respectively.

Chapter I: det.

Let X be a scheme. We denote by $\mathscr{P}_X$ the category of graded invertible $\mathcal{O}_X$-modules. An object of $\mathscr{P}_X$ is a pair (L,α) where L is an invertible $\mathcal{O}_X$-module and α is a continouus function:

$$\alpha : X \to \mathbf{Z} .$$

A homomorphism $h:(L,\alpha) \to (M,\beta)$ is a homomorphism of $\mathcal{O}_X$-modules such that for each $x \in X$ we have:

$$\alpha(x) \neq \beta(x) \Rightarrow h_x = 0 .$$

We denote by $\mathscr{P}is_X$ the subcategory of $\mathscr{P}_X$ whose morphisms are isomorphisms only.

The tensor product of two objects in $\mathscr{P}_X$ is given by:

$$(L,\alpha) \otimes (M,\beta) = (L\otimes M, \alpha+\beta) .$$

For each pair of objects $(L,\alpha),(M,\beta)$ in $\mathscr{P}_X$ we have an isomorphism:

$$\psi_{(L,\alpha),(M,\beta)}\colon (L,\alpha) \otimes (M,\beta) \xrightarrow{\sim} (M,\beta) \otimes (L,\alpha)$$

defined as follows: If $l \in L_x$ and $m \in M_x$ then

$$\psi(l\otimes m) = (-1)^{\alpha(x)+\beta(x)}\cdot m \otimes l .$$

Clearly:

$$\psi_{(M,\beta),(L,\alpha)}\cdot \psi_{(L,\alpha),(M,\beta)} = 1_{(L,\alpha)\otimes(M,\beta)} .$$

We denote by 1 the object $(\mathcal{O}_X,0)$. A right inverse of an object (L,α) in $\mathscr{P}_X$ will be an object (L',α') together with an isomorphism

$$\delta : (L,\alpha) \otimes (L',\alpha') \xrightarrow{\sim} 1 .$$

Of course $\alpha' = -\alpha$.

A right inverse will be considered as a left inverse via:

$$(L',\alpha') \otimes (L,\alpha) \xrightarrow[\sim]{\psi} (L,\alpha) \otimes (L',\alpha') \xrightarrow[\sim]{\delta} 1 .$$

We denote by $\mathscr{C}_X$ the category of finite locally free $\mathcal{O}_X$-modules, and by $\mathscr{C}is_X$ the subcategory whose morphisms are isomorphisms only.

If $F \in \operatorname{ob}(\mathscr{C}_X)$ we define:

$$\det{}^*(F) = (\Lambda^{\max}F, \operatorname{rank}F)$$

$\left(\text{where}(\Lambda^{\max}F)_x = \Lambda^{\operatorname{rank}F_x}F_x\right)$.

It is well known that det* is a functor from $\mathscr{C}is_X$ to $\mathscr{P}is_X$.
For every short-exact sequence of objects in $\mathscr{C}_X$

$$0 \to F' \xrightarrow{\alpha} F \xrightarrow{\beta} F'' \to 0$$

we have an isomorphism:

$$\mathrm{i}^*(\alpha,\beta)\colon \det{}^*F' \otimes \det{}^*F'' \xrightarrow{\sim} \det{}^*F$$

such that locally,

$$\mathrm{i}^*(\alpha,\beta)((e_1 \wedge \ldots \wedge e_t) \otimes (\beta f_1 \wedge \ldots \wedge \beta f_s)) = \alpha e_1 \wedge \ldots \wedge \alpha e_t \wedge f_1 \wedge \ldots \wedge f_s$$

for $e_i \in \Gamma(U,F')$ and $f_j \in \Gamma(U,F)$.
The following proposition is well known:

Proposition 1. i) i* *is functorial*, i.e., *given a diagram*:

$$\begin{array}{ccccccccc} 0 & \to & F' & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & F'' & \to & 0 \\ & & \wr\downarrow\lambda' & & \wr\downarrow\lambda & & \wr\downarrow\lambda'' & & \\ 0 & \to & G' & \xrightarrow{\gamma} & G & \xrightarrow{\delta} & G'' & \to & 0 \end{array}$$

where the rows are short-exact sequences of objects in $\mathscr{C}_X$, and the columns are isomorphisms, the diagram:

$$\begin{array}{ccc} \det{}^*F' \otimes \det{}^*F'' & \xrightarrow[\sim]{\mathrm{i}^*(\alpha,\beta)} & \det{}^*F \\ \wr\downarrow \det{}^*\lambda' \otimes \det{}^*\lambda'' & & \wr\downarrow \det{}^*\lambda \\ \det{}^*G' \otimes \det{}^*G'' & \xrightarrow[\mathrm{i}^*(\gamma,\delta)]{\sim} & \det{}^*G \end{array}$$

commutes.

ii) *Given a commutative diagram of objects in $\mathscr{C}_X$*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F' & \xrightarrow{\alpha'} & G' & \xrightarrow{\beta'} & H' & \to & 0 \\ & & \downarrow\gamma' & & \downarrow\gamma & & \downarrow\gamma'' & & \\ 0 & \to & F & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & H & \to & 0 \\ & & \downarrow\delta' & & \downarrow\delta & & \downarrow\delta'' & & \\ 0 & \to & F'' & \xrightarrow{\alpha''} & G'' & \xrightarrow{\beta''} & H'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where each row and each column is a short-exact sequence, the diagram

$$\begin{array}{ccc} \det^*F' \otimes \det^*F'' \otimes \det^*H' \otimes \det^*H'' & \xrightarrow[\sim]{i^*(\gamma',\delta')\otimes i^*(\gamma'',\delta'')} & \det^*F \otimes \det^*H \\ \wr\downarrow{\scriptstyle i^*(\alpha',\beta')\otimes i^*(\alpha'',\beta'')\cdot(1\otimes\psi_{\det^*F'',\det^*H'}\otimes 1)} & & \wr\downarrow{\scriptstyle i^*(\alpha,\beta)} \\ \det^*G' \otimes \det^*G'' & \xrightarrow[i^*(\gamma,\delta)]{\sim} & \det^*G \end{array}$$

commutes.

iii) det* *and* i* *commute with base change.*

The isomorphism i* is a special case of a more general canonical isomorphism: suppose E is a locally free $\mathcal{O}_X$-module and:

$$(0) = F^0E \subset F^1E \subset \ldots \subset F^rE = E$$

is a filtration such that $F^iE/F^{i-1}E$ are all locally free. Then there is a canonical isomorphism:

$$i^*(\{FE\})\colon \otimes_{i=1}^r \det^*(F^iE/F^{i-1}E) \xrightarrow{\approx} \det^*(E)\,.$$

Moreover these isomorphisms satisfy the following basic compatibility generalizing (ii) above: suppose $\{F^0E\}$ and $\{G^0E\}$ are 2 filtrations on E such that for all i,j

$$G^{i,j} = F^iE \cap G^jE/(F^{i-1}E \cap G^jE) + (F^iE \cap G^{j-1}E)$$

is locally free. For each fixed i, the $G^{i,j}$ are the graded objects associated to a filtration on $F^iE/F^{i-1}E$, and for each fixed j, they are the graded objects associated to a filtration on $G^jE/G^{j-1}E$. Thus the i's give us a diagram:

$$\begin{array}{ccc} \otimes_{i,j}\det^*(G^{i,j}) & \xrightarrow{\sim} & \otimes_i \det^*(F^iE/F^{i-1}E) \\ \wr\downarrow & & \wr\downarrow \\ \otimes_j \det^*(G^jE/G^{j-1}E) & \xrightarrow{\sim} & \det^*E \end{array}$$

This then commutes. We will not enter into the details here however, because the general isomorphism i can be defined inductively as a composition of the special isomorphisms i associated to short filtrations:

$$(0) = F^0E \subset F^1E \subset F^2E = E\,,$$

which is then just the i associated to the exact sequence:

$$0 \to F^1E \to E \to E/F^1E \to 0\,.$$

Moreover, the general compatibility property is just a formal consequence of the special one – (ii) above.

Next we consider the category $\mathscr{C}^{\cdot}_X$ of bounded complexes of objects in $\mathscr{C}_X$, morphisms being all maps of complexes. A map of complexes which induces an isomorphism in cohomology will be called a *quasi-isomorphism*. The subcategory of $\mathscr{C}^{\cdot}_X$ whose maps are quasi-isomorphisms will be called $\mathscr{C}^{\cdot}is_X$.

DEFINITION 1. A determinant functor from $\mathscr{C}^{\cdot}is$ to $\mathscr{P}is$ consists of the following data:

I) For each scheme X a functor $\mathfrak{f}_X$ from $\mathscr{C}^{\cdot}is_X$ to $\mathscr{P}is_X$.

II) For each scheme X and for each short-exact sequence:

$$0 \to F^{\cdot\prime} \xrightarrow{\alpha} F^{\cdot} \xrightarrow{\beta} F^{\cdot\prime\prime} \to 0$$

in $\mathscr{C}^{\cdot}_X$ an isomorphism:

$$\mathrm{i}_X(\alpha,\beta) : \mathfrak{f}(F^{\cdot\prime}) \otimes \mathfrak{f}(F^{\cdot\prime\prime}) \xrightarrow{\sim} \mathfrak{f}(F^{\cdot}) .$$

This data is to satisfy the following requirements:

i) Given a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & F^{\cdot\prime} & \xrightarrow{\alpha} & F^{\cdot} & \xrightarrow{\beta} & F^{\cdot\prime\prime} & \to & 0 \\ & & \downarrow{\scriptstyle \lambda'} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda''} & & \\ 0 & \to & G^{\cdot\prime} & \xrightarrow{\gamma} & G^{\cdot} & \xrightarrow{\delta} & G^{\cdot\prime\prime} & \to & 0 \end{array}$$

where the rows are short-exact sequences of objects in $\mathscr{C}^{\cdot}_X$ and λ', λ and λ'' are quasi-isomorphisms, the diagram:

$$\begin{array}{ccc} \mathfrak{f}(F^{\cdot\prime}) \otimes \mathfrak{f}(F^{\cdot\prime\prime}) & \xrightarrow[\sim]{\mathrm{i}_X(\alpha,\beta)} & \mathfrak{f}(F^{\cdot}) \\ \wr\downarrow{\scriptstyle \mathfrak{f}(\lambda')\otimes\mathfrak{f}(\lambda'')} & & \wr\downarrow{\scriptstyle \mathfrak{f}(\lambda)} \\ \mathfrak{f}(G^{\cdot\prime}) \otimes \mathfrak{f}(G^{\cdot\prime\prime}) & \xrightarrow[\sim]{\mathrm{i}_X(\gamma,\delta)} & \mathfrak{f}(G^{\cdot}) \end{array}$$

commutes.

ii) Given a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & F^{\cdot\prime} & \xrightarrow{\alpha'} & G^{\cdot\prime} & \xrightarrow{\beta'} & H^{\cdot\prime} & \rightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma''} & & \\
0 & \rightarrow & F^{\cdot} & \xrightarrow{\alpha} & G^{\cdot} & \xrightarrow{\beta} & H^{\cdot} & \rightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta'} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta''} & & \\
0 & \rightarrow & F^{\cdot\prime\prime} & \xrightarrow{\alpha''} & G^{\cdot\prime\prime} & \xrightarrow{\beta''} & H^{\cdot\prime\prime} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where each row and each column is a short-exact sequence, the diagram:

$$\begin{array}{ccc}
\mathrm{f}(F^{\cdot\prime}) \otimes \mathrm{f}(F^{\cdot\prime\prime}) \otimes \mathrm{f}(H^{\cdot\prime}) \otimes \mathrm{f}(H^{\cdot\prime\prime}) & \xrightarrow[\sim]{\mathrm{i}_X(\gamma',\delta')\,\mathrm{i}_X(\gamma'',\delta'')} & \mathrm{f}(F^{\cdot}) \otimes \mathrm{f}(H^{\cdot}) \\
\wr\downarrow{\scriptstyle \mathrm{i}_X(\alpha',\beta') \otimes \mathrm{i}_X(\alpha'',\beta'') \otimes 1 \otimes \psi_{\mathrm{f}(F^{\cdot\prime\prime}),\mathrm{f}(H^{\cdot\prime})} \otimes 1} & & \wr\downarrow{\scriptstyle \mathrm{i}_X(\alpha,\beta)} \\
\mathrm{f}(G^{\cdot\prime}) \otimes \mathrm{f}(G^{\cdot\prime\prime}) & \xrightarrow[\mathrm{i}_X(\gamma,\delta)]{\sim} & \mathrm{f}(G^{\cdot})
\end{array}$$

commutes.

iii) f and i both commute with base change.
iv) f and i are normalized as follows:
a) $\mathrm{f}(0^{\cdot}) = 1$
b) For the exact sequence:

$$0 \rightarrow F^{\cdot} \xrightarrow{1_{F^{\cdot}}} F^{\cdot} \xrightarrow{0} 0^{\cdot} \rightarrow 0$$

the map

$$\mathrm{f}(F^{\cdot}) \otimes 1 \xrightarrow[\mathrm{i}_X(1,0)]{\sim} \mathrm{f}(F^{\cdot})$$

is the canonical one,

b′) For the exact sequence:

$$0 \rightarrow 0^{\cdot} \xrightarrow{0} F^{\cdot} \xrightarrow{1_{F^{\cdot}}} F^{\cdot} \rightarrow 0$$

the map

$$\mathrm{f}(F^{\cdot}) \otimes 1 \xrightarrow[\mathrm{i}_X(1,0)]{\sim} \mathrm{f}(F^{\cdot})$$

is the canonical one.

v) We consider $\mathscr{C}is$ as a full subcategory of $\mathscr{C}^{\cdot}is$ by viewing objects of $\mathscr{C}is$ as complexes with only one nonvanishing term, this term being placed in degree zero. Then for such objects:

$$\mathrm{f}(F) = \det{}^{*}F$$
$$\mathrm{i}_X(\alpha,\beta) = \mathrm{i}^{*}(\alpha,\beta)\ .$$

The main theorem of this chapter is

THEOREM 1. *There is one and, up to canonical isomorphism, only one determinant functor* (f, i), *which we will write* (det, i).

Let X be a scheme, $H^{\cdot}$ an acyclic object in $\mathscr{C}^{\cdot}_X$. If (f, i) is a determinant functor, we have an isomorphism:

$$f(0)\colon f(H^{\cdot}) \to 1\,.$$

If

$$0 \to H^{\cdot\prime} \xrightarrow{\alpha} H^{\cdot} \xrightarrow{\beta} H^{\cdot\prime\prime} \to 0$$

is an exact sequence of acyclic objects it follows from Definition 1, i) and iv a) that the diagram

$$\begin{array}{ccc} f(H^{\cdot\prime}) \otimes f(H^{\cdot\prime\prime}) & \xrightarrow[\sim]{i_X(\alpha,\beta)} & f(H^{\cdot}) \\ \wr\downarrow & & \wr\downarrow \\ 1 \otimes 1 & \xrightarrow[\sim]{\text{mult.}} & 1 \end{array}$$

commutes.

Let $\alpha\colon F^{\cdot} \to G^{\cdot}$ be an injective quasi-isomorphism such that the cokernel is again an object of $\mathscr{C}^{\cdot}_X$, i.e., we have a short-exact sequence:

$$0 \to F^{\cdot} \xrightarrow{\alpha} G^{\cdot} \xrightarrow{\beta} H^{\cdot} \to 0$$

such that $H^{\cdot}$ is acyclic.

From the diagram:

$$\begin{array}{ccccccccc} 0 & \to & F^{\cdot} & \longrightarrow & F^{\cdot} & \longrightarrow & 0^{\cdot} & \to & 0 \\ & & \downarrow 1_{F^{\cdot}} & & \downarrow \alpha & & \downarrow 0 & & \\ 0 & \to & F^{\cdot} & \xrightarrow{\alpha} & G^{\cdot} & \xrightarrow{\beta} & H^{\cdot} & \to & 0 \end{array}$$

we get a commutative diagram:

$$\begin{array}{ccc} f(F^{\cdot}) \otimes 1 & \xrightarrow[\sim]{\text{mult.}} & f(F^{\cdot}) \\ \wr\downarrow & & \wr\downarrow \\ f(F^{\cdot}) \otimes f(H^{\cdot}) & \xrightarrow[\sim]{i_X(\alpha,\beta)} & f(G^{\cdot}) \end{array}$$

hence we see that $f(\alpha)$ is determined by the maps $i_X(\alpha,\beta)$ and $f(0)\colon f(H^{\cdot}) \to 1$.

Let $\lambda\colon F^{\cdot} \to G^{\cdot}$ be an arbitrary quasi-isomorphism. We denote by $Z_\lambda^{\cdot}$ the following complex:

$$Z_\lambda{}^i = F^i \oplus G^i \oplus F^{i+1}$$

$$d^i{}_{Z_\lambda} = \begin{pmatrix} d^i & 0 & -1 \\ 0 & d^i & \lambda^{i+1} \\ 0 & 0 & -d^{i+1} \end{pmatrix}$$

Consider the diagram:

$$F^\cdot \xrightarrow{\alpha} Z_\lambda^\cdot \underset{\beta}{\overset{\beta'}{\rightleftarrows}} G^\cdot$$

where

$$\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \beta' = (\lambda, 1, 0)\,.$$

We leave to the reader to check that these are all quasi-isomorphisms and furthermore,

$$\beta' \circ \alpha = \lambda, \quad \beta' \circ \beta = 1_G\,.$$

Hence we have:

$$\begin{aligned} \mathrm{f}(\lambda) &= \mathrm{f}(\beta') \circ \mathrm{f}(\alpha) = \mathrm{f}(\beta') \circ \mathrm{f}(\beta) \circ \mathrm{f}(\beta)^{-1} \circ \mathrm{f}(\alpha) \\ &= \mathrm{f}(\beta' \circ \beta) \circ \mathrm{f}(\beta)^{-1} \circ \mathrm{f}(\alpha) = \mathrm{f}(\beta)^{-1} \circ \mathrm{f}(\alpha)\,. \end{aligned}$$

Hence, since both α and β are injective quasi-isomorphisms, the map $\mathrm{f}(\lambda)$ is determined by the maps i and f(0) from $\mathrm{f}(H^\cdot) \to 1$ for acyclic $H^\cdot$. We summarize this in the following:

LEMMA 1. *Let* (f,i) *and* (g,j) *be two determinant functors from* $\mathscr{C}^\cdot$*is to* $\mathscr{P}$*is. Suppose we are given* θ *as follows*:

i) *For each scheme* X *and each object* $F^\cdot$ *in* $\mathscr{C}^\cdot_X$ *we have an isomorphism*:

$$\theta_{X,F^\cdot} : \mathrm{f}(F^\cdot) \xrightarrow{\sim} \mathrm{g}(F^\cdot)\,.$$

ii) *For all acyclic* $H^\cdot$ *the diagram*:

$$\begin{array}{ccc} \mathrm{f}(H^\cdot) & \xrightarrow{\theta_{X,H^\cdot}} & \mathrm{g}(H^\cdot) \\ \wr\downarrow \mathrm{f}(0) & & \wr\downarrow \mathrm{g}(0) \\ 1 & = & 1 \end{array}$$

commutes.

iii) *For all short-exact sequences*:

$$0 \to F^\cdot \xrightarrow{\alpha} G^\cdot \xrightarrow{\beta} H^\cdot \to 0$$

with $H^\cdot$ *acyclic, the diagram*:

$$\begin{array}{ccc} \mathrm{f}(F^{\cdot}) \otimes \mathrm{f}(H^{\cdot}) & \xrightarrow[\sim]{\mathrm{i}_X(\alpha,\beta)} & \mathrm{f}(G^{\cdot}) \\ \wr\downarrow \theta_{X,F^{\cdot}} \otimes \theta_{X,H^{\cdot}} & & \wr\downarrow \theta_{X,G^{\cdot}} \\ \mathrm{g}(F^{\cdot}) \otimes \mathrm{g}(H^{\cdot}) & \xrightarrow[\sim]{\mathrm{j}_X(\alpha,\beta)} & \mathrm{g}(G^{\cdot}) \end{array}$$

commutes.

iv) θ *commutes with base change.*

Then for all quasi-isomorphisms $\lambda\colon F^{\cdot} \to G^{\cdot}$ *the diagram*:

$$\begin{array}{ccc} \mathrm{f}(F^{\cdot}) & \xrightarrow[\sim]{\mathrm{f}(\lambda)} & \mathrm{f}(G^{\cdot}) \\ \wr\downarrow \theta_{X,F^{\cdot}} & & \wr\downarrow \theta_{X,G^{\cdot}} \\ \mathrm{g}(F^{\cdot}) & \xrightarrow[\sim]{\mathrm{g}(\lambda)} & \mathrm{g}(G^{\cdot}) \end{array}$$

commutes.

As a side remark, notice that these methods prove:

PROPOSITION 2. *Let* (f, i) *be a determinant functor from* $\mathscr{C}^{\cdot}$*is to* $\mathscr{P}$*is, and let*

$$\lambda, \mu\colon F^{\cdot} \rightrightarrows G^{\cdot}$$

be two quasi-isomorphism such that locally on X, λ *is homotopic to* μ, *then*

$$f(\lambda) = \mathrm{f}(\mu) .$$

PROOF. Two maps being equal is a local property, and since f commutes with base change we may assume that X is affine. However in the affine case locally homotopic maps are homotopic so let H be such a homotopy, i.e.,

$$\lambda - \mu = dH + Hd .$$

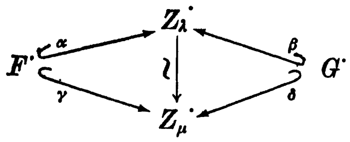
We leave to the reader to check that we have an isomorphism of complexes:

$$Z_{\lambda}^{\cdot} \xrightarrow{\sim} Z_{\mu}^{\cdot}$$

given by the matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & H \\ 0 & 0 & 1 \end{pmatrix}$$

such that the diagram



commutes. But we have already seen that

$$\mathrm{f}(\lambda) = \mathrm{f}(\beta)^{-1} \circ \mathrm{f}(\alpha) \quad \text{and} \quad \mathrm{f}(\mu) = \mathrm{f}(\delta)^{-1} \circ \mathrm{f}(\gamma)$$

hence the proposition.

LEMMA 2. *Suppose we are given a pair* (f, i) *satisfying all the axioms of definition* 1 *except*:

I) *is replaced by*:
I′) *For each scheme X we have a map*

$$\mathrm{f}_X : \mathrm{ob}(\mathscr{C}_X^\cdot) \to \mathrm{ob}(\mathscr{P}_X)$$

such that $\mathrm{f}_X(0^\cdot) = 1$ *and for each acyclic complex $H^\cdot$ on X an isomorphism*:

$$\mathrm{f}_X(0) : \mathrm{f}_X(H^\cdot) \xrightarrow{\sim} 1 .$$

i) *is replaced by*
i′) *For each scheme X and for each short-exact sequence of acyclic objects*:

$$0 \to H'^\cdot \xrightarrow{\alpha} H^\cdot \xrightarrow{\beta} H''^\cdot \to 0$$

the diagram

$$\begin{array}{ccc} \mathrm{f}(H'^\cdot) \otimes \mathrm{f}(H''^\cdot) & \xrightarrow[\sim]{\mathrm{i}_X(\alpha,\beta)} & \mathrm{f}(H^\cdot) \\ \wr\Big\downarrow \mathrm{f}(0)\otimes\mathrm{f}(0) & & \wr\Big\downarrow \mathrm{f}(0) \\ 1 \otimes 1 & \xrightarrow[\sim]{\text{mult.}} & 1 \end{array}$$

commutes.
(*The rest is left unaltered.*)
Then there exists up to caninical isomorphism a unique determinant functor $(\tilde{\mathrm{f}}, \mathrm{i})$ *such that for all $F^\cdot$ we have*

$$\tilde{\mathrm{f}}(F^\cdot) = \mathrm{f}(F^\cdot)$$

and for each quasi-isomorphism

$$H^\cdot \xrightarrow{0} 0 .$$

we have:

$$\tilde{\mathrm{f}}(0) = \mathrm{f}(0) .$$

PROOF. Uniqueness follows immediately from Lemma 1. Suppose we have defined $\tilde{\mathrm{f}}$ for all affine schemes, then since $\tilde{\mathrm{f}}$ commutes with base change, the maps patch together to give $\tilde{\mathrm{f}}$ on all schemes, hence we may assume that X is affine.

Let $F^{\cdot} \xrightarrow{\alpha} G^{\cdot}$ be an injective quasi-isomorphism. We will say that α is good if the cokernel of α is again in $\mathscr{C}^{\cdot}_X$. Let $H^{\cdot} =$ cokernel of α. Then we get a short-exact sequence of complexes

$$0 \to F^{\cdot} \xrightarrow{\alpha} G^{\cdot} \xrightarrow{\beta} H^{\cdot} \to 0$$

such that $H^{\cdot}$ is acyclic. We define $\mathrm{f}'(\alpha)$ via:

$$\begin{array}{ccccccc} \mathrm{f}(F^{\cdot}) & \xleftarrow[\sim]{\text{mult.}} & \mathrm{f}(F^{\cdot}) \otimes 1 & \xleftarrow[\sim]{1 \otimes \mathrm{f}(0)} & \mathrm{f}(F^{\cdot}) \otimes \mathrm{f}(H^{\cdot}) & \xleftarrow[\sim]{i_X(\alpha,\beta)} & \mathrm{f}(G^{\cdot}) \\ \big| & & & & & & \uparrow \\ \multicolumn{7}{c}{\xrightarrow[\sim]{\qquad\qquad\qquad \mathrm{f}'(\alpha) \qquad\qquad\qquad}} \end{array}$$

Let $\alpha: E^{\cdot} \to F^{\cdot}$ and $\beta: F^{\cdot} \to G^{\cdot}$. be two good injective quasi-isomorphisms. We have a commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E & \xrightarrow{\alpha} & F & \xrightarrow{\gamma} & H & \to & 0 \\ & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\xi} & & \\ 0 & \to & E & \xrightarrow{\beta\circ\alpha} & G & \xrightarrow{\delta} & K & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\zeta} & & \\ 0 & \to & 0 & \longrightarrow & L & = & L & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

so by axiom ii), iv) and i′) we have:

$$(**) \qquad \mathrm{f}'(\beta)\circ\mathrm{f}'(\alpha) = \mathrm{f}'(\beta\circ\alpha)\,.$$

If $\lambda: F^{\cdot} \to G^{\cdot}$ is an arbitrary quasi-isomorphism, we have a diagram:

$$F^{\cdot} \xrightarrow{\alpha} Z_{\lambda}^{\cdot} \xrightarrow{\beta} G^{\cdot}$$

where

$$\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Clearly α and β are both good injective quasi-isomorphisms, and we define

$$\tilde{\mathrm{f}}(\lambda) = \mathrm{f}'(\beta)^{-1}\cdot\mathrm{f}'(\alpha)\,.$$

To see that $\tilde{\mathrm{f}}$ is functorial, let

$$\lambda : E^\cdot \to F^\cdot \quad \text{and} \quad \mu : F^\cdot \to G^\cdot$$

be quasi-isomorphisms: we define a complex $W^\cdot$ as follows:

$$W^i = E^i \oplus F^i \oplus G^i \oplus E^{i+1} \oplus F^{i+1}$$

$$d_W = \begin{bmatrix} d & 0 & 0 & -1 & 0 \\ 0 & d & 0 & \lambda & -1 \\ 0 & 0 & d & 0 & \mu \\ 0 & 0 & 0 & -d & 0 \\ 0 & 0 & 0 & 0 & -d \end{bmatrix}$$

We then have a commutative diagram:

$$\begin{array}{ccccc} & Z_\lambda^\cdot & \longleftarrow & F^\cdot & \\ \nearrow & & \searrow^{p} & & \searrow \\ E^\cdot & & W^\cdot & \xleftarrow{\quad q \quad} & & Z_\mu^\cdot \\ \searrow & & \nearrow_{r} & & \nearrow \\ & Z^\cdot_{\mu\circ\lambda} & \longleftarrow & G^\cdot & \end{array}$$

where

$$p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

The fact that $\tilde{\mathrm{f}}(\mu\circ\lambda) = \tilde{\mathrm{f}}(\mu)\circ\tilde{\mathrm{f}}(\lambda)$ now follows from this diagram and the functoriality of f'. We leave to the reader to check axiom i): this is not hard. It is also easy to check that $\tilde{\mathrm{f}} = \mathrm{f}'$ where f' is defined, but this is not needed.

For each scheme X and each object L in $\mathscr{P}_X$ we fix a right inverse L^{-1} of L, i.e., an isomorphism

$$\delta_L : L \otimes L^{-1} \xrightarrow{\sim} 1 .$$

If $\alpha : L \xrightarrow{\sim} M$ is an isomorphism in $\mathscr{P}_X$ we denote by α^{-1} the unique isomorphism making the diagram:

$$\begin{array}{ccc} L \otimes L^{-1} & \xrightarrow{\sim} & 1 \\ \wr \downarrow {\scriptstyle \alpha \otimes \alpha^{-1}} & & \| \\ M \otimes M^{-1} & \xrightarrow{\sim} & 1 \end{array}$$

commutative.

For every pair of objects L, M we denote by $\theta_{L,M}$ the unique isomorphism making the diagram

$$\begin{array}{ccc} (M\otimes L)\otimes(M\otimes L)^{-1} & \xrightarrow[\sim]{\delta} & 1 \\ \wr\downarrow 1\otimes\theta_{M,L} & & \wr\downarrow(\delta\cdot\delta)^{-1} \\ (M\otimes L)\otimes(M^{-1}\otimes L^{-1}) & \xrightarrow[\sim]{1\otimes\psi_{L,M}\otimes 1} & M\otimes M^{-1}\otimes L\otimes L^{-1} \end{array}$$

Then $^{-1}$ is a functor from $\mathscr{P}is$ to $\mathscr{P}is$ which commutes with base change, and for each pair L, M the diagram:

$$\begin{array}{ccc} (M\otimes L)^{-1} & \xrightarrow[\sim]{\theta_{M,L}} & M^{-1}\otimes L^{-1} \\ \wr\downarrow(\psi_{M,L})^{-1} & & \wr\downarrow\psi_{M^{-1},L^{-1}} \\ (L\otimes M)^{-1} & \xrightarrow[\sim]{\theta_{L,M}} & L^{-1}\otimes M^{-1} \end{array}$$

commutes.

If F^i is an indexed object of $\mathscr{C}_X$ we define:

$$\det(F^i) = \begin{Bmatrix} \det{}^*(F^i) & \text{for } i \text{ even} \\ \det{}^*(F^i)^{-1} & \text{for } i \text{ odd} \end{Bmatrix}$$

If

$$0 \to F^{i\prime} \xrightarrow{\alpha^i} F^i \xrightarrow{\beta^i} F^{i\prime\prime} \to 0$$

is an indexed short-exact sequence of objects in $\mathscr{C}_X$, we define

$$\mathrm{i}(\alpha^i,\beta^i) = \begin{Bmatrix} \mathrm{i}^*(\alpha^i,\beta^i) & \text{for } i \text{ even} \\ \mathrm{i}^*(\alpha^i,\beta^i)^{-1} & \text{for } i \text{ odd} \end{Bmatrix}$$

If $F^\cdot$ is an object of $\mathscr{C}^\cdot_X$ we define

$$\det(F^\cdot) = \ldots\otimes\det(F^{i+1})\otimes\det(F^i)\otimes\det(F^{i-1})\otimes\ldots$$

Finally if

$$0 \to F^{\cdot\prime} \xrightarrow{\alpha} F^\cdot \xrightarrow{\beta} F^{\cdot\prime\prime} \to 0$$

is a short exact sequence of objects in $\mathscr{C}_X^\cdot$ we define

$$\mathrm{i}(\alpha,\beta):\ \det(F^{\cdot\prime})\otimes\det(F^{\cdot\prime\prime}) \xrightarrow{\sim} \det(F^\cdot)$$

to be the composite:

$$\begin{aligned} \det(F^{\cdot\prime})\otimes\det(F^{\cdot\prime\prime}) &= \ldots\otimes\det(F^{i\prime})\otimes\det(F^{i-1\prime})\otimes\ldots \\ \otimes\det(F^{i\prime\prime})\otimes\det(F^{i-1\prime\prime})\otimes\ldots &\xrightarrow{\sim} \ldots\otimes\det(F^{i\prime})\otimes\det(F^{i\prime\prime}) \\ \otimes\det(F^{i-1\prime})\otimes\det(F^{i-1\prime\prime})\otimes\ldots &\xrightarrow[\sim]{\otimes_i \mathrm{i}(\alpha^i,\beta^i)} \ldots\otimes\det(F^i) \\ \otimes\det(F^{i-1})\otimes\ldots &= \det(F^\cdot)\,. \end{aligned}$$

the most amazing thing is that we can construct for each acyclic object $H^{\cdot}$ in $\mathscr{C}^{\cdot}_X$ an isomorphism:

$$\det_X(0): \det_X(H^{\cdot}) \xrightarrow{\sim} 1$$

such that all the axioms of lemma 2 holds.

These axioms are all trivially verified execpt for I′) and i′). We will verify these simultaneously and we use induction with respect to the length of the complexes.

STEP 1. Complexes of length 2.

Consider first an acyclic complex

$$H^{\cdot} = \ldots \to 0 \to H^i \xrightarrow{d} H^{i+1} \to 0 \to \ldots$$

with i an odd integer. Since d is an isomorphism we get an isomorphism:

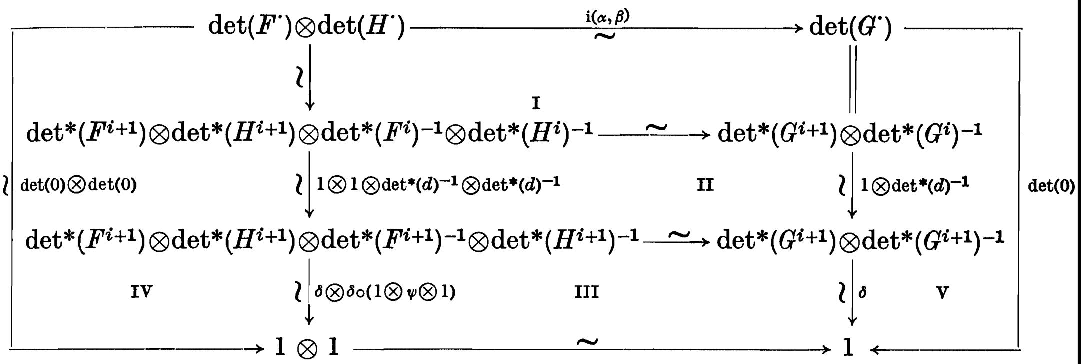
$$\det(H^{\cdot}) = \det{}^*H^{i+1})\otimes\det{}^*(H^i)^{-1} \xrightarrow[\sim]{1\otimes\det^*(d)^{-1}} \det{}^*(H^{i+1})\otimes\det{}^*(H^{i+1})^{-1} \xrightarrow{\sim} 1\,.$$

We define this isomorphism to be det(0).

Given a short exact sequence of acyclic length 2-complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F^i & \xrightarrow{\alpha^i} & G^i & \xrightarrow{\beta^i} & H^i & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \to & F^{i+1} & \longrightarrow & G^{i+1} & \xrightarrow{\beta^i} & H^{i+1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

we get a diagram:



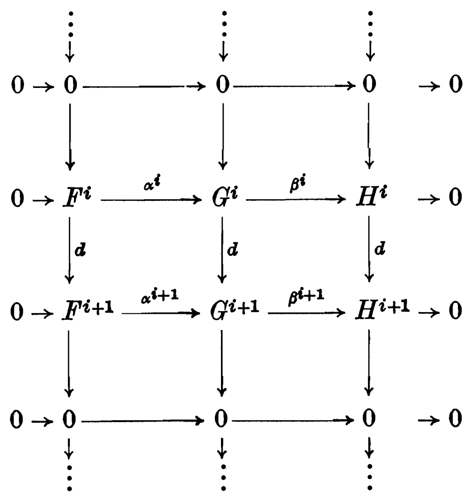
The square I is commutative by the definition of i, and the squares IV and V are commutative by the definition of det(0). The square III is commutative by the definition of i^{-1} and finally II is commutative by axioms iii) of definition 1. Hence the whole diagram commutes. If

$$H^{\cdot} = \ldots \to 0 \to H^i \xrightarrow{d} H^{i+1} \to 0 \to \ldots$$

is an acyclic complex with i even we define det(0) to be the composite

$$\det(H^{\cdot}) = \det{}^*(H^{i+1})^{-1} \otimes \det{}^*(H^i) \xrightarrow[\sim]{1\otimes\det^*(d)} \det{}^*(H^{i+1})^{-1} \otimes \det{}^*(H^{i+1})$$
$$\xrightarrow[\sim]{\psi} \det{}^*(H^{i+1}) \otimes \det{}^*(H^{i+1})^{-1} \xrightarrow[\sim]{\delta} 1 \; \ldots .$$

Given a short-exact sequence of acyclic length 2-complexes



with i even we get just as before a commutative diagram.

$$\begin{array}{ccc}
\det(F^{\cdot})\otimes\det(H^{\cdot}) & \xrightarrow{i(\alpha,\beta)} & \det(G^{\cdot}) \\
\Big\downarrow{\scriptstyle \det(0)\otimes\det(0)} & & \Big\downarrow{\scriptstyle \det(0)} \\
1\otimes 1 & \xrightarrow{\sim} & 1
\end{array}$$

Hence I′) and i′) holds for all acyclic complexes of length 2.

Step 2. Suppose I′) and i′) hold for all acyclic complexes of length $\leqq n$, and let

$$H^{\cdot} = \ldots \to 0 \to H^{i} \to H^{i+1} \to H^{i+2} \to \ldots \to H^{i+n} \to 0 \to \ldots$$

be an acyclic complex of length $n+1$. We then get a short-exact sequence of complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^{i} & = & H^{i} & \to & 0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^{i} & \to & H^{i+1} & \to & H^{i+1}/H^{i} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & H^{i+2} & \to & H^{i+2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & H^{i+n} & \to & H^{i+n} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \| & & \| & & \| & & \\
0 & \to & H^{\cdot}_{\mathrm{I}} & \xrightarrow{\alpha} & H^{\cdot} & \xrightarrow{\beta} & H^{\cdot}_{\mathrm{II}} & \to & 0
\end{array}$$

Since $H^{\cdot}_{\mathrm{I}}$ and $H^{\cdot}_{\mathrm{II}}$ are of length $\leqq n$ we define det(0) so as to make the diagram

$$\begin{array}{ccc}
\det(H^{\cdot}_{\mathrm{I}})\otimes\det(H^{\cdot}_{\mathrm{II}}) & \xrightarrow[\sim]{i(\alpha,\beta)} & \det(H^{\cdot}) \\
\Big|{\scriptstyle \det(0)\otimes\det(0)} & & \Big|{\scriptstyle \det(0)} \\
1\otimes 1 & \xrightarrow[\sim]{\text{mult.}} & 1
\end{array}$$

commutative. It is then easy to check that i′) follows from axiom ii) of definition 2. Now by Lemma 2, the pair (det,i) is a determinant functor $\mathscr{C}is$ to $\mathscr{P}is$. Now say (f,j) is any determinant functor. If $E^{\cdot}$ is any complex we define an isomorphism

$$\theta_1 : \mathrm{f}(E^{\cdot}) \xrightarrow{\sim} \mathrm{f}(TE^{\cdot})^{-1}$$

in such a way that the diagram:

$$\begin{array}{ccccc} \mathrm{f}(E^{\cdot})^{-1}\otimes \mathrm{f}(E^{\cdot}) & \xrightarrow[\sim]{\psi} & \mathrm{f}(E^{\cdot})\otimes \mathrm{f}(E^{\cdot})^{-1} & \xrightarrow[\sim]{\delta} & 1 \\ \downarrow{\scriptstyle 1\otimes\theta_1} & & & & \| \\ \mathrm{f}(E^{\cdot})^{-1}\otimes \mathrm{f}(TE^{\cdot})^{-1} & \xrightarrow{\sim} & \mathrm{f}(C^{\cdot}_{1_{E^{\cdot}}})^{-1} & \xrightarrow{\mathrm{f}'(0)} & 1 \end{array}$$

commutes. Here T stands for the shift operator defined by

$$(TE^{\cdot})^n = E^{n+1} \quad \text{and} \quad Td = -d\,.$$

$C^{\cdot}$ is the mapping cylinder complex. Inductively we define

$$\theta_n(E^{\cdot}) = \theta_{n-1}(TE^{\cdot})^{-1}\cdot\theta_1(E^{\cdot}) \quad \text{and} \quad \theta_{-n}(E^{\cdot}) = \theta_n(T^{-n}E^{\cdot})^{-1}$$

(note this $^{-1}$ is the functor mentioned on p. 31.)

It is straightforward to check that given a short-exact sequence

$$0 \to E^{\cdot} \xrightarrow{\alpha} F^{\cdot} \xrightarrow{\beta} G^{\cdot} \to 0$$

the diagram

$$\begin{array}{ccc} \mathrm{f}(E^{\cdot})\otimes \mathrm{f}(G^{\cdot}) & \xrightarrow[\sim]{\mathrm{i}(\alpha,\beta)} & \mathrm{f}(F^{\cdot}) \\ \wr\downarrow{\scriptstyle \theta_1\otimes\theta_1} & & \wr\downarrow{\scriptstyle \theta_1} \\ \mathrm{f}(TE^{\cdot})^{-1}\otimes \mathrm{f}(TG^{\cdot})^{-1} & \xrightarrow[\sim]{\mathrm{i}(T\alpha,T\beta)^{-1}} & \mathrm{f}(TF^{\cdot})^{-1} \end{array}$$

commutes.

And for any quasi-isomorphism

$$\lambda : E^{\cdot} \to F^{\cdot}$$

the diagram

$$\begin{array}{ccc} \mathrm{f}(E^{\cdot}) & \xrightarrow[\sim]{\mathrm{f}(\lambda)} & \mathrm{f}(F^{\cdot}) \\ \wr\downarrow{\scriptstyle \theta_1} & & \wr\downarrow{\scriptstyle \theta_1} \\ \mathrm{f}(TE^{\cdot})^{-1} & \xrightarrow[\sim]{\mathrm{f}(T\lambda)^{-1}} & \mathrm{f}(TF^{\cdot})^{-1} \end{array}$$

commutes.

We proceed to define an isomorphism of functors:

$$\eta : (\mathrm{f},\mathrm{j}) \xrightarrow{\sim} (\det,\mathrm{i})\,.$$

First consider a complex $E^{\cdot}$ concentrated in degree i.

We define η as follows:

$$\begin{array}{ccccc} \mathfrak{f}(E^{\cdot}) & \xrightarrow[\sim]{\theta_i} & \mathfrak{f}(T^iE^{\cdot})^{(-1)^i} & = & \det(E^{\cdot}) \\ \downarrow & & \xrightarrow[\sim]{\eta} & & \uparrow \end{array}$$

It is then obvious that restricted to all complexes which are concentrated in a single degree, η is an isomorphism of functors. If

$$E^{\cdot} = \ldots \to 0 \to E^i \to E^{i+1} \to \ldots \to E^{i+n} \to 0$$

is a complex of length $n+1$, we get a short-exact sequence of complexes

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & 0 & \to & E^i & \to & E^i & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E^{i+1} & \to & E^{i+1} & \to & 0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E^{i+2} & \to & E^{i+2} & \to & 0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E^{i+n} & \to & E^{i+n} & \to & 0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \\ & & \| & & \| & & \| & & \\ 0 & \to & E^{\cdot}_{\mathrm{I}} & \xrightarrow{\alpha} & E^{\cdot} & \xrightarrow{\beta} & E^{\cdot}_{\mathrm{II}} & \to & 0 \end{array}$$

Inductively we can define η such that the diagram

$$\begin{array}{ccc} \mathfrak{f}(E^{\cdot}_{\mathrm{I}}) \otimes \mathfrak{f}(E^{\cdot}_{\mathrm{II}}) & \xrightarrow[\sim]{\mathfrak{j}(\alpha,\beta)} & \mathfrak{f}(E^{\cdot}) \\ \wr\downarrow \eta\otimes\eta & & \wr\downarrow \eta \\ \det(E^{\cdot}_{\mathrm{I}})\otimes\det(E^{\cdot}_{\mathrm{II}}) & \xrightarrow[\sim]{\mathfrak{i}(\alpha,\beta)} & \det(E^{\cdot}) \end{array}$$

commutes.

Using axiom ii) it is easy to check that for all short-exact sequences of complexes

$$0 \to E^{\cdot} \xrightarrow{\alpha} F^{\cdot} \xrightarrow{\beta} G^{\cdot} \to 0$$

the diagram

$$\begin{array}{ccc} f(E^\cdot)\otimes f(G^\cdot) & \xrightarrow[\sim]{j(\alpha,\beta)} & f(F^\cdot) \\ \wr\downarrow \eta\otimes\eta & & \downarrow \eta \\ \det(E^\cdot)\otimes\det(G^\cdot) & \xrightarrow[\sim]{i(\alpha,\beta)} & \det(F^\cdot) \end{array}$$

commutes.

Finally we want to show that for each acyclic complex $H^\cdot$ the diagram

$$\begin{array}{ccc} f(H^\cdot) & \xrightarrow[\sim]{\eta} & \det(H^\cdot) \\ \wr \mid f(0) & & \det(0) \mid \wr \\ & 1 & \end{array}$$

commutes.

By induction we only have to prove this in case of a length 2 acyclic complex. Note that any such complex is the mapping cylinder of an isomorphism of pointed complexes, say:

$$H^\cdot = C_\lambda^\cdot \quad \text{where} \quad \lambda : A^\cdot \xrightarrow{\simeq} B^\cdot .$$

We have then a short-exact sequence

$$0 \to B^\cdot \xrightarrow{\alpha} H^\cdot \xrightarrow{\beta} TA^\cdot \to 0$$

and f(0) is given as the composite:

$$f(H^\cdot) \xrightarrow[\sim]{j(\alpha,\beta)} f(B^\cdot)\otimes f(TA^\cdot) \xrightarrow{1\otimes\theta_{-1}} f(B)\otimes f(A)^{-1} \xrightarrow{1\otimes f(\lambda)^{-1}}$$
$$f(B)\otimes f(B)^{-1} \xrightarrow{\delta} 1$$

The same formula holds for det, and so by Lemma 1 η is an isomorphism of functors.

We can in fact extend det even further. We need some preliminaries concerning derived categories for this.

Let $\mathscr{A}$ be an abelian category; we denote by $\mathscr{A}_3$ the following category.

i) The objects of $\mathscr{A}_3$ are sequences of the form

$$E'' \xrightarrow{\alpha} E \xrightarrow{\beta} E'$$

such that $\beta\cdot\alpha = 0$.

ii) The morphisms in $\mathscr{A}_3$ are triples of maps in $\mathscr{A}$ making the resulting diagram commute.

DEFINITION 2. The subcategory of $D(\mathscr{A}_3)$ whose objects are short-exact sequences of complexes will be denoted by $VT(\mathscr{A})$ and, we will call it the category of true triangles of $D(\mathscr{A})$.

Remark. Let

$$X = 0 \to E^{\cdot\prime\prime} \xrightarrow{\alpha} E^{\cdot} \xrightarrow{\beta} E^{\cdot\prime} \to 0$$

be a true triangle. Taking the mapping cylinder of the first map we get an ordinary triangle

$$\ldots \to E^{\cdot\prime\prime} \to E^{\cdot} \to C_{\alpha}^{\cdot} \to TE^{\cdot\prime\prime} \to TE^{\cdot} \to TC_{\alpha}^{\cdot} \to \ldots .$$

If $1_{E^{\cdot\prime\prime}}$ is the identity map on $E^{\cdot\prime\prime}$ we have a short-exact sequence

$$0 \to C^{\cdot}_{1_{E^{\cdot\prime\prime}}} \to C_{\alpha}^{\cdot} \xrightarrow{u} E^{\cdot\prime} \to 0 .$$

But $C_{1_{E^{\cdot\prime\prime}}}$ is acyclic so u is a quasi-isomorphism, and hence the composition

$$E^{\cdot\prime} \xrightarrow{u^{-1}} C_{\alpha}^{\cdot} \to TE^{\cdot\prime\prime}$$

gives us a triangle which we call

$$\delta(X) = \to E^{\cdot\prime\prime} \xrightarrow{\alpha} E^{\cdot} \xrightarrow{\beta} E^{\cdot\prime} \to TE^{\cdot\prime\prime} \to \ldots .$$

In fact δ is a functor from true triangles to the category $\mathrm{TD}(\mathscr{A})$ of triangles in $\mathrm{D}(\mathscr{A})$. Note that the homomorphism

$$\delta : \mathrm{Hom}_{\mathrm{VT}(\mathscr{A})}(X\,Y) \to \mathrm{Hom}_{\mathrm{TD}(\mathscr{A})}(\delta(X), \delta(Y))$$

is in general neither injective nor surjective.

Proposition 3. *Let $f : X \to Y$ be a morphism of schemes and let* $\mathrm{Mod}(X)$, $\mathrm{Mod}(Y)$ *be the category of $\mathcal{O}_X$- and $\mathcal{O}_Y$-modules. Then left and right derived functors*

$$\mathrm{L}f^* : \mathrm{VT}(\mathrm{Mod}(Y))^- \to \mathrm{VT}(\mathrm{Mod}(X))^-$$
$$\mathrm{R}f_* : \mathrm{VT}(\mathrm{Mod}(X))^+ \to \mathrm{VT}(\mathrm{Mod}(Y))^+$$

exists.

Proof. According to Hartshorne: Residues and Duality, Chapter I, Theorem 5.1, the proposition follows if each true triangle bounded below allows a quasi-isomorphism into a true triangle consisting of injective $\mathcal{O}_Y$-modules, respectively each true triangle bounded above is quasi-isomorphic to a true triangle consisting of flat $\mathcal{O}_X$-modules. The fact that such quasi-isomorphisms exist follows from the following:

i) A short-exact sequence of injective $\mathcal{O}_X$-modules is an injective object in the category $\mathrm{Mod}(X)_3$, and every object of $\mathrm{Mod}(X)_3$ with the first map injective admits an embedding into a short-exact sequence of injectives.
ii) Every object of $\mathrm{Mod}(X)_3$ with the last map surjective is the quotient of a short-exact sequence of flat $\mathcal{O}_X$-modules.

This proves the proposition.

Recall the definition of a perfect complex $\mathscr{F}^{\cdot}$ on a scheme X [2]. This means that $\mathscr{F}^{\cdot}$ is a complex of $\mathcal{O}_X$-modules (not necessary quasi coherent) such that locally on X there exists a bounded complex $\mathscr{G}^{\cdot}$ of finite free $\mathcal{O}_X$-modules and a quasi isomorphism:

$$\mathscr{G}^{\cdot} \to \mathscr{F}^{\cdot}|_U$$

We denote by Parf_X the full subcategory of $\mathrm{D}(\mathrm{Mod}\,X)$ whose objects are perfect complexes. We leave the proof of the following result to the reader.

PROPOSITION 4. a) *Let X be any affine scheme and $\mathscr{F}^{\cdot}$ a perfect complex on X. Then there exists a bounded complex of locally free, finitely generated $\mathcal{O}_X$-modules $\mathscr{G}^{\cdot}$ and a quasi-isomorphism:*

$$\mathscr{G}^{\cdot} \to \mathscr{F}^{\cdot}$$

(i.e., globally on X:)

Let $\alpha : \mathscr{F}^{\cdot} \to \mathscr{F}^{\cdot\prime}$ be a map in the category Parf_X, *and suppose we are given quasi-isomorphisms:*

$$\mathrm{p} : \mathscr{G}^{\cdot} \to \mathscr{F}^{\cdot} \quad \text{and} \quad \mathrm{p}' : \mathscr{G}^{\cdot\prime} \to \mathscr{F}^{\cdot\prime}$$

where $\mathscr{G}^{\cdot}$ and $\mathscr{F}^{\cdot\prime}$ are bounded complexes of locally free $\mathcal{O}_X$-modules, then there exists up to homotopy a unique map

$$\beta : \mathscr{G}^{\cdot} \to \mathscr{G}^{\cdot\prime}$$

such that $\mathrm{p}'\beta = \alpha\mathrm{p}$ *in* Parf_X.

b) *If*

$$0 \to \mathscr{F}^{\cdot\prime} \to \mathscr{F}^{\cdot} \to \mathscr{F}^{\cdot\prime\prime} \to 0$$

is a true triangle of perfect complexes there exists a true triangle of bounded complexes of finite locally free $\mathcal{O}_X$-modules.

$$0 \to \mathscr{G}^{\cdot\prime} \to \mathscr{G}^{\cdot} \to \mathscr{G}^{\cdot\prime\prime} \to 0$$

and an isomorphism in the category $\mathrm{VT}(\mathrm{Parf}_X)$

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{G}^{\cdot\prime} & \to & \mathscr{G}^{\cdot} & \to & \mathscr{G}^{\cdot\prime\prime} & \to & 0 \\ & & \downarrow{\scriptstyle \mathrm{p}'} & & \downarrow{\scriptstyle \mathrm{p}} & & \downarrow{\scriptstyle \mathrm{p}''} & & \\ 0 & \to & \mathscr{F}^{\cdot\prime} & \to & \mathscr{F}^{\cdot} & \to & \mathscr{F}^{\cdot\prime\prime} & \to & 0 \end{array}$$

Moreover if

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{F}^{\cdot\prime} & \to & \mathscr{F}^{\cdot} & \to & \mathscr{F}^{\cdot\prime\prime} & \to & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\ 0 & \to & \mathscr{H}^{\cdot\prime} & \to & \mathscr{H}^{\cdot} & \to & \mathscr{H}^{\cdot\prime\prime} & \to & 0 \end{array}$$

is any morphism in $\mathrm{VT}(\mathrm{Parf}_X)$ *and*

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{K}^{\cdot\prime} & \to & \mathscr{K}^{\cdot} & \to & \mathscr{K}^{\cdot\prime\prime} & \to & 0 \\ & & \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q''} & & \\ 0 & \to & \mathscr{H}^{\cdot\prime} & \to & \mathscr{H}^{\cdot} & \to & \mathscr{H}^{\cdot\prime\prime} & \to & 0 \end{array}$$

is an isomorphism with $\mathscr{K}^{\cdot\prime}, \mathscr{K}^{\cdot}$ *and* $\mathscr{K}^{\cdot\prime\prime}$ *bounded complexes of locally free* $\mathcal{O}_X$*-modules. Then there exists up to homotopy a unique map*:

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{G}^{\cdot\prime} & \to & \mathscr{G}^{\cdot} & \to & \mathscr{G}^{\cdot\prime\prime} & \to & 0 \\ & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta''} & & \\ 0 & \to & \mathscr{K}^{\cdot\prime} & \to & \mathscr{K}^{\cdot} & \to & \mathscr{K}^{\cdot\prime\prime} & \to & 0 \end{array}$$

such that $\alpha'\mathrm{p}' = \mathrm{q}'\beta', \alpha\mathrm{p} = \mathrm{q}\beta$ *and* $\alpha''\mathrm{p}'' = \mathrm{q}''\beta''$ *in* Parf_X.

c) *Same for diagrams of the form*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{F}^{\cdot\prime} & \to & \mathscr{F}^{\cdot} & \to & \mathscr{F}^{\cdot\prime\prime} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{G}^{\cdot\prime} & \to & \mathscr{G}^{\cdot} & \to & \mathscr{G}^{\cdot\prime\prime} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{H}^{\cdot\prime} & \to & \mathscr{H}^{\cdot} & \to & \mathscr{H}^{\cdot\prime\prime} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

DEFINITION 4. An extended determinant functor (f,i) from Parf-is to $\mathscr{P}is$ consists of the following data:

I) For every scheme X a functor

$$\mathrm{f}_X : \text{Parf-is}_X \to \mathscr{P}is_X$$

such that $\mathrm{f}_X(0) = 1$.

II) For every true triangle in Parf-is_X

$$0 \to F \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 0$$

we have an isomorphism:

$$\mathrm{i}_X(\alpha,\beta) : \mathrm{f}_X(F) \otimes \mathrm{f}_X(H) \xrightarrow{\sim} \mathrm{f}_X(G)$$

such that for the particular true triangles

$$0 \to H = H \to 0 \to 0$$

and

$$0 \to 0 \to H = H \to 0$$

we have:

$$\mathrm{i}_X(1,0) = \mathrm{i}_X(0,1) = 1_{\mathrm{f}_X(H)}\,.$$

We require that:

i) Given an isomorphism of true triangles*:

$$\begin{array}{ccccccccc}
0 \to & F & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & H & \to 0 \\
 & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & \\
0 \to & F' & \xrightarrow{\alpha'} & G' & \xrightarrow{\beta'} & H' & \to 0
\end{array}$$

the diagram

$$\begin{array}{ccc}
\mathrm{f}_X(F)\otimes\mathrm{f}_X(H) & \xrightarrow[\sim]{\mathrm{i}_X(\alpha,\beta)} & \mathrm{f}_X(G) \\
\wr\downarrow{\scriptstyle \mathrm{f}_X(u)\otimes\mathrm{f}_X(w)} & & \wr\downarrow{\scriptstyle \mathrm{f}_X(v)} \\
\mathrm{f}_X(F')\otimes\mathrm{f}_X(H') & \xrightarrow[\mathrm{i}_X(\alpha',\beta')]{\sim} & \mathrm{f}_X(G')
\end{array}$$

commutes.

ii) Given a true triangle of true triangles, i.e. a commutative diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & F & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & H & \to 0 \\
 & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u''} & \\
0 \to & F' & \xrightarrow{\alpha'} & G' & \xrightarrow{\beta'} & H' & \to 0 \\
 & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle v''} & \\
0 \to & F'' & \xrightarrow{\alpha''} & G'' & \xrightarrow{\beta''} & H'' & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

the diagram:

$$\begin{array}{ccc}
\mathrm{f}_X(F)\otimes\mathrm{f}_X(H)\otimes\mathrm{f}_X(F'')\otimes\mathrm{f}_X(H'') & \xrightarrow{\mathrm{i}_X(\alpha,\beta)\otimes\mathrm{i}_X(\alpha'',\beta'')} & \mathrm{f}_X(G)\otimes\mathrm{f}_X(G'') \\
\wr\downarrow{\scriptstyle \mathrm{i}_X(u,v)\otimes\mathrm{i}_X(u'',v'')\otimes(1\otimes\psi\otimes 1)} & & \wr\downarrow{\scriptstyle \mathrm{i}_X(u',v')} \\
\mathrm{f}_X(F')\otimes\mathrm{f}_X(H') & \xrightarrow[\mathrm{i}_X(\alpha'\beta')]{\sim} & \mathrm{f}_X(G')
\end{array}$$

---

* This means this diagram commutes as $\mathcal{O}_X$-modules and not just $v\cdot\alpha = \alpha'\cdot u$ in D(Mod X): in fact, even assuming $v\cdot\alpha$ and $w\cdot\beta$ homotopic to $\alpha'\cdot u$ and $\beta'\cdot v$ respectively and all sheaves locally free this property will *not* hold for det!

commutes.

iii) f and i commute with base change. Written out this means: For every morphism of schemes

$$g : X \to Y$$

we have an isomorphism

$$\eta(g) : \mathrm{f}_X \cdot Lg^* \xrightarrow{\sim} g^*\mathrm{f}_X$$

such that for every true triangle

$$0 \to F^\cdot \xrightarrow{u} G^\cdot \xrightarrow{v} H^\cdot \to 0$$

the diagram:

$$\begin{array}{ccc} \mathrm{f}_X(Lg^*F^\cdot)\otimes\mathrm{f}_X(Lg^*H^\cdot) & \xrightarrow[\sim]{\mathrm{i}_X(Lg^*(u,v))} & \mathrm{f}_X(Lg^*G^\cdot) \\ \wr\big\downarrow \eta\cdot\eta & & \wr\big\downarrow \eta \\ g^*\mathrm{f}_Y(F^\cdot)\otimes g^*\mathrm{f}_Y(H^\cdot) & \xrightarrow[\sim]{\mathrm{i}_Y(u,v)} & g^*\mathrm{f}_Y(G^\cdot) \end{array}$$

commutes. Moreover if

$$X \xrightarrow{g} Y \xrightarrow{h} Z$$

are two consecutive morphisms, the diagram:

$$\begin{array}{ccccc} \mathrm{f}_X(Lg^*Lh^*) & \xrightarrow[\sim]{\eta(g)} & g^*\mathrm{f}_Y Lh^* & \xrightarrow[\sim]{g^*\eta(h)} & g^*h^*\mathrm{f}_Z \\ \wr\big\downarrow \mathrm{f}_X(\theta) & & & & \wr\big\downarrow \\ \mathrm{f}_X(L(g\cdot h)^*) & & \xrightarrow{\sim} & & (g\cdot h)^*\mathrm{f}_Z \end{array}$$

commutes where θ is the canonical isomorphism

$$\theta : Lg^* \cdot Lh^* \xrightarrow{\sim} L(g\cdot h)^* ,$$

iv) On finite complexes of locally free $\mathcal{O}_X$-modules,

$$\mathrm{f} = \det \quad \text{and} \quad \mathrm{i} = i .$$

Then using Proposition 4, one proves easily:

THEOREM 2. There is one, and, up to canonical isomorphism, only one extended determinant functor (f,i), which we will write (det,i) again.

REMARK. If $\mathscr{F}$ is a perfect complex and you filter it with subcomplexes such that the successive quotients $\mathrm{gr}^n(\mathscr{F}^\cdot)$ are all perfect, then there is a canonical isomorphism:

$$\det(\mathscr{F}^\cdot) \xrightarrow{\approx} \otimes \det(\mathrm{gr}^n\mathscr{F}^\cdot) .$$

This is constructed easily by induction on the number of steps in the filtration, using the isomorphisms $i(\alpha,\beta)$ at each stage and it has the compatibility property described after Proposition 1 above for ordinary det*. In particular:

a) if each $\mathscr{F}^n$ is itself perfect, i.e., has locally a finite free resolution, then

$$\det(\mathscr{F}^{\cdot}) \cong \otimes_n \det{}^*(\mathscr{F}^n)^{(-1)^n}$$

b) if the cohomology sheaves $H^n(\mathscr{F}^{\cdot})$ of the complex are perfect — we call these complexes the objects of the subcategory $\mathrm{Parf}^0 \subset \mathrm{Parf}$ — then

$$\det(\mathscr{F}^{\cdot}) \cong \otimes_n \det{}^*(H^n(\mathscr{F}^{\cdot}))^{(-1)^n}.$$

This has various easy consequences:

Corollary 1. *Let $\mathscr{F}^{\cdot}$ and $\mathscr{G}^{\cdot}$ be two objects of* $\mathrm{Parf}^0{}_X$ *and suppose α and β*

$$\alpha,\beta : \mathscr{F}^{\cdot} \rightrightarrows \mathscr{G}^{\cdot}$$

are two quasi-isomorphisms such that $H^i(\alpha) = H^i(\beta)$ for each i. Then $\det(\alpha) = \det(\beta)$.

Corollary 2. *Let*

$$\to \mathscr{F}_1^{\cdot} \xrightarrow{u} \mathscr{F}_2^{\cdot} \xrightarrow{v} \mathscr{F}_3^{\cdot} \xrightarrow{w} T\mathscr{F}_1^{\cdot} \to$$

be an ordinary triangle in Parf_X *such that the $\mathscr{F}_j$ are in* $\mathrm{Parf}^{\circ}{}_X$. *We then have an isomorphism*

$$\det(\mathscr{F}_1^{\cdot}) \otimes \det(\mathscr{F}_3^{\cdot}) \xrightarrow[\sim]{i_X(u,v,w)} \det(\mathscr{F}_2^{\cdot})$$

which is functorial with respect to such triangles.

Proof.

$$\det(\mathscr{F}_1^{\cdot})\otimes\det(\mathscr{F}_3^{\cdot}) \cong [\otimes\det{}^*(H^n(\mathscr{F}_1^{\cdot}))^{(-1)^n}]\otimes[\otimes\det{}^*(H^n(\mathscr{F}_3^{\cdot}))^{(-1)^n}]$$

and

$$\det(\mathscr{F}_2^{\cdot}) \cong \otimes\det{}^*(H^n(\mathscr{F}_2^{\cdot}))^{(-1)^n}.$$

But the long exact cohomology sequence $H^{\cdot}(u,v,w)$ is an acyclic complex with perfect sheaves at each stage, so

$$\begin{aligned} 1_X &\xrightarrow{\sim} \det(H^{\cdot}(u,v,w)) \\ &\cong \otimes \det{}^*(n^{\text{th}} \text{ sheaf of } H^{\cdot}(u,v,w))^{(-1)^n} \\ &\cong [\otimes \det{}^*(H^n(\mathscr{F}_1^{\cdot}))^{(-1)^n}]\otimes[\otimes \det{}^*(H^n(\mathscr{F}_2^{\cdot}))^{(-1)^{n+1}}] \\ &\quad \otimes[\otimes \det{}^*(H^n(\mathscr{F}_3^{\cdot}))^{(-1)^n}]. \end{aligned}$$

We tried for some time to extend i to ordinary triangles, but in general this is not possible. It is true that for each ordinary triangle we can find an isomorphism but it is by no means functorial or unique (cf. footnote to Definition 4 above). We have seen that i extends when the complexes are good, we will now see that it also extends when the schemes are good (i.e., reduced).

PROPOSITION 6. *Let X be a reduced scheme $\mathscr{F}^\cdot$ and $\mathscr{G}^\cdot$ perfect complexes, α and β two quasi-isomorphism*

$$\alpha,\beta : \mathscr{F}^\cdot \rightrightarrows \mathscr{G}^\cdot$$

such that

a) *For each integer i there are finite filtrations*

$$F^\cdot(H^i(\mathscr{F}^\cdot)) \quad \textit{and} \quad F^\cdot(H^i(\mathscr{G}^\cdot))\ .$$

b) *For each generic point $x \in X$, the maps*

$$H^i(\alpha) \otimes 1_{\boldsymbol{k}(x)} \quad \textit{and} \quad H^i(\beta) \otimes 1_{\boldsymbol{k}(x)}$$

are compatible with the induced filtrations on $H^i(\mathscr{F}^\cdot)\otimes \boldsymbol{k}(x)$ and $H^i(\mathscr{G}^\cdot)\otimes \boldsymbol{k}(x)$. (Note that $\boldsymbol{k}(x) = \mathcal{O}_{X,x}$) and we have

$$\mathrm{gr}(H^i(\alpha)\otimes 1_{\boldsymbol{k}(x)}) = \mathrm{gr}(H^i(\beta)\otimes 1_{\boldsymbol{k}(x)})$$

for each i.
Then

$$\det(\alpha) = \det(\beta)\ .$$

PROOF. Since X is reduced and det commutes with base change, we may as well assume $X = \mathrm{Spec}(k)$ where k is a field. However in this case we have

$$\mathrm{Parf}_X = \mathrm{Parf}^0{}_X$$

and so the proposition follows from the last one.

PROPOSITION 7. *Let X be a reduced scheme, then for each triangle of perfect complexes:*

$$\mathscr{F}^\cdot \xrightarrow{u} \mathscr{G}^\cdot \xrightarrow{v} \mathscr{H}^\cdot \xrightarrow{w} T\mathscr{F}^\cdot \rightarrow$$

we have a unique isomorphism.

$$\mathrm{i}_X(u,v,w) : \det(\mathscr{F}^\cdot)\otimes\det(\mathscr{H}^\cdot) \xrightarrow{\sim} \det(\mathscr{G}^\cdot)$$

which is functorial with respect to isomorphisms of triangles.

PROOF. First we represent the mapping w by a diagram of real maps

$$T^{-1}(\mathscr{H}^\cdot) \xrightarrow{\ \mu\ } I^\cdot \longleftarrow \mathscr{F}^\cdot$$

where $I^\cdot$ is injective.

The mapping cylinder of μ gives us a true triangle:

$$\begin{array}{ccccccccccc}
\ldots T^{-1}(\mathscr{H}^\cdot) & \to & I^\cdot & \to & C_\mu^\cdot & \to & \mathscr{H}^\cdot & \to & TI^\cdot & \to & \ldots \\
\| & & \downarrow & & & & \| & & \downarrow & & \\
\ldots T^{-1}(\mathscr{H}^\cdot) & \to & \mathscr{F}^\cdot & \to & \mathscr{G}^\cdot & \to & \mathscr{H}^\cdot & \to & T\mathscr{F}^\cdot & \to & \ldots
\end{array}$$

By the second axiom for triangles there exists a map (necessarily an isomorphism) $\lambda\colon C^\cdot \to \mathscr{G}^\cdot$ making the diagram above into an isomorphism of triangles. By Proposition 6 the map $\det(\lambda)$ does not depend on the choice of λ. If we represent w by a different diagram say:

$$T^{-1}(\mathscr{H}^\cdot) \xrightarrow{\ \mu'\ } I'^\cdot \longleftarrow \mathscr{F}^\cdot$$

we get a homotopy commutative diagram

$$\begin{array}{ccc}
T^{-1}(\mathscr{H}^\cdot) & \xrightarrow{\ \mu\ } & I^\cdot \\
\| & & \downarrow{\scriptstyle \alpha} \\
T^{-1}(\mathscr{H}^\cdot) & \xrightarrow{\ \mu'\ } & I'^\cdot
\end{array}$$

If H is a homotopy we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & I^\cdot & \longrightarrow & C_\mu^\cdot & \longrightarrow & \mathscr{H}^\cdot & \to & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \begin{pmatrix} \alpha & H \\ 0 & 1 \end{pmatrix}} & & \|{\scriptstyle 1} & & \\
0 & \to & I'^\cdot & \longrightarrow & C_{\mu'}^\cdot & \longrightarrow & \mathscr{H}^\cdot & \to & 0
\end{array}$$

i.e., a map of true triangles. It follows that if λ' is a map from $C_{\mu'}^\cdot$ to $\mathscr{G}^\cdot$ making

$$\begin{array}{ccccc}
I'^\cdot & \longrightarrow & C_{\mu'}^\cdot & \longrightarrow & \mathscr{H}^\cdot \\
\downarrow & & \downarrow{\scriptstyle \lambda'} & & \| \\
\mathscr{F} & \longrightarrow & \mathscr{G}^\cdot & \longrightarrow & \mathscr{H}^\cdot
\end{array}$$

into an isomorphism of triangles, then the diagram:

$$\begin{array}{ccccccc}
\det(\mathscr{F}^\cdot)\otimes\det(\mathscr{H}^\cdot) & \longrightarrow & \det(I^\cdot)\otimes\det(\mathscr{H}^\cdot) & \longrightarrow & \det(C^\cdot_\mu) & \xrightarrow{\det(\lambda)} & \det(\mathscr{G}^\cdot) \\
\| & & \downarrow\wr & & \downarrow\wr & & \| \\
\det(\mathscr{F}^\cdot)\otimes\det(\mathscr{H}^\cdot) & \longrightarrow & \det(I'^\cdot)\otimes\det(\mathscr{H}^\cdot) & \longrightarrow & \det(C^\cdot_{\mu'}) & \xrightarrow{\det(\lambda')} & \det(\mathscr{G}^\cdot)
\end{array}$$

commutes.

The composite map above we define to be $\mathrm{i}(u,v,w)$. It is clearly functorial.

Let $p: X \to Y$ be a proper morphism of finite Tor-dimension with Y noetherian. Recall that if $\mathscr{F}^\cdot$ is a perfect complex on X then $\mathrm{R}^\cdot p_*\mathscr{F}^\cdot$ is again perfect (cf. Proposition 4.8, SGA 6, expose 3 (Lecture Notes in Mathematics 225, p. 257, Springer-Verlag, Berlin-Heidelberg-New York). Hence to every perfect complex on X we can associate a graded invertible sheaf on Y

$$\det(\mathrm{R}^\cdot p_*(\mathscr{F}^\cdot)) .$$

True triangles on X have injective resolutions so $\mathrm{R}^\cdot p^*$ maps true triangles to true triangles. Hence for every true triangle

$$0 \to \mathscr{F}^\cdot \xrightarrow{\alpha} \mathscr{G}^\cdot \xrightarrow{\beta} \mathscr{H}^\cdot \to 0$$

on X we have an isomorphism:

$$\mathrm{i}_Y(\alpha,\beta):\ \det(\mathrm{R}^\cdot p_*\mathscr{F}^\cdot)\otimes\det(\mathrm{R}p_*\mathscr{H}^\cdot) \xrightarrow{\sim} \det(\mathrm{R}p_*\mathscr{G}^\cdot)$$

which is functorial with respect to isomorphisms in $\mathrm{VT}(\mathrm{Parf}_X)$.

This operation commutes with base change too, i.e., given a morphism of noetherian schemes, $g: Y' \to Y$, let

$$\begin{aligned} X' &= X\times_Y Y' , \\ g' &= p_1:\ X\times_Y Y' \to X , \\ p' &= p_2:\ X\times_Y Y' \to Y' . \end{aligned}$$

Then there are canonical isomorphisms:

$$g^*\big(\det_Y(\mathrm{R}p_*(\mathscr{F}^\cdot))\big) \cong \det_{Y'}\big(\mathrm{L}g^*(\mathrm{R}p_*(\mathscr{F}^\cdot))\big) \cong \det_{Y'}(\mathrm{R}p'_*(\mathrm{L}g'^*\mathscr{F}^\cdot)) .$$

The last result of this chapter we state in the

PROPOSITION 8. *Let $p: X \to Y$ be a proper morphism of noetherian schemes and suppose that Y is a regular scheme. We then have a functorial isomorphism:*

$$\det(\mathrm{R}^\cdot p_*\mathscr{F}^\cdot) \xrightarrow{\sim} \otimes_{p,q}\det(\mathrm{R}^q p_* H^p(\mathscr{F}^\cdot))^{(-1)^{p+q}} .$$

PROOF. The proof is easy by observing that on a noetherian regular scheme we have:

$$\mathrm{Parf}_X = \mathrm{Parf}^0{}_X ,$$

and using the spectral sequence

$$\mathrm{R}^q p^*(H^p(\mathscr{F}^\cdot)) \Rightarrow \mathrm{R}^{p+q} p_*(\mathscr{F}^\cdot) .$$

Chapter II: Div and Chow.

Let X be a notherian scheme, and

$$\lambda : \mathscr{F}^\cdot \to \mathscr{G}^\cdot$$

a map of perfect complexes in the derived categorical sense. We define the open set $U(\lambda)$ as follows:

$$U(\lambda) = \{x \in X \mid \text{there exists a neighbourhood } V \text{ of } x \text{ in } X \text{ such that } \lambda \text{ restricted to } V \text{ is an isomorphism in } D(\mathrm{Mod}(V))\} .$$

We define the support of λ to be the closed set:

$$\mathrm{Supp}(\lambda) = X - U(\lambda) .$$

Finally we say that λ is a *good* map if $\mathrm{Supp}(\lambda)$ contains no points of depth 0 or equivalently $U(\lambda)$ contains all points of depth 0.

Let again $\lambda : \mathscr{F}^\cdot \to \mathscr{G}^\cdot$ be a *good* map of perfect complexes, and let x be a point in X. By the very definition of a perfect complex, we can find a neighbourhood V containing x and two bounded complexes of coherent free $\mathcal{O}_X$-modules, say $\mathscr{E}_1{}^\cdot$ and $\mathscr{E}_2{}^\cdot$ plus, restricted to V, quasi-isomorphisms

$$\mathscr{E}_1{}^\cdot|_V \xrightarrow{\alpha} \mathscr{F}^\cdot|_V \quad \text{and} \quad \mathscr{E}_2{}^\cdot|_V \xrightarrow{\beta} \mathscr{G}^\cdot|_V .$$

By choosing basis for the various $\mathscr{E}_j{}^i$'s we get an isomorphism:

$$\mathcal{O}_X|_{V\cap U(\lambda)} \xrightarrow{\sim} \det(\mathscr{E}_1{}^\cdot)|_{V\cap U(\lambda)} \xrightarrow[\sim]{\det(\alpha)} \det(\mathscr{F}^\cdot)|_{V\cap U(\lambda)} \xrightarrow[\sim]{\det(\lambda)}$$
$$\det(\mathscr{G}^\cdot)|_{V\cap U(\lambda)} \xrightarrow[\sim]{\det(\beta)^{-1}} \det(\mathscr{E}_2{}^\cdot)|_{V\cap U(\lambda)} \xrightarrow{\sim} \mathcal{O}_X|_{V\cap U(\lambda)}$$

and this isomorphism determines a section $s \in \Gamma(V \cap U(\lambda), \mathcal{O}_X{}^*)$.

Since $V\cap U(\lambda)$ contains all points of depth 0 in V, $s = 0$ defines a Cartier divisor $\delta(s)$ in V. Clearly $\delta(s)$ does not depend on the choice of $\mathscr{E}_1{}^\cdot$ and $\mathscr{E}_2{}^\cdot$, so we have defined a global divisor via the formula:

$$\mathrm{Div}(\lambda)|_V = \delta(s) .$$

It follows immediately from the definition that the canonical map on $U(\lambda)$

$$\det(\lambda):\ \det(\mathscr{F}^{\cdot})|_{U(\lambda)} \xrightarrow{\approx} \det(\mathscr{G}^{\cdot})|_{U(\lambda)}$$

extends to an isomorphism on the whole of X:

$$\det(\lambda):\ \det(\mathscr{F}^{\cdot})(\mathrm{Div}(\lambda)) \xrightarrow{\approx} \det(\mathscr{G}^{\cdot})\,.$$

In particular:

(i) $\mathrm{Supp}(\mathrm{Div}(\lambda)) \subset \mathrm{Supp}(\lambda)$
(ii) $\mathscr{O}(\mathrm{Div}(\lambda)) \approx \det(\mathscr{G}^{\cdot}) \otimes (\det(\mathscr{F}^{\cdot}))^{-1}$.

If $\mathscr{F}^{\cdot}$ is a perfect complex on X such that the zero map:

$$0^{\cdot} \to \mathscr{F}^{\cdot}$$

is a good map of complexes, we simply write

$$\mathrm{Div}(\mathscr{F}^{\cdot}) = \mathrm{Div}(0^{\cdot} \to \mathscr{F}^{\cdot})$$

and we have a canonical map:

$$\det(0):\ \mathscr{O}(\mathrm{Div}(\mathscr{F}^{\cdot})) \xrightarrow{\approx} \det(\mathscr{F}^{\cdot})\,.$$

This association of a divisor to every good map of perfect complexes satisfies some properties which we will summarize in the following:

THEOREM 3. (i) *Let* $\lambda: \mathscr{F}^{\cdot} \to \mathscr{G}^{\cdot}$ *and* $\mu: \mathscr{G}^{\cdot} \to \mathscr{H}^{\cdot}$ *be two good maps of perfect complexes, then the composition is good too and we have*:

$$\mathrm{Div}(\mu\cdot\lambda) = \mathrm{Div}(\mu) + \mathrm{Div}(\lambda)\,.$$

(ii) *Consider a strictly commutative diagram of short-exact sequences of perfect complexes*:

$$\begin{array}{ccccccccc} 0 \to & \mathscr{F}^{\cdot} & \longrightarrow & \mathscr{G}^{\cdot} & \longrightarrow & \mathscr{H}^{\cdot} & \to 0 \\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & \\ 0 \to & \mathscr{F}^{\cdot\prime} & \longrightarrow & \mathscr{G}^{\cdot\prime} & \longrightarrow & \mathscr{H}^{\cdot\prime} & \to 0\,. \end{array}$$

Then of any two if the vertical maps are good, so is the third and we have:

$$\mathrm{Div}(\alpha) - \mathrm{Div}(\beta) + \mathrm{Div}(\gamma) = 0\,.$$

(iii) *Let*

$$0 \to \mathscr{F}^{\cdot} \xrightarrow{\lambda} \mathscr{G}^{\cdot} \xrightarrow{\mu} \mathscr{H}^{\cdot} \to 0$$

be a short exact sequence of perfect complexes such that λ *is good, then* $0^{\cdot} \to \mathscr{H}^{\cdot}$ *is good and we have*:

$$\mathrm{Div}(\lambda) = \mathrm{Div}(\mathscr{H}^{\cdot})\,.$$

(v) *Let $f:X \to Y$ be a morphism of noetherian schemes, $\lambda:\mathscr{F}^{\cdot} \to \mathscr{G}^{\cdot}$ a good map of perfect complexes on Y. Suppose that for each $x \in X$ of depth 0, $f(x) \in U(\lambda)$, then the map*:

$$\mathrm{L}f^*(\lambda):\ \mathrm{L}f^*(\mathscr{F}^{\cdot}) \to \mathrm{L}f^{\cdot}(\mathscr{G}^{\cdot})$$

is good too, and we have:

$$\mathrm{Div}(\mathrm{L}f^*(\lambda)) = f^*(\mathrm{Div}(\lambda)) .$$

(vi) *Let X be a normal noetherian scheme and $\mathscr{F}^{\cdot}$ a good perfect complex on X. For every point x in X of depth 1 recall that $\mathcal{O}_X$ is a discrete rank 1 valuation ring, and since $\mathscr{F}^{\cdot}$ is good $H^i(\mathscr{F}^{\cdot})_X$ is a torsion $\mathcal{O}_X$-module of finite length, say*:

$$\mathrm{length}\,(H^i(\mathscr{F}^{\cdot})_x) = r_x{}^i(\mathscr{F}^{\cdot}) .$$

We define the number:

$$\mathrm{r}_x(\mathscr{F}^{\cdot}) = \textstyle\sum_{i=-\infty}^{\infty}(-1)^i \mathrm{r}_x{}^i(\mathscr{F}^{\cdot}) .$$

Since X is a normal noetherian scheme the group of Cartier-divisors injects into the group of Weil-divisors and we have

$$(*) \qquad \mathrm{Div}(\mathscr{F}^{\cdot}) = \sum_{\substack{x\in X\\ \mathrm{depth}(x)=1}} \mathrm{r}_x(\mathscr{F}^{\cdot})\cdot \overline{\{x\}} .$$

Proof. Everything is obvious except for *v*. Since a divisor is determined by its values at points of depth 1 we may assume that $X = \mathrm{Spec}(\mathcal{O})$ where $\mathcal{O}$ is a regular local ring of dimension 1.

For every good perfect complex $\mathscr{F}^{\cdot}$ on X we define:

$$\mathrm{Div}(\mathscr{F}^{\cdot}) = \mathrm{r}_x(\mathscr{F}^{\cdot})\cdot x$$

where x is the unique closed point of X. Clearly Div satisfies (i), (ii), and (iii). Since every coherent sheaf $\mathscr{F}$ on X with $\mathrm{Supp}(\mathscr{F}) \subset \{x\}$ can be considered as a perfect complex, it follows by induction that we can reduce the proof of the equality (*) to the case where $\mathscr{F}^{\cdot}$ is a complex of length 1, that is $\mathscr{F}^{\cdot} = \tilde{M}$ in degree 0 and 0 otherwise where M is a torsion $\mathcal{O}$-module. By the structure theorem for such modules we can find integers n_i, $1 \leqq i \leqq s$ such that

$$M \approx \textstyle\sum_{i=1}^{s} \mathcal{O}/\pi^{n_i}\mathcal{O} .$$

We then have a free resolution of M

$$0 \to \mathcal{O}_s \xrightarrow{\ d\ } \mathcal{O}_s \longrightarrow M \to 0$$

where d is given by the matrix

$$\begin{pmatrix} \pi^{n_1} & 0 & 0 & 0 \\ 0 & \pi^{n_2} & 0 & 0 \\ 0 & 0 & \pi^{n_3} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 \ldots & \pi^{n_s} \end{pmatrix}.$$

It follows that the local equation of $\mathrm{Div}(\tilde{M})$ is $\det(d) = \pi^{\sum n_i}$. Since length $M = \sum n_i$, the equality (*) follows.

Let $f\colon X \to Y$ be a morphism of noetherian schemes, and $\mathscr{F}^\cdot$ a perfect complex on X. We put:

$$\mathrm{Supp}(\mathscr{F}^\cdot) = \bigcup_i \mathrm{Supp}(H^i(\mathscr{F}^\cdot)) .$$

For any point $y \in Y$ consider the fibre product

$$\begin{array}{ccc} \mathrm{Supp}(\mathscr{F}^\cdot)_Y & \longrightarrow & \mathrm{Supp}(\mathscr{F}^\cdot) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\,k(y)) & \longrightarrow & Y \end{array}$$

Definition. Let f, X, Y and $\mathscr{F}^\cdot$ be as above. We will say that $\mathscr{F}^\cdot$ satisfies condition $Q_{(r)}$ if the following holds:

1) For each point $y \in Y$ of depth 0

$$\dim(\mathrm{Supp}(\mathscr{F}^\cdot)_y) \leq r .$$

2) For each point $y \in Y$ of depth 1

$$\dim(\mathrm{Supp}(\mathscr{F}^\cdot)_y) \leq r+1 .$$

Proposition 9. *Let $f\colon X \to Y$ be a proper morphism of finite* Tor-*dimension. If $\mathscr{F}^\cdot$ is a perfect complex on X satisfying condition $Q_{(-1)}$ for the morphism f, then*

a) $\mathrm{Div}(\mathrm{R}f_*(\mathscr{F}^\cdot))$ *is defined,*
b) *for all line bundles $\mathscr{H}$ on X,*

$$\mathrm{Div}(\mathrm{R}f_*(\mathscr{F}^\cdot)) = \mathrm{Div}(\mathrm{R}f_*(\mathscr{F}^\cdot \otimes \mathscr{H})) .$$

Proof. a) is clear and to prove (b), we may make a base change and replace Y by $\mathrm{Spec}\,\mathcal{O}_{y,Y}$, where $y \in Y$ has depth 0 or 1. Then $\mathrm{Supp}(\mathscr{F}^\cdot)$ is finite over Y, hence there is an open neighborhood U

$$\mathrm{Supp}(\mathscr{F}^\cdot) \subset U \subset X$$

and an isomorphism of $\mathscr{H}|_U$ with $\mathcal{O}_U$. Therefore there is a sheaf of ideals $\mathscr{I} \subset \mathcal{O}_X$ such that $\operatorname{Supp}\mathcal{O}_X/\mathscr{I} \subset X - U$ and a homomorphism φ as follows:

$$0 \to \mathscr{I} \to \mathscr{H} \to \mathscr{K} \to 0$$

$$\operatorname{Supp}(\mathscr{K}) \subset X - U .$$

Then $\mathscr{F}^{\cdot} \otimes^L \mathcal{O}_X/\mathscr{I}$ and $\mathscr{F}^{\cdot} \otimes^L \mathscr{K}$ are acyclic, hence $\mathscr{F}^{\cdot} \otimes \mathscr{H}$ is quasi-isomorphic first to $\mathscr{F} \otimes^L \mathscr{I}$, and second to $\mathscr{F}^{\cdot}$. This proves (b).

Let $f: X \to Y$ be a morphism of noetherian schemes, $\mathscr{F}^{\cdot}$ a perfect complex on X and consider the function $Y \to \mathbf{Z}$ given by

$$y \to \dim(\operatorname{Supp}(\mathscr{F}^{\cdot})_y) .$$

It will be convenient to compute this function in a slightly different manner. Consider the fibre product:

$$\begin{array}{ccc} X_y & \xrightarrow{\;i\;} & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\boldsymbol{k}(y)) & \longrightarrow & Y \end{array}$$

Lemma 1. *With the notations as above we have*:

$$\dim(\operatorname{Supp}(\mathscr{F}^{\cdot})_y) = \dim \operatorname{Supp}(\mathrm{L}i^*\mathscr{F}^{\cdot}) .$$

Proof. We may assume X and Y affine, so let $X = \operatorname{Spec}(S)$, $Y = \operatorname{Spec}(R)$, and let $Y = [p]$. Let k be the field $\boldsymbol{k}(y)$

$$\boldsymbol{k}(y) = R_p/p \cdot R_p$$

we then have:

$$X_y = \operatorname{Spec}(S \otimes_R k) .$$

Also we may assume that $\mathscr{F}^{\cdot} = \tilde{M}^{\cdot}$ where $M^{\cdot}$ is a bounded complex of finite free S-modules, hence $\mathrm{L}i^*\mathscr{F}^{\cdot}$ is represented by $M^{\cdot} \otimes_S (S \otimes_R k) = M^{\cdot} \otimes_R k$. But there is a spectral sequence:

$$E_2^{-p,q} = \operatorname{Tor}^S{}_p(H^q(M^{\cdot}), S \otimes_R k) \Rightarrow H^n(M^{\cdot} \otimes_R k) .$$

If $x \in \bigcup \operatorname{Supp}(H^i(M^{\cdot}) \otimes_R k)$, let i_0 be the maximum of the indexes i such that:

$$x \in \operatorname{Supp}(H^i(M^{\cdot}) \otimes_R k) .$$

Then

$$x \in \operatorname{Supp}\big(\operatorname{Tor}^S{}_p(H^i(M), S \otimes_R k)\big)$$

for $i > i_0$. Consequently we have

$$x \in \bigcup_{p+q=i_0} \mathrm{Supp}(E_r^{p,q})$$

for all r, and hence

$$x \in \mathrm{Supp}(H^{i_0}(M^{\cdot} \otimes_R k)) .$$

Conversely, if $x \in \bigcup \mathrm{Supp}(H^i(M^{\cdot} \otimes_R k))$ we have $x \in \bigcup_{p,q} \mathrm{Supp}(E_r^{p,q})$ for all r, Since

$$\mathrm{Supp}\big(\mathrm{Tor}^S{}_r(H^q(M^{\cdot}), S \otimes_R k)\big) \subset \mathrm{Supp} H^q(M^{\cdot}) \otimes_R k$$

we are done.

Now we come to the main application of our techniques, namely to *"Chow points"*. Let Y be a noetherian scheme, and E a locally free rank $n+1$ sheaf of $\mathcal{O}_Y$-modules. These define:

$P = \mathsf{P}(E)$, a P^n-bundle over Y,

$\pi : P \to Y$ the projection,

$\mathcal{O}_p(1)$, the "tautological" line bundle (s.t. $\pi_* \mathcal{O}_p(1) = E$),

$\check{P} = \mathsf{P}(\check{E})$ the dual, $\mathcal{O}_{\check{P}}(1)$ its tautological line bundle,

$H \subset P \times_Y \check{P}$ the universal hyperplane, i.e.,

$$E \otimes \check{E} \cong \mathcal{O}_Y \oplus [\text{trace zero subsp. of } E \otimes \check{E}] \quad \text{canonically,}$$

and if $1 \in \Gamma(\mathcal{O}_Y)$ corresponds to

$$\delta \in \Gamma(Y, E \otimes \check{E}) = \Gamma(P \times_Y \check{P}, p_1{}^*\mathcal{O}(1) \otimes p_2{}^*\mathcal{O}(1))$$

then $H = V(\delta)$.

$\mathcal{K}^{\cdot}_{(1)}$: the complex on $P \times_Y \check{P}$:

$$\begin{array}{ccccc} 0 \to & p_1{}^*\mathcal{O}_P(-1) \otimes p_2{}^*\mathcal{O}_{\check{P}}(-1) & \xrightarrow{\otimes \delta} & \mathcal{O}_{P \times \check{P}} & \to 0 \\ & \| & & \| & \\ & \mathcal{K}^{-1}_{(1)} & & \mathcal{K}^{0}_{(1)} & \end{array}$$

which resolves $\mathcal{O}_H$.

$P \times_Y (\check{P})^k =$ the fibre product over Y,

$\mathcal{K}^{\cdot}_{(k)}$: the complex $\otimes_{i=2}^{k+1} p_{1,i}^*(\mathcal{K}^{\cdot}_{(1)})$ on $P \times_Y \check{P}^k$.

This complex is a resolution of $\mathcal{O}_{H_k}$, where

$$H_k = \bigcap_{i=2}^{k+1} p_{1,i}^{-1}(H) .$$

So much for the "universal" elements of our construction. Now say $\mathcal{F}^{\cdot}$ is a perfect complex on P and define:

$$\mathcal{F}^{\cdot}_{(k)}(n) = \mathrm{L}p_1{}^*(\mathcal{F}^{\cdot}(n)) \otimes^L \mathcal{K}^{\cdot}_{(k)}, \quad \text{on } P \times_Y \check{P}^k ,$$

$$\mathcal{L}_{(k)}(n) = \det(\mathrm{R}p_{2*}\mathcal{F}^{\cdot}_{(k)}(n)), \quad \text{on } \check{P}^k,$$
$$\mathcal{L}(n) = \det(\mathrm{R}\pi_*\mathcal{F}^{\cdot}(n)), \quad \text{on } Y.$$

Lemma 2. *If $\mathcal{F}^{\cdot}$ satisfies condition $Q_{(r)}$ for the morphism $\pi: P \to Y$, and $r \geqq k-1$, then $\mathcal{F}^{\cdot}_{(k)}$ satisfies condition $Q_{(r-k)}$ for the morphism $p_2: P \times_Y \check{P}^k \to \check{P}^k$.*

Proof. By induction it is sufficient to prove the Lemma in case $k=1$ (with $\check{P}^{k-1}$ as the new Y and $\mathcal{F}^{\cdot}_{(k-1)}$ as the new $\mathcal{F}^{\cdot}$). If x is a point of $\hat{P}$, let $y=\pi(x) \in Y$ and let $k = \boldsymbol{k}(y)$.
Identifying the fibre of $\check{P}$ over y with $\check{\mathsf{P}}_k^n$, we get the diagram:

$$\begin{array}{ccccc} \mathrm{Spec}(\boldsymbol{k}(x)) & \longrightarrow & \check{\mathsf{P}}_k^n & \longrightarrow & \check{P} \\ & & \downarrow & & \downarrow{\scriptstyle p_2} \\ & & \mathrm{Spec}(k) & \longrightarrow & Y \end{array}$$

Since p_2 is flat, it follows from E.G.A., Chapitre IV, Proposition 6.3.1 that:
$$\mathrm{depth}(\mathcal{O}_{Y,y}) + \mathrm{depth}(\mathcal{O}_{\check{\mathsf{P}}_k^n,x}) = \mathrm{depth}(\mathcal{O}_{\check{P},x}).$$
From this and the previous lemma it follows that we may assume $Y=\mathrm{Spec}(k), P=\mathsf{P}_k^n, k$ a field, in which case the Lemma is straight-forward.

Corollary-Definition. *If $\mathcal{F}^{\cdot}$ satisfies condition $Q_{(r)}$, then $\mathcal{F}^{\cdot}_{(r+1)}$ satisfies $Q_{(-1)}$, hence we can define the Chow divisor*
$$\mathrm{Chow}(\mathcal{F}^{\cdot}) = \mathrm{Div}(\mathrm{R}p_{2*}\mathcal{F}^{\cdot}_{(r+1)})$$
on $\check{P}^{r+1}$. Then $\mathrm{Chow}(\mathcal{F}^{\cdot}(n)) = \mathrm{Chow}(\mathcal{F}^{\cdot})$ *and there is a canonical isomorphism:*
$$\mathcal{O}_{\check{P}^{r+1}}(\mathrm{Chow}(\mathcal{F}^{\cdot})) \cong \mathcal{L}_{(r+1)}(n), \quad \textit{for every } n.$$

Next, we would like to compute $\mathcal{L}_{(k)}(n)$ in another way: since $\mathcal{K}^{\cdot}_{(k)}$ is locally free, each term $\mathcal{F}^{\cdot}\otimes\mathcal{K}_{(k)}^{l}$ is perfect, hence there is a canonical isomorphism
$$\mathcal{L}_{(k)}(n) = \det(\mathrm{R}p_{2*}\mathcal{F}^{\cdot}_{(k)}(n)) \cong \otimes_{l=0}^{k} \det(\mathrm{R}p_{2*}\mathrm{L}p_1{}^*\mathcal{F}^{\cdot}(n)\otimes^{L}\mathcal{K}_{(k)}^{-l})^{(-1)^l}.$$
On $\check{P}^k$, let $\mathcal{H}_i$ be the invertible sheaf $\mathcal{O}_{\check{P}}(1)$ pulled up from the i^{th} factor. Then by definition:
$$\mathcal{K}_{(k)}^{-l} = p_1{}^*(\mathcal{O}_P(-l))\otimes p_2{}^* \textstyle\sum_{1<i_1<\ldots<i_l\leq k}\mathcal{H}_{i_1}^{-1}\otimes\ldots\otimes\mathcal{H}_{i_l}^{-1}$$

hence if $\check{\pi}: \check{P}^k \to Y$ denotes the projection:

$$\begin{aligned}\mathscr{L}_{(k)}(n) &\cong \otimes_{l=0}^{k}\otimes_{1<i_1<\ldots<i_l\leq k}\det\big(\mathrm{R}p_{2*}(\mathrm{L}p_1{}^*\mathscr{F}^{\cdot}(n-l)\\ &\quad\otimes p_2{}^*(\mathscr{H}_{i_1}^{-1}\otimes\ldots\otimes\mathscr{H}_{i_l}^{-1}))\big)^{(-1)^l}\\ &\cong \otimes_{l=0}^{k}\otimes_{1<i_1<\ldots<i_l\leq k}\det\big(\mathrm{L}\check{\pi}^*(\mathrm{R}\pi_*\mathscr{F}^{\cdot}(n-l))\\ &\quad\otimes\mathscr{H}_{i_1}^{-1}\otimes\ldots\otimes\mathscr{H}_{i_l}^{-1}\big)^{(-1)^l}\end{aligned}$$

On the other hand, it is easy to check that for any perfect complex and invertible sheaf:

$$\det(\mathscr{G}^{\cdot}\otimes\mathscr{L}) \cong \det(\mathscr{G}^{\cdot})\otimes\mathscr{L}\mathrm{rk}(\mathscr{G}^{\cdot}) .$$

Note that

$$\mathrm{rk}(\mathscr{L}(n)) = \chi(\mathscr{F}^{\cdot}(n))$$

i.e. = the continuous function $Y \to \mathbf{Z}$ given by

$$y \to \sum(-1)^i\dim_{k(y)}H^i(\mathscr{F}^{\cdot}\otimes^L P_{k(y)}) .$$

We abbreviate this to $\chi(n)$. Therefore we have canonical isomorphisms:

$$\mathscr{L}_{(k)}(n) \cong \otimes_{l=0}^{k}\otimes_{1\leq i_1<\ldots<i_l\leq k}\check{\pi}^*\mathscr{L}(n-l)^{(-1)^l}\otimes(\mathscr{H}_{i_1}\otimes\ldots\otimes\mathscr{H}_{i_l})^{(-1)^{l+1}\chi(n-l)} .$$

Now defined by induction:

a) "difference" sheaves:

$$\begin{aligned}\Delta\mathscr{L}(n) &= \mathscr{L}(n)\otimes\mathscr{L}(n-1)^{-1}\\ \Delta^k\mathscr{L}(n) &= \Delta^{k-1}\mathscr{L}(n)\otimes\Delta^{k-1}\mathscr{L}(n-1)^{-1}\\ &\cong \Delta^{k-2}\mathscr{L}(n)\otimes\Delta^{k-2}\mathscr{L}(n-1)^{-2}\otimes\Delta^{k-2}\mathscr{L}(n-2)\\ &\quad\ldots\\ &\cong \otimes_{l=0}^{k}\mathscr{L}(n-l)^{(-1)^l\binom{k}{l}}\end{aligned}$$

b) difference functions:

$$\begin{aligned}\chi_1(n) &= \chi(n)-\chi(n-1)\\ \chi_k(n) &= \chi_{k-1}(n)-\chi_{k-1}(n-1)\\ &\quad\ldots\\ &= \textstyle\sum_{l=0}^{k}(-1)^l\binom{k}{l}\chi(n-l)\end{aligned}$$

Then it follows easily that:

$$\mathscr{L}_{(k)}(n) \cong \check{\pi}^*(\Delta^k\mathscr{L}(n))\otimes(\mathscr{H}_1\otimes\ldots\otimes\mathscr{H}_k)^{\chi_{k-1}(n-1)} .$$

Combining this with above Corollary, if $\mathscr{F}^{\cdot}$ satisfies $Q_{(r)}$, then we find

$\chi_r(n)$ is independent of n
Up to canonical isomorphisms, $\pi^*(\Delta^{r+1}\mathscr{L}(n))$ is independent of n.

Since $\check{\pi}_*(\mathcal{O}_{\check{P}_k}) = \mathcal{O}_Y$, this implies that:

Up to canonical isomorphisms, $\Delta^{r+1}\mathscr{L}(n)$ independent of n. Going backwards, this implies that χ is a polynomial of degree at most r and that $\mathscr{L}(n)$ can be expanded as in the following final Theorem:

THEOREM 4. *Let Y be a noetherian scheme, E a locally free sheaf of rank $n+1$ on $Y, P = \mathsf{P}(E)$ and $\mathscr{F}^\cdot$ a perfect complex on P satisfying condition $Q_{(r)}$ for $\pi\colon P \to Y$. Then there are sheaves $\mathscr{M}_0, \ldots, \mathscr{M}_{r+1}$ on Y and canonical and functorial isomorphisms:*

$$\det(\mathrm{R}\pi_*\mathscr{F}^\cdot(n)) \cong \otimes_{k=0}^{r+1}\mathscr{M}_k^{\binom{n}{k}}.$$

Moreover the leading term $\mathscr{M}_{r+1}$ is related to the Chow divisor by a canonical isomorphism:

$$\check{\pi}^*\mathscr{M}_{r+1} \otimes (\mathscr{H}_1 \otimes \ldots \otimes \mathscr{H}_{r+1})^d \cong \mathcal{O}_{\check{P}^{r+1}}(\mathrm{Chow}(\mathscr{F}^\cdot))$$

where $\check{\pi}\colon \check{P}^{r+1} \to Y$ is the projection,

$$\mathscr{H}_i = i^{\text{th}} \textit{ sheaf } \mathcal{O}_{\check{P}}(1) \textit{ on } \check{P}^{r+1},$$

$d\binom{n}{r}$ = leading term of the Hilbert polynomial $\chi(\mathscr{F}^\cdot(n))$.

REFERENCES

1. P. Deligne and D. Mumford, *The irreducibility of the space of curves of a given genus*, Inst. Hautes Études Sci. Publ. Math. 36 (1969), 75–109.
2. P. Berthelot, A. Grothendieck and L. Illusie, SGA 6, *Théorie des intersectiones et Théoreme de Riemann-Roch*, Lecture Notes in Mathematics 225, Springer-Verlag, Berlin, Heidelberg New York, 1969.
3. D. Mumford, *Geometric Invariant Theory*, Ergebnisse der Math. 34, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1965.
4. J. Fogarty, *Truncated Hilbert functors*, J. Reine und Angew. Math. 234 (1969), 65–88.

UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA, U.S.A.
AND
HARVARD UNIVERSITY, CAMBRIDGE, MASSACHUSETTS, U.S.A.

MATH. SCAND 52 (1983), 161–199

THE PROJECTIVITY OF THE MODULI SPACE OF STABLE CURVES, II: THE STACKS $M_{g,n}$

FINN F. KNUDSEN

Introduction.

This paper is the second in a sequence of three papers by D. Mumford and myself, containing the results of my thesis and leading to a proof of the projectivity of the moduli space of stable curves. The story is as follows: after investigating the stack $M_{g,0}$ with Deligne, Mumford got interested in the question of whether or not it was projective. His original idea was to study the Torelli map:

$$t : M_{g,0} \to \begin{Bmatrix} \text{Satake's compactification of the} \\ \text{moduli space of abelian varieties.} \end{Bmatrix}$$

and use the fact that Satake's compactification was a projective variety (defined over $\mathbf{Z}$ by use of θ-functions in [12]). In my thesis, I then investigated the line bundles on $M_{g,0}$ and showed that the line bundle $\mathfrak{F}^{-1}$ (defined in section 4) was ample on all fibres of t. The map t, however, has only been constructed in characteristic 0. Seshadri then suggested that the problem could be attacked directly without the use of Jacobians by using instead the stability of Chow points of curves proven in [9], theorem 4.5. Mumford realized that it was necessary for this proof to introduce curves with basepoints, i.e. the stacks $M_{g,n}$ (cf. section 1 for definition). In this paper we study the stacks $M_{g,n}$, and certain maps between these stacks, that is:

1) contraction: $M_{g,n+1} \to M_{g,n}$

2) clutching: $\begin{cases} M_{g,n+2} \to M_{g+1,n} \\ M_{g_1,n_1+1} \times M_{g_2,n_2+1} \to M_{g_1+g_2,n_1+n_2} . \end{cases}$

In the first three sections of this paper we investigate these maps and prove that they are representable. The crucial point is to prove that $M_{g,n+1} \approx Z_{g,n}$, the universal curve over $M_{g,n}$, hence contraction is representable. The clutching maps factor through the contraction map.

Received August 6, 1979; in revised form November 23, 1981.

In sections 1 and 2 we prove that we have an isomorphism of functors $M_{g,n+1} \approx Z_{g,n}$, where $Z_{g,n}$ is the universal curve over $M_{g,n}$ (i.e. the functor of n-pointed stable curves with one extra section without any smoothness condition). The main steps of this proof are Lemma 1.6, Theorem 1.8, and the results in the appendix. We then use this result and an inductive argument to prove that $M_{g,n}$ is an algebraic stack, proper and smooth over Spec $(\mathbf{Z})$, and that the substack $S_{g,n}$ consisting of singular curves is a divisor with normal crossings relative to Spec (Z) (cf. Theorem 2.7.).

In section 3 we study the clutching morphisms β^0 and $\beta_{g_1,g_2,H,K}$, and prove that they are representable (Lemma 3.7), finite and unramified and almost always closed immersions (Corollary 3.9).

The clutching sequence Theorem 3.5 is used in section 4 to compute the pullback of the basic line bundles on $M_{g,n}$ by β. Actually what we are doing here is computing the self-intersection of the divisor at infinity $S_{g,n}$ of $M_{g,n}$ (section 4).

1. n-pointed stable curves.

Let S be a scheme, and let g, n be non-negative integers such that $2g-2+n > 0$.

DEFINITION 1.1. An n-pointed stable curve of genus g over S is a flat and proper morphism $\pi: C \to S$ together with n distinct sections $s_i: S \to C$ such that

i) The geometric fibres C_s of π are reduced and connected curves with at most ordinary double points.
ii) C_s is smooth at $P_i = s_i(s)$ $(1 \leqq i \leqq n)$.
iii) $P_i \neq P_j$ for $i \neq j$.
iv) The number of points where a nonsingular rational component E of C_s meets the rest of C_s plus the number of points P_i on E is at least 3.
v) $\dim H^1(C_s, \mathcal{O}_{C_s}) = g$.

Note that if $n=0$ and $g \geqq 2$, then C is a stable curve in the sense of [3].

Before we start with the technicalities, we briefly state some facts about the basic sheaves on a stable curve. Let $\pi: C \to S$ be a stable curve with sections s_i, $1 \leqq i \leqq n$. Since π is flat and the geometric fibres are reduced with only ordinary double points, π is locally a complete intersection morphism [8]. Therefore there is a canonical invertible dualizing sheaf $\omega_{C/S}$ on C. For reference, see [6], where $\omega_{C/S}$ is also denoted by $\pi^! \mathcal{O}_S$.

A way of constructing $\omega_{C/S}$ is via the theory of determinants [7]. The sheaf $\Omega_{C/S}$ of relative Kähler differentials is flat over S (see section 3, Proposition 3.2) and locally on C we can find a two-term free resolution

$$(0) \to \mathscr{E}^1 \to \mathscr{E}^0 \to \Omega_{C/S}|_U \to (0) .$$

This means that $\Omega_{C/S}$ considered as a complex supported only in degree 0 is a perfect complex, so we may form its determinant. We then have canonical isomorphisms

$$\Lambda^{\max}\mathscr{E}^0 \otimes \Lambda^{\max}(\mathscr{E}^1)\check{} \approx \det \Omega_{C/S}|_U \approx \omega_{C/S}|_U .$$

Since the fibres of π are reduced and one-dimensional, $\operatorname{rank}\mathscr{E}^0 = \operatorname{rank}\mathscr{E}^1 + 1 = k+1$. Let $\bar{v}$ be an element of $\Omega_{C/S,x}$, where $x \in U$ and let $v \in \mathscr{E}^0_x$ be an element which maps to $\bar{v}$. Choose a basis $w_1, w_2, \ldots, w_k$ of $\mathscr{E}^1$ and let $w'_1, w'_2, \ldots, w'_k$ be the dual basis. Considering the elements w_i as elements of $\mathscr{E}^0$ as well, we get an element

$$v \wedge w_1 \wedge w_2 \wedge \ldots \wedge w_k \otimes w'_1 \wedge w'_2 \wedge \ldots \wedge w'_k \in (\Lambda^{\max}\mathscr{E}^0 \oplus \Lambda^{\max}(\mathscr{E}^1)\check{}\,)_x$$

which is independent of the choice of $\bar{v}$ and the w_i's. Composing with the isomorphisms above, we see that we have a canonical homomorphism

$$\psi : \Omega_{C/S} \to \omega_{C/S} .$$

In general, this homomorphism is neither injective nor surjective, but it is an isomorphism near every point of C where π is smooth. Since $\Omega_{C/S}$ is flat over S, we have compactibility with any base change; i.e.,

a) For any morphism $T \to S$ we have a commutative diagram

$$\begin{array}{ccc} p_1^*(\Omega_{C/S}) & \xrightarrow{\sim} & \Omega_{C\times_S T/T} \\ {\scriptstyle p_1^*\psi}\downarrow & & \psi\downarrow \\ p_1^*(\omega_{C/S}) & \xrightarrow{\sim} & \omega_{C\times_S T/T} \end{array}$$

b) If $S = \operatorname{Spec}(k)$, where k is an algebraically closed field, $f\colon \tilde{C} \to C$ is the normalization of C, and $x_1, \ldots, x_m, y_1, \ldots, y_m$ are the points of $\tilde{C}$ such that $z_i = f(x_i) = f(y_i)$, $1 \leqq i \leqq m$, are the double points of C, then $\omega_{C/S}$ is the sheaf of 1-forms η regular on $\tilde{C}$ except for simple poles at the x's and the y's and such that

$$\operatorname{Res}_{x_i}(\eta) + \operatorname{Res}_{y_i}(\eta) = 0 \quad \text{for } 1 \leqq i \leqq m .$$

c) If S is locally noetherian and of finite Krull dimension, and $\mathscr{F}$ is a coherent sheaf on C, then

$$\operatorname{Hom}_{\mathcal{O}_S}(R^1\pi_*\mathscr{F}, \mathcal{O}) \approx \operatorname{Hom}_{\mathcal{O}_C}(\mathscr{F}, \omega_{C/S})$$

(cf. [6, VII, Corollary 4.3.])

DEFINITION 1.2. We denote by $M_{g,n}$ the category of n-pointed stable curves. Morphisms in this category are diagrams of the form

$$\begin{array}{ccc} C' & \xrightarrow{f} & C \\ {\scriptstyle s_i'}\uparrow\downarrow{\scriptstyle \pi'} & & {\scriptstyle s_i}\uparrow\downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{g} & S \end{array}$$

where

(i) $$fs_i' = s_i g \quad \text{for } 1 \leqq i \leqq n .$$

(ii) f, π' induce an isomorphism

$$C' \xrightarrow{\sim} C \times_S S' .$$

We denote by $Z_{g,n}$ the category of n-pointed stable curves with an extra section Δ. Morphisms in $Z_{g,n}$ are diagrams as above such that $f\Delta' = \Delta g$.

The category $M_{g,n}$ is a stack fibred in groupoids over the category of schemes. For a definition of stack, see [3, Definition 4.1]. In the next paragraph we prove that $M_{g,n}$ is a separated algebraic stack, smooth and proper over Spec ($\mathbf{Z}$).

The following definition plays a central role in this whole paper.

DEFINITION 1.3. A morphism of pointed stable curves over S:

$$\begin{array}{ccc} C & \xrightarrow{f} & C' \\ {\scriptstyle s_i}\uparrow\downarrow{\scriptstyle \pi} & & {\scriptstyle s_i'}\uparrow\downarrow{\scriptstyle \pi'} \\ S & = & S \end{array}$$

is called a contraction of

(i) C is an $n+1$-pointed curve, C' is an n-pointed curve and $fs_i = s_i'$ for $1 \leqq i \leqq n$.
(ii) If we consider the induced morphism on a geometric fibre C_s, we have one of two possible cases:

a) f_s: $C_s \to C_s'$ is an isomorphism.
b) There is a rational component $E \subset C_s$ such that $s_{n+1}(s) \in E$, $f_s(E) = x$ is a closed point of C_s', and

$$f_s : C_s \setminus E \to C_s' \setminus \{x\}$$

is an isomorphism.

The picture in Figure 1 below shows the only two non-trivial contractions over Spec (k), where k is an algebraically closed field.

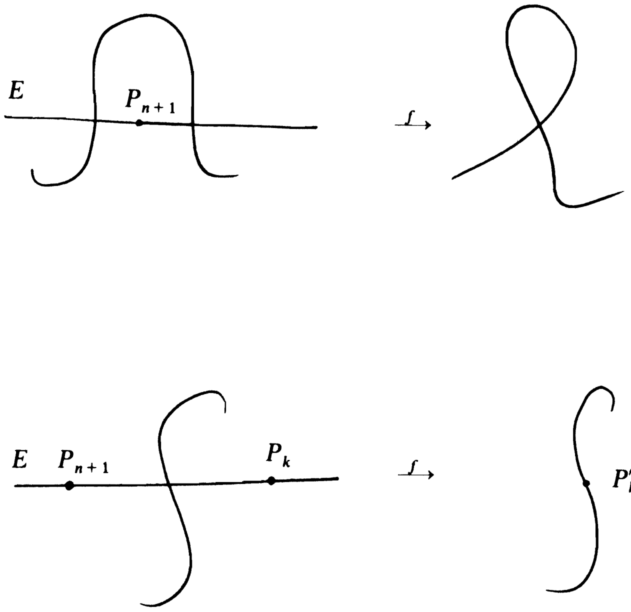


Figure 1.

REMARK. We leave to the reader to verify that when $S = \mathrm{Spec}(k)$, k an algebraically closed field, then for every C over S, there is one, and up to a unique isomorphism, only one contraction $C \to C'$. In fact, we have an equivalence of categories:

$$M_{g,n+1}(k) \xrightarrow{\sim} Z_{g,n}(k) .$$

In order to prove that there is an isomorphism of stacks $M_{g,n+1} \xrightarrow{\sim} Z_{g,n}$, we need the following results, which are corollaries of [5, III 4.6.1].

LEMMA 1.4. *Let Y be a locally noetherian scheme, f: $X \to Y$ a proper morphism, $\mathscr{F}$ a coherent sheaf on X, and y a point on Y. Suppose $f^{-1}(y) = X \times_Y \mathrm{Spec}(k(y))$ is an n-dimensional scheme and that*

$$H^n(f^{-1}(y), \mathscr{F} \otimes_{\mathcal{O}_Y} k(y)) = (0) .$$

Then:

i) *There exists a neighbourhood U of y in Y such that $R^n f_* \mathscr{F}|_U = (0)$.*
ii) *For each integer $p \geq 0$ the canonical morphism*

$$(R^{n-1} f_* \mathscr{F})_y \to H^{n-1}(f^{-1}(y), \mathscr{F} \otimes_{\mathcal{O}_y} \mathcal{O}_y / m_y^{p+1})$$

is surjective.

PROOF. Consider the diagram

$$\begin{array}{ccccc} f^{-1}(y) & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}(k(y)) & \rightarrow & \operatorname{Spec}(\mathcal{O}_y) & \xrightarrow{i} & Y. \end{array}$$

Since i is flat, we can reduce the proof of the lemma to the case, where Y is affine and y is a closed point of Y. Since $R^n f_* \mathscr{F}$ is coherent, the first assertion is equivalent to $(R^n f_* \mathscr{F})_y = (0)$.

By [5, III 4.2.1], it suffices to prove that

$$H^n(f^{-1}(y), \mathscr{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_y/m_y^{p+1}) = (0) \qquad \text{for all } p .$$

It is true for $p=0$, so we proceed to prove it by induction.

Let $X_p = X \times_Y \operatorname{Spec}(\mathcal{O}_y/m_y^{p+1})$. Then X_{p-1} is a closed subscheme of X_p with the same underlying topological space. Hence by induction hypothesis we have

$$H^n(X_p, \mathscr{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_y/m_y^p) = H^n(X_{p-1}, \mathscr{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_y/m_y^p) = (0) .$$

On X_p we have an exact sequence of sheaves

$$(0) \rightarrow m_y^p \mathscr{F}/m_y^{p+1} \mathscr{F} \rightarrow \mathscr{F}/m_y^{p+1} \mathscr{F} \rightarrow \mathscr{F}/m_y^p \mathscr{F} \rightarrow (0) .$$

So from the long exact cohomology sequence it suffices to prove that for each p

$$(*) \qquad H^n(X_p, m_y^p \mathscr{F}/m_y^{p+1} \mathscr{F}) = (0) .$$

If we denote by Z the fibre $f^{-1}(y) = X_0$, then $m_y^p \mathscr{F}/m_y^{p+1} \mathscr{F}$ may be considered as an $\mathcal{O}_Z$-module and we have:

$$H^n(X_p, m_y^p \mathscr{F}/m_y^{p+1} \mathscr{F}) = H^n(Z, m_y^p \mathscr{F}/m_y^{p+1} \mathscr{F}) .$$

Let Q_p denote the kernel of the surjection

$$\mathscr{F}/m_y \mathscr{F} \otimes_{k(y)} m_y^p/m_y^{p+1} \rightarrow m_y^p \mathscr{F}/m_y^{p+1} \mathscr{F} .$$

We then have an exact sequence

$$\ldots \rightarrow H^n(Z, \mathscr{F}/m_y \mathscr{F} \otimes_{k(y)} m_y^p/m_y^{p+1}) \rightarrow H^n(Z, m_y^p \mathscr{F}/m_y^{p+1} \mathscr{F}) \rightarrow H^{n+1}(Z, Q_p) .$$

The sheaf $\mathscr{F}/m_y \mathscr{F} \otimes_{k(y)} m_y^p/m_y^{p+1}$ is just a direct sum of $\mathscr{F}/m_y \mathscr{F}$'s and therefore its nth cohomology group vanishes by the hypothesis. The last term vanishes since Z is n-dimensional. This proves the first assertion. For the second assertion we replace $\mathscr{F}$ by $\mathscr{G} = m_y^k \mathscr{F}$. By (*) we see that $\mathscr{G}$ satisfies the condition of the lemma so by the first assertion we have $(R^n f_* \mathscr{G})_y = (0)$.

From the long exact sequence of cohomology sheaves we get

$$(R^{n-1} f_* \mathscr{F})_y \rightarrow (R^{n-1} f_*(\mathscr{F}/m_y^p \mathscr{F}))_y \rightarrow (R^n f_*(m_y^p \mathscr{F}))_y = (0) .$$

But $R^{n-1}f_*(\mathscr{F}/m_y^p\mathscr{F})$ is a skyscraper sheaf, whose stalk at y is $H^{n-1}(f^{-1}(y), \mathscr{F}/m_y^p\mathscr{F})$. This completes the proof of the lemma.

COROLLARY 1.5. *Let S be a scheme, X and Y S-schemes and $f\colon X \to Y$ a proper S-morphism, whose fibres are at most one-dimensional. Let $\mathscr{F}$ be a coherent sheaf on X, flat over S, and consider the following two conditions:*

(1) *For each closed point $y \in Y$,*

$$H^1(f^{-1}(y), \mathscr{F} \otimes_{\mathcal{O}_Y} k(y)) = (0) .$$

(2) *For each closed point $y \in Y$, the sheaf $\mathscr{F} \otimes_{\mathcal{O}_Y} k(y)$ is generated by its global sections.*

Then if $\mathscr{F}$ satisfies (1), *we have*

i) $$R^1f_*\mathscr{F} = (0) .$$

ii) $$f_*\mathscr{F} \text{ is } S\text{-flat} .$$

iii) *For any morphism $T \to S$ we have canonical isomorphisms*

$$f_*\mathscr{F} \otimes_{\mathcal{O}_S} \mathcal{O}_T \xrightarrow{\sim} (f \times 1)_*(\mathscr{F} \otimes_{\mathcal{O}_S} \mathcal{O}_T) .$$

If $\mathscr{F}$ satisfies both (1) *and* (2) *we have*

iv) *The canonical map $f^*f_*\mathscr{F} \to \mathscr{F}$ is surjective.*

PROOF. We may assume Y and S affine. Let $\mathfrak{U}$ be a finite affine covering of X. The first three assertions follows immediately by considering the sheaves $f_*\mathscr{C}_a^i(\mathfrak{U}, \mathscr{F})$ of alternating Čech cochains on Y. By the second part of the previous lemma and condition (2) it follows that, for each closed point $x \in X$, we have a surjection

$$f^*f_*\mathscr{F} \to \mathscr{F} \otimes_{\mathcal{O}_X} k(x) ,$$

so iv) follows by Nakayama's lemma.

Let $f\colon C \to C'$ be a contraction of an $n+1$-pointed stable curve over S, s_i $(1 \leqq i \leqq n+1)$ the sections of C over S, and $t_i = s_i f$ $(1 \leqq i \leqq n)$ the sections of C' over S. For every open U in C', a regular function on U with at least simple zeros at the sections t_i pulls back to a regular function on $f^{-1}(U)$ with at least simple zeros at the s_i's $(1 \leqq i \leqq n)$. Hence we have a canonical morphism of sheaves on C'

$$u\colon \mathcal{O}_{C'}(-t_1 - \ldots - t_n) \to f_*\mathcal{O}_C(-s_1 - \ldots - s_n) .$$

By definition, the geometric fibres of $C \to C'$ are at worst $\boldsymbol{P}^1$'s. Since they all have at least one rational point via one of the compositions $s_i \circ \pi'$ $(1 \leqq i \leqq n+1)$, all fibres are at worst $\boldsymbol{P}^1$'s. Each such fibre has at most one s_i $(1 \leqq i \leqq n)$ on it, so $\mathcal{O}_C(-s_1 - \ldots - s_n)$ satisfies condition (1) of Corollary 1.5. Therefore the formation of f_* commutes with base change over S. When $S = \operatorname{Spec}(k)$, u is easily seen to be an isomorphism. Since an extension of fields is faithfully flat, u is an isomorphism at every point of C', so by Nakayama's lemma u is always surjective. By Corollary 1.5 ii), $f_*\mathcal{O}_C(-s_1 - \ldots - s_n)$ is flat over S, so by the same reasoning u is an isomorphism. The inverse of the isomorphism u induces an isomorphism of sheaves on C':

$$\mathcal{H}om_{\mathcal{O}_{C'}}(\Omega_{C'/S}, f_*\mathcal{O}_C(-s_1 - \ldots - s_n)) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{C'}}(\Omega_{C'/S}, \mathcal{O}_{C'}(-t_1 - \ldots - t_n)).$$

By the general theory of sheaves of modules there is a canonical isomorphism

$$\mathcal{H}om_{\mathcal{O}_{C'}}(\Omega_{C'/S}, f_*\mathcal{O}_C(-s_1 - \ldots - s_n)) \xrightarrow{\sim} f_*\mathcal{H}om_{\mathcal{O}_C}(f^*\Omega_{C'/S}, \mathcal{O}_C(-s_1 - \ldots - s_n)).$$

Combining this with the natural map $f^*\Omega_{C'/S} \to \Omega_{C/S}$ we get a morphism

$$f_*\mathcal{H}om_{\mathcal{O}_C}(\Omega_{C/S}, \mathcal{O}_C(-s_1 - \ldots - s_n)) \to \mathcal{H}om_{\mathcal{O}_{C'}}(\Omega_{C'/S}, \mathcal{O}_{C'}(-t_1 - \ldots - t_n)).$$

For a stable curve $\pi\colon X \to S$, let $\mathcal{F}$ be the cokernel of the morphism $\psi\colon \Omega_{X/S} \to \omega_{X/S}$. We have

$$\operatorname{Ass} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) = \operatorname{Supp} \mathcal{F} \cap \operatorname{Ass} \mathcal{O}_X$$

[2, Chapt. IV, § 1, Prop. 10.]

Since X is flat over S, the associated points of $\mathcal{O}_X$ lie over the associated points of $\mathcal{O}_S$. However, π is smooth at the associated points of the fibres and $\mathcal{F}$ is supported on the closed set where π is not smooth, so $\check{\omega}_{X/S} \to \check{\Omega}_{X/S}$ is injective. Consider the diagram

$$\begin{array}{ccc} f_*\mathcal{H}om_{\mathcal{O}_C}(\Omega_{C/S}, \mathcal{O}_C(-s_1 - \ldots - s_n)) & \to & \mathcal{H}om_{\mathcal{O}_{C'}}(\Omega_{C'/S}, \mathcal{O}_{C'}(-t_1 - \ldots - t_n)) \\ \uparrow & & \uparrow \\ f_*(\omega_{C/S}(s_1 + \ldots + s_n)^{\check{}}) & & \omega_{C'/S}(t_1 + \ldots + t_n)^{\check{}} \end{array}$$

and let $\mathcal{P}$ denote the subsheaf of $\mathcal{H}om_{\mathcal{O}_{C'}}(\Omega_{C'/S}, \mathcal{O}_{C'}(-t_1 - \ldots - t_n))$ generated by $\omega_{C'/S}(t_1 + \ldots + t_n)^{\check{}}$ and the image of $f_*(\omega_{C/S}(s_1 + \ldots + s_n)^{\check{}})$. On the nontrivial fibres of $C \to C'$, $\omega_{C/S}(s_1 + \ldots + s_n)^{\check{}}$ is non-canonically isomorphic to $\mathcal{O}_{\boldsymbol{P}^1}$, so it satisfies both conditions of Corollary 1.5. Therefore, on the geometric fibres of $C' \to S$, the map $f_*(\omega_{C/S}(s_1 + \ldots + s_n)^{\check{}}) \to \mathcal{P} \otimes_{\mathcal{O}_S} k$ factors through $\omega_{C'/S}(t_1 + \ldots + t_n)^{\check{}}$. By Nakayama's lemma, then $\omega_{C'/S}(t_1 + \ldots + t_n)^{\check{}} \approx \mathcal{P}$, so we get a global factorization which again by Nakayama's lemma is an isomorphism. Pullback by f gives us

$$f^*(\omega_{C'/S}(t_1 + \ldots + t_n)^{\check{}}) \xleftarrow{\sim} f^*f_*(\omega_{C/S}(s_1 + \ldots + s_n)^{\check{}}) \twoheadrightarrow \omega_{C/S}(s_1 + \ldots + s_n)^{\check{}}.$$

By checking on the geometric fibres we see that the surjection on the right is an isomorphism. Since for locally free sheaves f^* commutes with dualization we get an isomorphism

$$f^*\omega_{C'/S}(t_1+\ldots+t_n) \xrightarrow{\sim} \omega_{C/S}(s_1+\ldots+s_n)\,.$$

By the general theory of sheaves there is an induced map

$$\omega_{C'/S}(t_1+\ldots+t_n) \to f_*\omega_{C/S}(s_1+\ldots+s_n)\,.$$

Again $\omega_{C/S}(s_1+\ldots+s_n)$ is trivial on the fibres of f so we may apply Corollary 1.5 and Nakayama's lemma to show that this map, too, is an isomorphism. We sum this up in

LEMMA 1.6 (MAIN LEMMA). *Consider a contraction $f\colon C \to C'$ as in Definition 1.3. We denote by $\mathscr{F}$ and $\mathscr{F}'$ the sheaves $\omega_{C/S}(s_1+\ldots+s_n)$ and $\omega_{C'/S}(s'_1+\ldots+s'_n)$ respectively. Then for all $k>0$ we have*:

a) *There are canonical isomorphisms*

$$\mathscr{F}'^{\otimes k} \xrightarrow{\sim} f_*(\mathscr{F}^{\otimes k})$$

and

$$f^*\mathscr{F}'^{\otimes k} \xrightarrow{\sim} \mathscr{F}^{\otimes k}\,.$$

b) $$R^1f_*(\mathscr{F}^{\otimes k}) = (0)\,.$$

c) $$R^i\pi_*(\mathscr{F}^{\otimes k}) \approx R^i\pi'_*(\mathscr{F}'^{\otimes k}) \quad \textit{for } i \geqq 0\,.$$

PROOF. The isomorphism $f^*\mathscr{F}'^{\otimes k} \xrightarrow{\sim} \mathscr{F}^{\otimes k}$ follows from what we have just done. Pushing this down we at least get a map

$$\mathscr{F}'^{\otimes k} \to f_*f^*(\mathscr{F}'^{\otimes k}) \xrightarrow{\sim} f_*(\mathscr{F}^{\otimes k})\,.$$

Since $\mathscr{F}$ is trivial on the fibres of f, so is $\mathscr{F}^{\otimes k}$ and Corollary 1.5 applies to $\mathscr{F}^{\otimes k}$. On the geometric fibres the above composition is an isomorphism and $f_*(\mathscr{F}^{\otimes k})$ is flat over S. The same reasoning as before then proves a). b) is exactly the first assertion of Corollary 1.5 and c) follows from a) and the Leray spectral sequence which is degenerate by b).

DEFINITION 1.7. Let $\mathscr{F}$ be a coherent sheaf on a scheme X. We will say that $\mathscr{F}$ is *normally generated* if the canonical map

$$\Gamma(X,\mathscr{F})^{\otimes k} \to \Gamma(X,\mathscr{F}^{\otimes k})$$

is surjective for $k \geqq 1$.

For any pair of $\mathcal{O}_X$-modules $\mathscr{F},\mathscr{G}$, let $S(\mathscr{F},\mathscr{G})$ denote the cokernel of the map

$$\Gamma(X,\mathscr{F})\otimes\Gamma(X,\mathscr{G}) \to \Gamma(X,\mathscr{F}\otimes\mathscr{G}) .$$

Clearly, $\mathscr{F}$ is normally generated if and only if $S(\mathscr{F},\mathscr{F}^{\otimes k})=(0)$ for all $k\geqq 1$. We shall need the following result of [13].

GENERALIZED LEMMA OF CASTELNUOVO. *Suppose $\mathscr{L}$ is an invertible sheaf on a complete scheme X of finite type over a field k such that $\Gamma(X,\mathscr{L})$ has no base points and suppose $\mathscr{F}$ is a coherent sheaf on X such that*

$$H^i(X,\mathscr{F}\otimes\mathscr{L}^{-i}) = (0) \quad \textit{for } i\geqq 1 .$$

Then

(a) $$H^i(X,\mathscr{F}\otimes\mathscr{L}^{j}) = (0) \quad \textit{for } i+j\geqq 0,\ i\geqq 1 .$$

(b) $$S(\mathscr{F}\otimes\mathscr{L}^{i},\mathscr{L}) = (0) \quad \textit{for } i\geqq 0 .$$

THEOREM 1.8. *Let C be an n-pointed stable curve over* Spec (k) *with distinguished k-valued points $P_1,\ldots,P_n$. We denote by $\mathscr{L}$ the invertible sheaf*

$$\mathscr{L} = \omega_{C/k}(D) ,$$

where $D=P_1+P_2+\ldots+P_n$.
Then we have

a) $H^1(C,\mathscr{L}^{\otimes m})=(0)$ *for $m\geqq 2$.*
b) $\Gamma(C,\mathscr{L}^{\otimes m})$ *is base-point-free for $m\geqq 2$.*
c) $\mathscr{L}^{\otimes m}$ *is normally generated for $m\geqq 3$.*

PROOF. By the Künneth theorem we may assume that k is algebraically closed. Let x be a node of C. We will call x a disconnecting node if $B_x(C)$, the blowing up of C with center x, is disconnected. We first prove the theorem in case C has no disconnecting nodes. Let x be a node of C. From the exact sequence

$$(0) \to \mathcal{O}_C \to p_*\mathcal{O}_{B_x(C)} \xrightarrow{-} k \to (0) ,$$

we get $\chi(\mathcal{O}_C)=\chi(\mathcal{O}_{B_x(C)})-1$. Since $B_x(C)$ is connected, we have

$$\dim H^1(B_x(C),\mathcal{O}_{B_x(C)}) = \dim H^1(C,\mathcal{O}_C)-1 .$$

From this formula we see that a curve of genus 0 without disconnecting nodes is nonsingular, i.e., $\boldsymbol{P}^1$ and a curve of genus 1 is either nonsingular or a "ring" of $\boldsymbol{P}^1$'s as in Figure 2 below.
For $g=0$, $C=\boldsymbol{P}^1$ and the theorem is clear. When $g=1$ and C is nonsingular, the theorem is classical. Consider, then, a "ring" of $\boldsymbol{P}^1$'s. In a noncanonical way, $\omega_{C/k}\approx\mathcal{O}_C$, and since there are lots of distinguished points spread around, $\mathscr{L}$

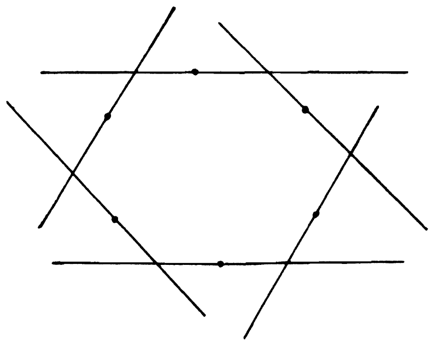


Figure 2.

restricted to each P^1 has degree $\geqq 1$. Hence $\Gamma(C, \mathscr{L})$ is base-point-free and $H^1(C, \mathscr{L}) = (0)$. The theorem then follows from Castelnuovo's Lemma. When $g \geqq 2$, we have the following result. If E is an effective divisor on C, then

$$\Gamma(\omega_C(E)) \text{ is base-point-free for } \deg E = 0 \text{ or } \deg E \geqq 2 \,,$$
$$H^1(\omega_C(E)) = 0 \quad \text{if } \deg E \geqq 1 \,.$$

PROOF. Let x be a k-rational (closed) point of C. From the short exact sequence

$$(0) \to m_x\omega(E) \to \omega(E) \to \omega(E) \otimes k \to (0)$$

we get the long exact sequence

$$\to \Gamma(\omega(E)) \to \Gamma(\omega(E) \otimes k) \to H^1(m_x\omega(E)) \xrightarrow{\alpha} H^1(\omega(E)) \to (0) \,.$$

Hence $\Gamma(\omega(E))$ is base-point-free if and only if α is an isomorphism for all points $x \in C$. By duality, α is an isomorphism, if and only if

$$\dim \mathscr{H}om_{\mathcal{O}_C}(m_x, \mathcal{O}(-E)) = \dim \Gamma(C, \mathcal{O}(-E)) \,.$$

But we have

$$\dim \mathscr{H}om_{\mathcal{O}_C}(m_x, \mathcal{O}(-E)) = \begin{cases} \dim \Gamma(C, \mathcal{O}(-E+x)), & x \text{ nonsingular} \\ \dim \Gamma(B_x(C), \mathcal{O}_{B_x(C)}(-E)), & x \text{ singular} \,. \end{cases}$$

The result follows, since $B_x(C)$ is connected.

This proves a) and b) of the theorem. To prove c), consider the diagram

$$\begin{array}{ccc} \Gamma(\omega^{km}(kmD)) \otimes \Gamma(\omega) \otimes \Gamma(\omega^{m-1}(mD)) & \longrightarrow & \Gamma(\omega^{km}(kmD)) \otimes \Gamma(\omega^m(mD)) \\ \downarrow \alpha & & \downarrow \gamma \\ \Gamma(\omega^{km+1}(kmD)) \otimes \Gamma(\omega^{m-1}(mD)) & \xrightarrow{\beta} & \Gamma(\omega^{(k+1)m}((k+1)mD)) \,. \end{array}$$

By Castelnuovo's Lemma, α is surjective, since $\Gamma(\omega)$ is base-point-free and

$$H^1(\omega^{km-1}(kmD)) = (0) \quad \text{for } k \geqq 1 \text{ and } m \geqq 3\,.$$

β is surjective, since $\Gamma(\omega^{m-1}(mD))$ is base-point-free for $m \geqq 2$ and

$$H^1(\omega^r((k-1)mD)) = (0) \quad \text{for } r \geqq 2\,.$$

Hence γ is surjective for all $k \geqq 1$ and $m \geqq 3$.

We now prove the theorem by induction on the number of disconnecting nodes. Let x be a disconnecting node of C, C_1, and C_2 the two connected components of $B_x(C)$. If x_1 (respectively x_2) is the point of C_1 (respectively C_2) which maps to x, and if we take x_1 (respectively x_2) to be an extra distinguished point on C_1 (respectively C_2), we see that C_1 is an $l+1$-pointed stable curve and C_2 is an l_2+1-pointed stable curve, where $l_1+l_2=n$. Let $\mathscr{L}_1$ (respectively $\mathscr{L}_2$) be the sheaf $\omega_{C_1}(D_1)$ (respectively $\omega_{C_2}(D_2)$), where D_1 (respectively D_2) is the distinguished divisor of C_1 (respectively C_2). If i_1 and i_2 are the closed immersions of C_1 and C_2 into C, we have by property b) of the dualizing sheaves

$$i_\beta^*(\mathscr{L}^{\otimes m}) \approx \mathscr{L}_\beta^{\otimes m}, \quad \beta = 1,2\,.$$

Moreover, both C_1 and C_2 have fewer disconnecting nodes than C, so the theorem holds for $\mathscr{L}_1$ and $\mathscr{L}_2$ by the induction hypothesis.

We have an exact sequence

$$(0) \to \Gamma(\mathscr{L}^m) \to \Gamma(\mathscr{L}_1^m) \oplus \Gamma(\mathscr{L}_2^m) \xrightarrow{\alpha_m} k(x) \to H^1(\mathscr{L}^m) \to H^1(\mathscr{L}_1^m) \oplus H^1(L_2^m) \to (0)\,.$$

For $m \geqq 2$, α_m is surjective by part b) of the theorem and $H^1(\mathscr{L}_\beta^m) = (0)$ for $\beta = 1,2$ by part a) of the theorem. This proves a) for $\mathscr{L}$. Part b) of the theorem is clear, since a section of $\mathscr{L}^m$ is the same as a pair of sections (s,t) with $s \in \Gamma(\mathscr{L}_1^m)$ and $t \in \Gamma(\mathscr{L}_2^m)$ such that $s(x_1) = t(x_2)$. To prove part c) let $m \geqq 3$ and $k \geqq 1$ and consider a section s of $\Gamma(\mathscr{L}^{(k+1)m})$ such that $s(y) = 0$ for all points $y \in C_2$. Let $s_1,\ldots,s_r$ be sections of $\Gamma(\mathscr{L}_1^{km})$ and $t_1,\ldots,t_r$ be sections of $\Gamma(\mathscr{L}_1^m)$ such that $s|_{C_1}$ is the image of $s_1 \otimes t_1 + \ldots + s_r \otimes t_r$ by the canonical map. Since $\Gamma(\mathscr{L}_2^m)$ is base-point-free we can find sections u of $\Gamma(\mathscr{L}_2^{km})$ and v of $\Gamma(\mathscr{L}_2^m)$ such that $u \otimes v\ (x_2) \neq (0)$. Hence there are scalars $a_1,\ldots,a_r$, $b_1,\ldots,b_r$ such that

$$a_i u(x_2) = s_i(x_1) \quad \text{and} \quad b_i v(x_2) = t_i(x_1), \quad 1 \leqq i \leqq r\,.$$

The sections $\bar{s}_i$ defined as s_i on C_1 and as $a_i u$ on C_2 and the sections $\bar{t}_i$ defined as t_i on C_1 and as $b_i v$ on C_2 are global sections of $\mathscr{L}^{km}$ and $\mathscr{L}^m$, respectively. By the canonical map the section

$$\bar{s}_1 \otimes \bar{t}_1 + \ldots + \bar{s}_r \otimes \bar{t}_r \in \Gamma(\mathscr{L}^{km}) \otimes \Gamma(\mathscr{L}^m)$$

maps to s because $\sum a_i b_i = 0$. This argument holds just as well for a section s of $\Gamma(\mathscr{L}^{(k+1)m})$ which vanishes on C_1. It follows that the image of the map

$$\Gamma(\mathscr{L}^{km}) \otimes \Gamma(\mathscr{L}^{m}) \to \Gamma(\mathscr{L}^{(k+1)m})$$

contains all sections which vanish at x. The theorem will follows, if we can show that there is at least one section in the image that does not vanish at x, but this is clear since $\mathscr{L}_C^m$ is base-point-free.

COROLLARY 1.9. *If C and $\mathscr{L}$ are as in Theorem* 1.8, *then $\mathscr{L}^m$ is very ample for* $k \geqq 3$.

PROOF. $\mathscr{L}$ restricted to each irreducible component has positive degree and is therefore ample. Furthermore, it is clear that a normally generated ample sheaf is very ample.

COROLLARY 1.10. *Let C be an $n+1$-pointed stable curve over a field k with distinguished points $P_1, \ldots, P_{n+1}$ and such that $2g-2+n>0$. Then the sheaf $\omega_{C/k}(P_1 + \ldots + P_n)$ satisfies* a), b), *and* c) *of Theorem* 1.8.

PROOF. Clear by Lemma 1.6 and the theorem.

COROLLARY 1.11. *Let $\pi\colon C \to S$ be an $n+1$-pointed stable curve with $2g-2+n>0$. Then the sheaf $\pi_*(\omega_{C/S}(s_1 + \ldots + s_n)^{\otimes m})$ is locally free of rank $(2g-2+n)m-g+1$ for $m \geqq 2$.*

PROOF. Clear, since C is flat over S and

$$R^1\pi_*(\omega_{C/S}(s_1 + \ldots + s_n)^{\otimes m}) = (0) \quad \text{for } m \geqq 2\,.$$

2. Contraction and stabilization.

In this paragraph we will constantly make use of the following fact:

Let $\mathscr{F}$ be a coherent sheaf on a scheme Y, then we have a one to one correspondence:

$$\left\{\begin{matrix}S\text{-valued points}\\ \text{of } \mathscr{P}roj(\operatorname{Sym}\mathscr{F})\end{matrix}\right\} \leftrightarrow \left\{\begin{array}{l}\text{Triples consisting of:}\\ \text{1) a map } f\colon S \to Y,\\ \text{2) an invertible sheaf } \mathscr{L} \text{ on } S\\ \text{3) an epimorphism } \alpha\colon f^*\mathscr{F} \to \mathscr{L}\end{array}\right\}\Big/\sim$$

Two triples $(f, \mathscr{L}, \alpha)$ and $(f', \mathscr{L}', \alpha')$ are equivalent, if $f=f'$ and $\ker\alpha = \ker\alpha'$.

PROPOSITION 2.1. *Given any $n+1$-pointed stable curve X over S with $2g-2+n>0$, there is one and up to canonical isomorphism only one contraction.*

PROOF. Let $\pi\colon X \to S$ be the curve and define

$$\mathcal{S}^i = \pi_*(\omega_{X/S}(s_1+\ldots+s_n)^{\otimes i})$$
$$\mathcal{S} = \bigoplus_{i\geqq 0} \mathcal{S}^i .$$

By Corollary 1.11, $\mathcal{S}^i$ is a locally free sheaf on S for $i\geqq 2$.

We define

$$X^c = \mathcal{P}roj(\mathcal{S})$$
$$Y = \mathcal{P}roj(\operatorname{Sym}\mathcal{S}^3) .$$

By Theorem 1.8. and Corollary 1.5. we have a surjection:

$$\pi^*(\mathcal{S}^3) \to \omega_{C/S}(s_1+\ldots+s_n)^{\otimes 3}$$

i.e. a morphism $p\colon X \to Y$. But since $\omega_{X/S}(s_1+\ldots+s_n)^{\otimes 3}$ is normally generated by Corollary 1.9, X^c is exactly the image of this morphism, and X^c is flat over S, since $\mathcal{S}^i$ is flat for $i\geqq 2$.

For uniqueness consider a diagram:

$$\begin{array}{ccc} X & & \\ {\scriptstyle p}\downarrow & \searrow & \\ X' & \xrightarrow{f} & X^c \\ {\scriptstyle s_i'}\curvearrowleft{\scriptstyle \pi'}\downarrow & & \downarrow \\ S & = & S \end{array}$$

where q is a contraction. We have to prove that a map f exists making the diagram commutative.

By Lemma 1.6 c) we have an isomorphism:

$$\pi_*\omega_{X/S}(s_1+\ldots+s_n)^{\otimes k} \approx \pi'_*\omega_{X'/S}(s_1'+\ldots+s_n')^{\otimes k} .$$

Hence by Corollary 1.5 and Theorem 1.8 a surjection:

$$\pi'^*(\pi_*\omega_{X/S}(s_1+\ldots+s_n)^{\otimes k}) \to \omega_{X'/S}(s_1'+\ldots+s_n')^{\otimes k}$$

and this is f.

In the language of stacks, the proposition says that contraction is a 1-morphism of stacks:

$$c : M_{g,n+1} \to Z_{g,n} .$$

The rest of this paragraph will be devoted to the construction of an inverse to c which we call stabilization:

$$s : Z_{g,n} \to M_{g,n+1}$$

LEMMA 2.2. *Let S be a noetherian scheme, and $\pi: X \to S$ a flat family of reduced curves with at most ordinary double points. Let $\Delta: S \to X$ be a section defined by an $\mathcal{O}_X$-ideal $\mathcal{J}$. Then*

1) *$\mathcal{J}$ is stably reflexive with respect to π [see appendix].*
2) *The subsheaf $\mathcal{J}'$ of the total quotient ring sheaf K_X consisting of sections that multiply $\mathcal{J}$ into $\mathcal{O}_X$ is isomorphic to the dual of $\mathcal{J}$, that is $\mathcal{J}' \approx \mathcal{J}^{\vee} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}, \mathcal{O}_X)$.*
3) *$\Delta^*(\mathcal{J}^{\vee}/\mathcal{O}_X)$ is an invertible sheaf on S.*

PROOF. The theorem is of local nature so let s be a point of S such that $x = \Delta(s)$ is an ordinary double point of the fibre $\pi^{-1}(s)$, $\mathfrak{o}$ the completion of the local ring $\mathcal{O}_{S,s}$, with residue field k. We consider the category Λ of Artin local $\mathfrak{o}$-algebras with residue field k and the functor G on Λ defined by:

$$G(A) = \left\{ \begin{array}{l} \text{Cartesian diagrams of the form} \\ \begin{array}{ccc} k[[x,y]]/(x \cdot y) & \xleftarrow{q} & B \\ {\scriptstyle g}\downarrow\uparrow & & {\scriptstyle h}\downarrow\uparrow \\ k & \xleftarrow{p} & A \end{array} \\ \text{where } g(x)=g(y)=0 \text{ and } gp=ph \end{array} \right\} \Big/ \begin{array}{l}\text{modulo} \\ \text{isomorphisms}\end{array}$$

It follows from the general theory of deformations that there exists a versal deformation

$$\begin{array}{ccc} k[[x,y]]/(x \cdot y) & \xleftarrow{q} & \mathcal{B} \\ {\scriptstyle g}\downarrow\uparrow & & {\scriptstyle h}\downarrow\uparrow \\ k & \xleftarrow{p} & \mathcal{A} \end{array}$$

where $\mathcal{B} = \mathfrak{o}[[s,t,x,y]]/(xy - st)$, $\mathcal{A} = \mathfrak{o}[[s,t]]$, and $h(x) = s$, $h(y) = t$.

This means that there exist two element b, c in $\mathfrak{o}$ such that $\hat{\mathcal{O}}_{X,x} \approx \mathfrak{o}[[x,y]]/(xy - bc)$ and $\hat{\mathcal{J}}$ corresponds to the ideal generated by $x - b$ and $y - c$. The Lemma now follows from Proposition 6 of the appendix and the example.

DEFINITION 2.3 (The stabilization morphism). Consider an S-valued point of $Z_{g,n}$ i.e. an n-pointed stable curve $\pi: X \to S$ together with n sections $s_1, \ldots, s_n$ and an extra section Δ. Let $\mathscr{J}$ be the $\mathcal{O}_X$-ideal defining Δ, and define the sheaf $\mathscr{K}$ on X via the exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\delta} \mathscr{J}^{\vee} \oplus \mathcal{O}_X(s_1 + s_2 + \ldots + s_n) \to \mathscr{K} \to 0$$

where δ is the "diagonal" $\delta(t) = (t, t)$.

Then

$$X^s = \mathscr{P}roj(\mathrm{Sym}\,\mathscr{K}) .$$

It is clear by Lemma 2.2 that for any $T \to S$ we have a canonical isomorphism

$$\psi_{T,S} : (X^s)_T \xrightarrow{\sim} (X_T)^s$$

and that the ψ's satisfy the "cocycle" condition for any pair of morphisms $U \to T$, $T \to S$.

THEOREM 2.4. *With notations as in Definition* 2.3 *the sections* $s_1, \ldots, s_n$ *and* Δ *have unique liftings* $s_1', s_2', \ldots, s_{n+1}'$ *to* X^s *making* X^s *an* $n+1$*-pointed stable curve and* p: $X^s \to X$ *a contraction,* i.e. *the assignment* X^s *to* X *is a* 1*-morphism of stacks*

$$s : Z_{g,n} \to M_{g,n+1} .$$

PROOF. The theorem is local on S. We must study the map p in the neighbourhood of points, where Δ meets non-smooth points of the fibre and in the neighbourhood of points, where Δ meets one of the other sections. Since π is smooth near the s_i's we may study these cases separately.

CASE I. Δ meets a non-smooth point x in the fibre.

In this case we have a completed fibre-product diagram

$$\begin{array}{ccc} X^s|_{p^{-1}(x)} & \hookrightarrow & X^s \\ \hat{p}\downarrow & & p\downarrow \\ \hat{X}|_x & \hookrightarrow & X \end{array}$$

$$\hat{\mathcal{O}}_x \approx \mathfrak{o}[[x,y]]/(xy - bc)$$

and we have an exact sequence

$$\hat{\mathcal{O}}_x \oplus \hat{\mathcal{O}}_x \xrightarrow{\alpha} \hat{\mathcal{O}}_x \oplus \hat{\mathcal{O}}_x \to (\mathscr{J}_x^{\vee})^{\wedge} \to 0 ,$$

where

$$\alpha = \begin{pmatrix} -b & y \\ -x & c \end{pmatrix}$$

(note that in this case we have $\mathscr{I}^{\vee} \approx \mathscr{K}$).

Therefore $\hat{X}^s|_{p^{-1}(x)}$ is covered by two affines $\operatorname{Spec} f(R_1)$ and $\operatorname{Spec} f(R_2)$, where

$$R_1 = \hat{\mathcal{O}}_x\{s\}/(x+bs, c-ys) = \mathfrak{o}[[y]]\{s\}/(ys-c)$$

$$R_2 = \hat{\mathcal{O}}_x\{t\}/(xt+b, ct-y) = \mathfrak{o}[[x]]\{t\}/(xt+b)\ .$$

Hence we see that $X^s \to S$ is a flat family of reduced curves with at most ordinary double points.

The surjection $\Delta^*(\mathscr{K}) \to \Delta^*(\mathscr{K}/\mathcal{O}_X) \cong \Delta^*(\mathscr{I}^{\vee}/\mathcal{O}_X)$ gives us a lifting of the section Δ. Recall Lemma 2.2 that $\Delta^*(\mathscr{I}^{\vee}/\mathcal{O}_X)$ is an invertible sheaf on S.

In the coordinates we have chosen $s'_{n+1}(p(x))$ is the point given by $s=t=-1$. In particular $X^s \to S$ is smooth at $s'_{n+1}(p(x))$.

Case II. In this case Δ is a divisor. Assuming only one section s we have

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(\Delta) \oplus \mathcal{O}_X(s) \to \mathscr{K} \to 0\ .$$

It is therefore clear that the fibres of p are at most projective lines, so by Corollary 1.5, $X^s \to S$ is a flat family of curves with at most ordinary double points.

The composition

$$\mathscr{I}^{\vee} \xrightarrow{(1,0)} \mathscr{I}^{\vee} \oplus \mathcal{O}_X(s_1 + \ldots + s_n) \to \mathscr{K} \to 0$$

gives us an injection $\mathscr{I}^{\vee} \hookrightarrow \mathscr{K}$ and it is clear that the cokernel is simply $\mathcal{O}_X(s_1 + s_2 + \ldots + s_n)|_{\bigcup_{i=1}^n s_i}$. Hence for each i we have surjections $s_i^*(\mathscr{K}) \to s_i^*\mathcal{O}_X(s_1 + \ldots + s_n)$ and this defines the liftings of the s_i's. The picture for $S = \operatorname{Spec}(k)$ looks as follows in Figure 3.

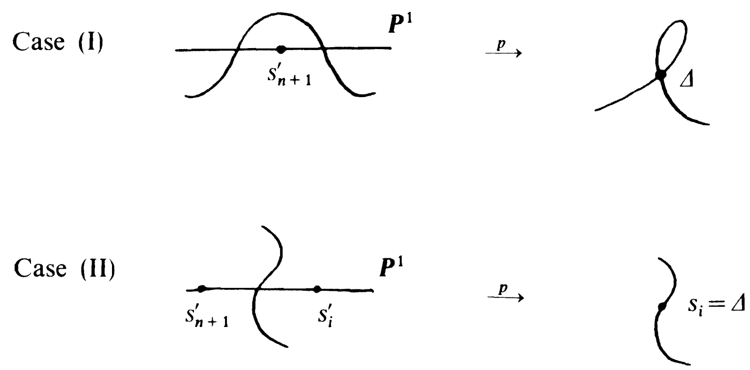


Figure 3.

LEMMA 2.5. *Consider a diagram*

$$
t'_i\left[\begin{array}{ccc} Y & \xrightarrow{f} & X^s \\ {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow \\ X & = & X \\ \downarrow & & \downarrow \\ S & = & S \end{array}\right]s'_i
$$

where q is a contraction and p is as in Theorem 2.4. Then there is a unique isomorphism $f\colon Y \to X^s$ making the diagram commutative.

PROOF. By Corollary 1.5 we have isomorphisms

$$\mathcal{O}_X \xrightarrow{\sim} q_*\mathcal{O}_Y$$

and

$$\mathcal{O}_X(-s_1-\ldots-s_n) \xrightarrow{\sim} q_*\mathcal{O}_Y(-t_1-\ldots-t_n)\,.$$

Let $x \in X$, $a \in \mathcal{I}_x$, $b \in q_*\mathcal{O}_Y(t_{n+1}-t_1-\ldots-t_n)$. Then a may be considered as an element of $q_*\mathcal{O}_Y(-t_{n+1})$ and so

$$ab \in q_*\mathcal{O}_Y(-t_1-\ldots-t_n) \approx \mathcal{O}_X(-s_1-\ldots-s_n)\,,$$

i.e. we have a morphism of sheaves

$$q_*\mathcal{O}_Y(t_{n+1}-t_1-\ldots-t_n) \to \mathcal{I}^{\vee}(-s_1-\ldots-s_n)\,.$$

We leave to the reader to check that this is an isomorphism on the geometric fibres of π so by flatness it is an isomorphism. We may put this together in a commutative diagram with exact rows.

$$
\begin{array}{ccccccccc}
(0) \to & q_*\mathcal{O}_Y(-t_1-\ldots-t_n) & \xrightarrow{(1,1)} & q_*\mathcal{O}_Y(t_{n+1}-t_1-\ldots-t_n)\otimes q_*\mathcal{O}_Y & \xrightarrow{\binom{1}{-1}} & q_*\mathcal{O}_Y(t_{n+1}) & \to (0) \\
& \wr\wr & & \wr\wr & & \wr\wr & \\
(0) \to & \mathcal{O}_X(-s_1-\ldots-s_n) & \longrightarrow & \mathcal{I}^{\vee}(-s_1-\ldots-s_n)\oplus\mathcal{O}_X & \longrightarrow & \mathcal{K} & \to (0)\,.
\end{array}
$$

By Corollary 1.5 the composition

$$q^*\mathcal{K} \to q^*q_*\mathcal{O}_Y(t_{n+1}) \to \mathcal{O}_Y(t_{n+1})$$

is a surjection and this defines a morphism

$$f\colon\; Y \to X^s\,.$$

f is easily seen to be an isomorphism on the geometric fibres of π so by flatness f is an isomorphism everywhere. For uniqueness we simply mention that an automorphism of $\boldsymbol{P}^1$ fixing three distinct points is the identity.

COROLLARY 2.6. *Contraction and stabilization are inverse to each other.*

THEOREM 2.7. *For all relevant g, n, $M_{g,n}$ is an algebraic stack, proper and smooth over* Spec ($\mathbf{Z}$). *The substack $S_{g,n}$ consisting of singular curves is a divisor with normal crossings relative to* Spec ($\mathbf{Z}$). (*We refer to* [3] *for definitions.*)

PROOF. For $g \geqq 2$, $n=0$ the result is proved in Theorem 5.2 of [3]. We consider first the cases $g=0$, $n=3$, and $g=1$, $n=1$, $M_{0,3} = \text{Spec}(\mathbf{Z})$, so here is nothing to prove. For $g=1$, $n=1$ we make use of the clutching morphism of the next paragraph. Consider the 3-pointed elliptic curve E having three rational components as in Figure 4 below.

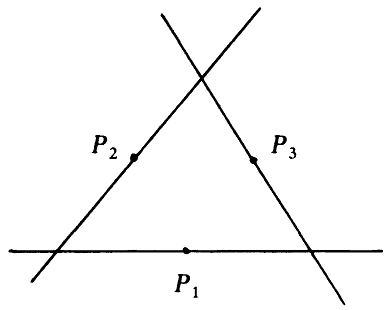


Figure 4.

Clearly E has no non-trivial automorphisms leaving the dstinguished points fixed, and so clutching defines a closed immersion $M_{1,1} \hookrightarrow M_{2,2}$.

Assuming for the moment that the theorem is proved for $M_{2,2}$, we see that $M_{1,1}$ in $M_{2,2}$ is the intersection of four branches of $S_{2,2}$, and $S_{1,1}$ is the intersection with a fifth branch. See the example at the end of section 3. This proves the theorem for $M_{1,1}$. We then proceed by induction with respect to n, having in mind that $M_{g,n+1}$ is the universal n-pointed curve. The divisor

$$S_{g,n+1} = \pi^{-1}(S_{g,n}) \cup \bigcup_{i=1}^{n} S_{g,n+1}^{\{i,n+1\}},$$

where a "point" in $S_{g,n+1}^{\{i,n+1\}}$ corresponds to a curve of the type in Figure 5. Since $\pi\colon M_{g,n+1} \to M_{g,n}$ is smooth near the sections $S_{g,n+1}^{\{i,n+1\}}$, we only have to prove that $\pi^{-1}(S_{g,n})$ has normal crossings. Near a singular point of $\pi^{-1}(S_{g,n})$ the morphism π looks formally like

$$\mathfrak{o}[[t_1 \ldots t_k]] \to \mathfrak{o}[[x, y, t_1, \ldots \hat{t}_i \ldots, t_k]],$$

where $t_i \to x \cdot y$. $S_{g,n}$ has local equation $t_1 \cdot t_2 \ldots t_k$ so $\pi^{-1}(S_{g,n})$ has local equation $x \cdot y \cdot t_1 \ldots \hat{t}_i \ldots t_k$.

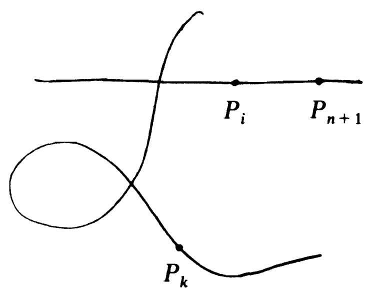


Figure 5.

3. The clutching morphism.

In this section we will study families of curves without stability conditions.

DEFINITION 3.1. A prestable curve is a flat and proper morphism $\pi\colon X \to S$ such that the geometric fibres of π are reduced curves with at most ordinary double points. No connectedness is assumed.

Recall that for any morphism $X \to S$ Lichtenbaum and Schlessinger [8] have defined the notion of a cotangent complex $L.(X/S)$. In general this complex is defined only locally on X, but it is unique up to homotopy, so it defines cotangent sheaves. For any coherent $\mathscr{F}$ on X, we have for $0 \leqq i \leqq 2$

$$T^i(X/S, \mathscr{F}) = H^i(\mathscr{H}om_{\mathcal{O}_X}(L.(X/S), \mathscr{F}))$$
$$T_i(X/S, \mathscr{F}) = H_i(L.(X/S) \otimes_{\mathcal{O}_X} \mathscr{F}).$$

PROPOSITION 3.2. *If $\pi\colon X \to S$ is a prestable curve, then*

a) $T_0(X/S, \mathcal{O}_X) \cong \Omega^1_{X/S}$ *is flat over* S,
b) $T_i(X/S, \mathscr{F}) \cong \mathscr{T}or_i^X(\Omega_{X/S}, \mathscr{F})$,
c) $T_2(X/S, \mathscr{F}) = (0)$ *for all* $\mathscr{F}$.

PROOF. Since a reduced curve with at most ordinary double points is locally a complete intersection and the morphism π is flat, it follows that π is locally a complete intersection morphism. That means that we have factorisations

$$\begin{array}{ccccc} X & \supset & U & \overset{i}{\hookrightarrow} & A^n_V \\ {}^{\pi}\downarrow & & \downarrow & & \\ S & \supset & V & & \end{array}$$

where the ideal defining U, $\mathscr{I}$, is generated by a regular sequence. By [8] we have $T^2 \equiv T_2 \equiv 0$ and

$$(*) \qquad (0) \to i^*(\mathscr{I}) \xrightarrow{d} i^*\Omega^1_{\mathbf{A}^n_V/V} \to (0)$$

is a cotangent complex for the morphism $U \to V$.

To prove a) and b) we have to show that d is universally injective. $i^*(\mathscr{I})$ and
$i^*\Omega^i_{\mathbf{A}^n_V/V}$ are flat over $\mathcal{O}_V$ and the cotangent complex commutes with base change, since π is flat so in fact all we have to prove is that d is injective.

Let $\mathscr{K} = T_1(U/V, \mathcal{O}_U) = \ker d$. By the Jacobian criterion of smoothness, $\mathscr{K}_x = (0)$ for all $x \in U$, where π is smooth. So if $\Delta \subset U$ is the closed subset of U, where π is not smooth, we have

$$\operatorname{Supp}(\mathscr{K}) \subset \Delta .$$

Since $i^*(\mathscr{I})$ is locally free and $\mathscr{K} \subset i^*(\mathscr{I})$, we have

$$\operatorname{Ass}(\mathscr{K}) \subset \operatorname{Ass}(U) .$$

Since U is flat over V we have [2, Chap. IV, 2.6.2]

$$\operatorname{Ass}(U) = \bigcup_{y \in \operatorname{Ass}(V)} \operatorname{Ass}(U_y) .$$

Therefore since the fibres of π are reduced

$$\Delta \cap \operatorname{Ass}(U) = \emptyset .$$

Hence $\mathscr{K} = (0)$ and d is injective.

COROLLARY 3.3. *Let $\pi: X \to S$ be a prestable curve, $\mathscr{E}_0$ a locally free sheaf on X, and $d_0: \mathscr{E}_0 \to \Omega^1_{X/S}$ a surjection. Then $\mathscr{E}_1 = \ker(d_0)$ is locally free and $d_1: \mathscr{E}_1 \to \mathscr{E}_0 \to (0)$ is a cotangent complex for the morphism π.*

Let $\pi: X \to S$ be a prestable curve, and $s_1, s_2: S \to X$ two non-crossing sections such that π is smooth at all points $s_i(t)$ $(t \in S)$.

THEOREM 3.4. *With the notations above there is a diagram*

$$\begin{array}{ccc} X & \xrightarrow{p} & X' \\ {\scriptstyle s_i}\uparrow\downarrow{\scriptstyle \pi} & & {\scriptstyle \pi'}\downarrow\uparrow{\scriptstyle s} \\ S & = & S \end{array}$$

such that

(1) *$ps_1 = ps_2$ and p is universal with respect to this property.*

(2) *p is a finite morphism.*

(3) *If t is a geometric point of S, the fibre X'_t is obtained from X_t by identifying the two points $s_1(t)$ and $s_2(t)$ in such a way that the image point is an ordinary double point.*

(4) *As a topological space, X' is the quotient of X under the equivalence relation $s_1(t) \sim s_2(t)$ for all $t \in S$.*

(5) *If U is open in X' and $V = p^{-1}(U)$, then*

$$\Gamma(U, \mathcal{O}_{X'}) = \{h \in \Gamma(V, \mathcal{O}_X) \mid s_1^*(h) = s_2^*(h)\} .$$

(6) *The morphism $\pi'\colon X' \to S$ is flat, so by* (3), *π' is again a prestable curve.*

PROOF. Properties (4) and (5) determine X' as a ringed space. To show that X' is a scheme satisfying (1), (4), and (5), all we have show is that X' is locally affine.

If $x \in X'$, and $p^{-1}(x)$ does not meet any of the sections, we can clearly find an affine neighbourhood of x.

Suppose therefore that $x = p_1 s_1(t)$. Since π is flat and any curve is projective, we may assume that π is locally projective. Hence we can find an affine $U \subset X$ containing $s_1(t)$ and $s_2(t)$. Let $V \subset S$ be an open affine contained in $s_1^{-1}(U) \cap s_2^{-1}(U)$ and containing t. The restriction of π to $W = U \cap \pi^{-1}(V)$ yields an affine morphism $W \to V$ with two disjoint regular sections. Localizing further, if necessary, we may assume that $W = \operatorname{Spec}(B)$, $V = \operatorname{Spec}(A)$, and that the sections are defined by two principal ideals (f_1) and (f_2) of B.

We have two split-exact sequences of A-modules

$$(0) \to B \xrightarrow{f_i} B \to A \to (0), \qquad i = 1, 2 .$$

Since the sections do not cross, we have $(f_1) + (f_2) = B$, so $(f_1) \cap (f_2) = (f_1 f_2)$, and the ring of invariants is given by

$$B' = A \oplus (f_1 f_2) B .$$

B' is isomorphic to B as an A-module, hence B' is flat over A. It is easy to check that $\operatorname{Spec}(B')$ satisfies (1), (4), (6) of the theorem for the morphism $W \to V$.

Suppose $B = A[x_1, \ldots, x_n]$ and write

$$x_i = a_{1i} + f_1 b_{1i} = a_{2i} + f_2 b_{2i}$$

with $a_{ij} \in A$, $b_{ij} \in B$.

The elements $y_i = x_i^2 - (a_{1i} + a_{2i}) x_i$ are in B', hence

$$A[y_1, \ldots, y_n] = B'' \subset B' \subset B .$$

B is a finite B''-module, and since A is noetherian, B' is a finitely generated A-algebra and B is a finite B'-module. This proves (2).

To show (3), note first that the construction of B' commutes with base change, hence we may assume that $A = k$ is an algebraically closed field. If $\hat{B}$ denotes the completion of B with respect to the ideal $(f_1 \cdot f_2)$ we have an isomorphism.

$$k[[x]] \oplus k[[y]] \approx \hat{B}$$

sending x to f_1 and y to f_2. The completion of B' with respect to the ideal $f_1 \cdot f_2 B$, corresponds then to the kernel of the map

$$k[[x]] \oplus k[[y]] \to k$$

sending (a, b) to $\bar{a} - \bar{b}$ and this is just $k[[x,y]]/(x \cdot y)$.

THEOREM 3.5. (The clutching sequence). *We consider the diagram as in* 3.4

$$\begin{array}{ccc} X & \xrightarrow{p} & X' \\ {\scriptstyle s_i}\uparrow\downarrow{\scriptstyle \pi} & & {\scriptstyle \pi'}\downarrow\uparrow{\scriptstyle s} \\ S & = & S \end{array}$$

and denote $\varkappa^{(i)}$ *the conormal bundle of* S *in* X *via the section* s_i.

$$\varkappa^{(i)} = s_i^*(\Omega_{X/S}) \approx s_i^*(\omega_{X/S}) \approx s_i^*(\mathcal{O}_X(-D_i)),$$

where D_i *is the divisor on* X *defined by* s_i.

Then on X' *we have a short exact sequence (the clutching sequence)*

$$(0) \to s_*(\varkappa^{(1)} \otimes \varkappa^{(2)}) \to \Omega_{X'/S} \to p_*\Omega_{X/S} \to (0).$$

PROOF. It turns out to be a bit messy to define the map $s_*(\varkappa^{(1)} \otimes \varkappa^{(2)}) \to \Omega_{X'/S}$ so we take double coverings

$$\begin{array}{ccc} Y & \xrightarrow{q} & Y' \\ {\scriptstyle \sigma}\downarrow & & {\scriptstyle \sigma'}\downarrow \\ X & \xrightarrow{p} & X' \end{array}$$

where $Y = X \coprod X$,

$$Y' = X \coprod X \Big/ \begin{pmatrix} s_1 \text{ in first factor } \sim s_2 \text{ in second} \\ s_2 \text{ in first factor } \sim s_1 \text{ in second} \end{pmatrix}.$$

The section $s\colon S \to X'$ lifts to two sections t_1 and t_2 in Y'.

The picture is given by Figure 6.

Assuming everything to be affine we have

$$S = \operatorname{Spec}(T), \quad X = \operatorname{Spec}(R), \quad Y = \operatorname{Spec}(R \oplus R).$$

The $\mathbf{Z}/2\mathbf{Z}$ action simply interchanges the factors. The four sections of Y are defined by non-zero-divisors $(f_i, 1)$ and $(1, f_i)$, $i = 1, 2$.

The affine sets

$$Y_1 = \operatorname{Spec}(R_{f_1} \oplus R_{f_2})$$
$$Y_2 = \operatorname{Spec}(R_{f_2} \oplus R_{f_1})$$

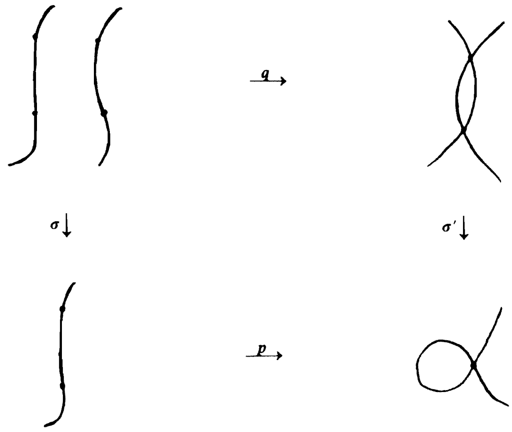


Figure 6.

are invariant sets for the map q. Clutching the sections of Y_1 and Y_2 gives us an affine covering Y_1' and Y_2' of Y', where

$$Y_i' = \operatorname{Spec}(S_i) \qquad i=1,2$$

and

$$S_1 = \{(u,v) \in R_{f_1} \oplus R_{f_2} \mid s_2^*(u) = s_1^*(v)\}$$

$$S_2 = \{(u,v \in R_{f_2} \oplus R_{f_1} \mid s_1^*(u) = s_2^*(v)\} \,.$$

S_1 is an R_{f_1}-algebra via $u \to (u, \pi^* s_2^* u)$ and an R_{f_2}-algebra via $v \to (\pi^* s_1^* v, v)$.
Hence we have a homomorphism

$$. \qquad R_{f_1} \otimes_T R_{f_2} \to S_1 \,.$$

We leave to the reader to check that

$$(0) \to (f_2 \otimes f_1) \to R_{f_1} \otimes_T R_{f_2} \to S_1 \to (0)$$

We leave to the reader to check that

$$(0) \to (f_2 \otimes f_1) \to R_{f_1} \otimes_T R_{f_2} \to S_1 \to (0)$$

is exact.
Similarly we have the exact sequence

$$(0) \to (f_1 \otimes f_2) \to R_{f_2} \otimes_T R_{f_1} \to S_2 \to (0) \,.$$

By flatness $f_1 \otimes f_2$ is not a zero-divisor in $R_{f_1} \otimes R_{f_2}$, so we have

$$(*) \qquad (0) \to (f_2 \otimes f_1)/(f_2 \otimes f_1)^2 \to \Omega_{R_{f_1} \otimes R_{f_2}/T} \otimes_{R_{f_1} \otimes R_{f_2}} S_1 \to \Omega_{S_1/T} \to 0 \,.$$

Using the canonical isomorphism

$$\Omega_{R_{f_1}\otimes R_{f_2}/T} \approx R_{f_1}\otimes_T \Omega_{R_{f_2}/T}\oplus \Omega_{R_{f_1}/T}\otimes_T R_{f_2},$$

where $d(u\otimes v) = (u\otimes dv,\ du\otimes v)$.

We may write (*) in the form

$$(**)\quad (0) \to (f_2\otimes f_1)/(f_2\otimes f_1)^2 \xrightarrow{d} (S_1\otimes_{R_{f_2}} \Omega_{R_{f_2}/T})\oplus (\Omega_{R_{f_1}/T}\otimes_{R_{f_1}} S_1) \to \Omega_{S_1/T} \to (0),$$

where $d(f_2\otimes f_1) = ((f_2,0)\otimes df_1, df_2\otimes (0,f_1))$.

Hence in $\Omega_{S_1/T}$ we have

$$(f_2,0)d(0,f_1) = -(0,f_1)d(f_2,0).$$

From the canonical isomorphism

$$\Omega_{R_{f_1}\oplus R_{f_2}/T} \approx \Omega_{R_{f_1}/T}\oplus \Omega_{R_{f_2}/T}$$

(module multiplication componentwise), we see from (**) that the natural map

$$\Omega_{S_1/T} \to \Omega_{R_{f_1}\oplus R_{f_2}/T}$$

is surjective and that the kernel is generated by the element $(0,f_1)d(f_2,0) = -(f_2,0)d(0,f_1)$ that is we have a right exact sequence

$$f_1R_{f_2}/f_1^2R_{f_2}\otimes f_2R_{f_1}/f_2^2R_{f_1} \xrightarrow{\alpha_1} \Omega_{S_1/T} \to \Omega_{R_{f_1}\oplus R_{f_2}/T} \to (0),$$

where $\alpha_1(u\otimes v) = (0,u)d(v,0)$.

The kernel of $\Omega_{S_1/T} \to \Omega_{R_{f_1}\oplus R_{f_2}/T}$ is a flat T-module and the left hand side of the sequence above is a locally free rank 1 T-module. Hence α_1 is injective.

On Y_2' we have a similar exact sequence

$$(0) \to (f_1R_{f_2}/f_1^2R_{f_2})\otimes (f_2R_{f_1}/f_2^2R_{f_1}) \xrightarrow{\alpha_2} \Omega_{S_2/T} \to \Omega_{R_{f_2}\oplus R_{f_1}/T} \to (0),$$

where $\alpha_2(u\otimes v) = (u,0)d(0,v)$.

Since both α_1 and α_2 vanish on $Y_1'\cap Y_2'$, they patch up to give a global map

$$(0) \to \underbrace{(t_1)_*(\varkappa^{(1)}\otimes\varkappa^{(2)})\oplus (t_2)_*(\varkappa^{(1)}\otimes\varkappa^{(2)})}_{\sigma^*s_*(\varkappa^{(1)}\otimes\varkappa^{(2)})} \xrightarrow{\alpha} \Omega_{Y'/S} \to q_*\Omega_{Y/S} \to (0).$$

By definition we see that α is $\mathbf{Z}/2\mathbf{Z}$ invariant and is therefore induced by a map

$$\alpha' : s_*(\varkappa^{(1)}\otimes\varkappa^{(2)}) \to \Omega_{X'/S}.$$

Since étale morphisms are faithfully flat

$$(0) \to s_*(\varkappa^{(1)}\otimes\varkappa^{(2)}) \to \Omega_{X'/S} \to p_*\Omega_{X/S} \to (0)$$

is exact.

REMARK. Let $\mathscr{I}$ be the sheaf of ideals on X' defining the section s. We have natural maps

$$p_1 : \mathscr{I}/\mathscr{I}^2 \to p_*(\mathscr{I}_1/\mathscr{I}_1^2) = s_*\varkappa^{(1)}$$
$$p_2 : \mathscr{I}/\mathscr{I}^2 \to p_*(\mathscr{I}_2/\mathscr{I}_2^2) = s_*\varkappa^{(2)} .$$

One checks that these maps give an isomorphism

$$\mathscr{I}/\mathscr{I}^2 \approx s_*(\varkappa^{(1)} \oplus \varkappa^{(2)}) .$$

Hence we have an isomorphism

$$\Lambda^2(\mathscr{I}/\mathscr{I}^2) \approx s_*(\varkappa^{(1)} \otimes \varkappa^{(2)})$$

defined by

$$\bar{u} \wedge \bar{v} \mapsto p_1\bar{u} \otimes p_2\bar{v} - p_1\bar{v} \otimes p_2\bar{u} .$$

The map α' is the composition

$$s_*(\varkappa^{(1)} \otimes \varkappa^{(2)}) \approx \Lambda^2(\mathscr{I}/\mathscr{I}^2) \xrightarrow{\beta} \Omega_{X'/S} ,$$

where $\beta(\bar{u} \wedge \bar{v}) = udv$.

DEFINITION 3.6. Let

$$H = \{h_1, h_2, h_3, \ldots, h_{n_1}\}, \quad h_1 < h_2 < \ldots < h_{n_1}$$

and

$$K = \{k_1, k_2, \ldots, k_{n_2}\}, \quad k_1 < k_2 < \ldots < k_{n_2}$$

be disjoint subsets of $\{1, 2, \ldots, n\}$, $n_1 + n_2 = n$.

Let g_1 and g_2 be two non negative integers with $g_1 + g_2 = g$. Then for each quadruple g_1, g_2, H, K we have a morphism of stacks

$$\gamma_{g_1, g_2, H, K} : M_{g_1, n_1+1} \times M_{g_2, n_2+1} \to M_{g, n+2} .$$

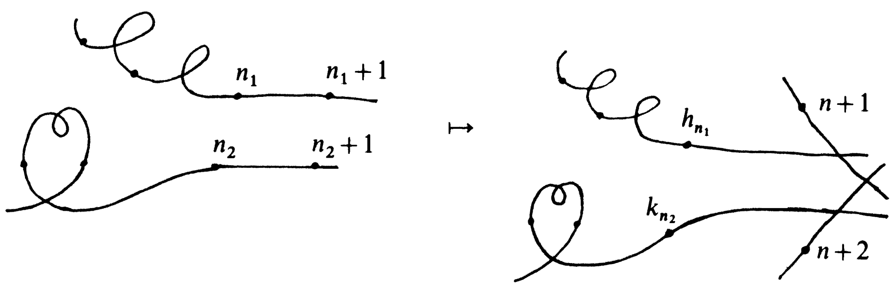


Figure 7.

This is obtained by attaching a pair of projective lines and renumbering the sections as best described by the picture in Figure 7.

We define

$$\gamma_0 : M_{g-1,n+2} \to M_{g,n+2}$$

as described by Figure 8.

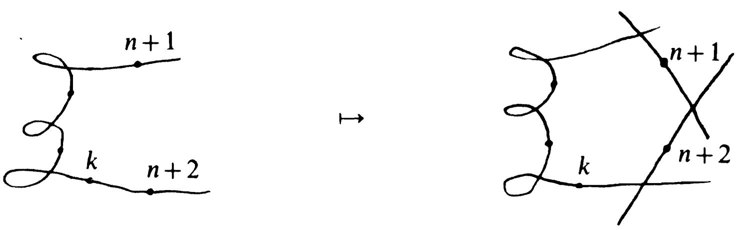


Figure 8.

Contracting the two last sections gives a morphism

$$\pi_{n+1,n+2} : M_{g,n+2} \to M_{g,n} .$$

We denote by β the composition $\beta = \pi_{n+1,n+2} \circ \gamma$.

THEOREM 3.7. *γ is a closed immersion and β is finite.*

PROOF. Clearly β is quasifinite and $\pi_{n+1,n+2}$ is proper, so it suffices to prove that γ is a closed immersion.

Let $\pi : C \to S$ be an $n+2$-pointed stable curve with sections $s_1, \ldots, s_{n+2}$.

Contracting s_{n+1}, s_{n+2} and both s_{n+1} and s_{n+2} gives us a diagram

$$\begin{array}{ccccc} C & \xrightarrow{p} & C' & & \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p''} & & \\ C'' & \xrightarrow{p'''} & C''' & \xrightarrow{\pi'''} & S . \end{array}$$

On C' and C'' we have extra sections $\Delta' = p \circ s_{n+2}$ and $\Delta'' = p' \circ s_{n+1}$, respectively. On C''' we have two extra sections namely

$$p'' p s_{n+1} = p''' p' s_{n+1} = p''' \Delta''$$

and

$$p'' p s_{n+2} = p''' p' s_{n+2} = p'' \Delta' .$$

We define three closed subschemes of S via the cartesian diagrams:

$$\begin{array}{ccc} T' & \hookrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle \Delta'} \\ C'_{\text{sing}} & \hookrightarrow & C' \end{array} \qquad \begin{array}{ccc} T'' & \hookrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle \Delta''} \\ C''_{\text{sing}} & \hookrightarrow & C'' \end{array} \qquad \begin{array}{ccc} T''' & \hookrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle p''\Delta'} \\ S & \underset{p'''\Delta''}{\hookrightarrow} & C''' \end{array}$$

and finally

$$T = T' \times_S T'' \times_S T''' .$$

Note that for any stable curve $X \to S$, X_{sing} is defined by the sheaf of ideals image of $\Omega_{X/S} \otimes \check{\omega}_{X/S}$ in $\mathcal{O}_X$.

Let Δ''' denote the section of $C''' \times_S T$, and let $\mathcal{J}'$, $\mathcal{J}''$, and $\mathcal{J}'''$ be the sheaf of ideals defining the sections Δ', Δ'', and Δ''' over T, respectively.

By uniqueness of stabilization we have

$$E_1 = p^{-1}(\Delta') \approx \operatorname{Proj}(\mathcal{S}ym(\mathcal{J}'^{\vee}/\mathcal{J}'\mathcal{J}'^{\vee}))$$

$$E_2 = p'^{-1}(\Delta'') \approx \operatorname{Proj}(\mathcal{S}ym(\mathcal{J}''^{\vee}/\mathcal{J}''\mathcal{J}''^{\vee}))$$

$$E_3 = p'''^{-1}(\Delta''') \approx E_4 = p'''^{-1}(\Delta''') \approx \operatorname{Proj}(\mathcal{S}ym(\mathcal{J}'''^{\vee}/\mathcal{J}'''\mathcal{J}'''^{\vee})) .$$

Over T we have the picture in Figure 9.

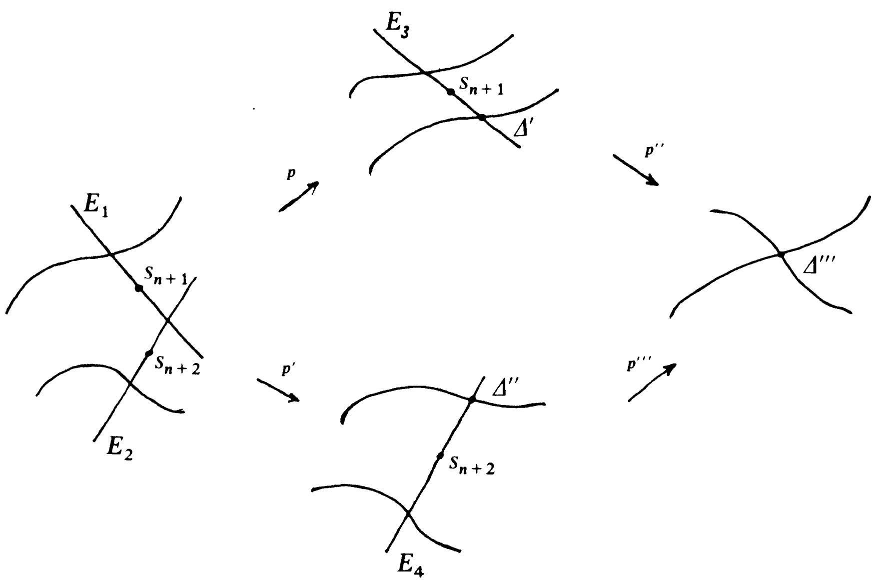


Figure 9.

Consider the general situation where $X \to S$ is a stable curve with a section Δ. Let s be a point in S and $x = \Delta(s)$ a point in X such that X_s has a double point with rational tangents at x. If $\mathcal{J}$ is the ideal defining the section in the local ring $R = \mathcal{O}_{X,x}$, we have a map

$$\mathcal{J} \otimes_R \mathcal{J}^{\vee} \to R .$$

We denote by $\mathcal{J}\mathcal{J}^{\vee}$ the image of this map. Clearly $\mathcal{J} \subset \mathcal{J}\mathcal{J}^{\vee}$.

If we take completions with respect to the maximal ideal $m = m_x$ in R we have by general theory of Zariski rings

$$(\mathscr{I}\mathscr{I}^{\vee})^{\wedge} \approx \mathscr{I}^{\vee}.(\mathscr{I}^{\vee})^{\wedge} \approx \mathscr{I}^{\wedge}(\mathscr{I}^{\wedge})^{\vee}.$$

So we have a diagram

$$\begin{array}{ccccc} \mathscr{I}^{\wedge} & \subset & \mathscr{I}^{\wedge}(\mathscr{I}^{\wedge})^{\vee} & = & (\mathscr{I}\mathscr{I}^{\vee})^{\wedge} \\ \cup & & \cup & & \\ \mathscr{I} & \subset & \mathscr{I}\mathscr{I}^{\vee}. & & \end{array}$$

By the example in the appendix $\mathscr{I}^{\wedge} = \mathscr{I}^{\wedge}(\mathscr{I}^{\wedge})^{\vee}$ and so since $\hat{R}$ is a faithfully flat R-algebra $\mathscr{I} = \mathscr{I}\mathscr{I}^{\vee}$. The exact sequence

$$(*) \qquad (0) \to R/\mathscr{I} \to \mathscr{I}^{\vee}/\mathscr{I}\mathscr{I}^{\vee} \to \mathscr{I}^{\vee}/R \to (0)$$

shows that $\mathscr{I}^{\vee}/\mathscr{I}\mathscr{I}^{\vee}$ is locally free of rank 2. If X_s does not have rational tangents at x, there is an étale neighbourhood $S' \to S$ of S such that the sequence (*) is exact on $X_{S'}$; hence (*) is exact in any case, and the E's are flat over T. p and p' induce proper maps $E_1 \to E_3$ and $E_2 \to E_4$. On the geometric fibres they are isomorphism, so by Nakayama's lemma they are closed immersions. By flatness they are isomorphisms.

The isomorphisms $E_1 \approx E_3 \approx E_4 \approx E_2$ shows that they all have exactly three sections. Hence they are all isomorphic to $\boldsymbol{P}^1_T$. In particular we have three extra sections t_1, t_2, and t_3 in C_T, that is in Figure 10.

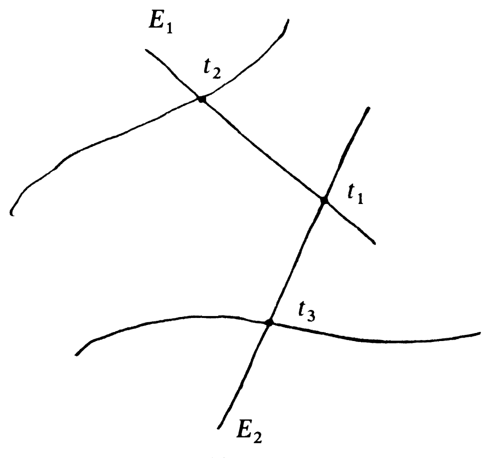


Figure 10.

The three schemes $C_T \setminus t_1(T)$, E_1, and E_2 patched along their common open sets yields a stable curve $\tilde{C}_T$ over T with $n+4$ sections. Contracting the two last sections in $\tilde{C}_T$ gives us a morphism:

$$T \to M = M_{g-1,n+2} \cup \bigcup M_{g_1,n_1+1} \times M_{g_2,n_2+1}$$

and the diagram

$$\begin{array}{ccc} T & \hookrightarrow & S \\ \downarrow & & \downarrow \\ M & \xrightarrow{\gamma} & M_{g,n+2} \end{array}$$

commutes by the universal property (1) of the clutching construction. It is cartesian by the very definition of T.

DEFINITION 3.8. Let $H=\{h_1,h_2,\ldots,h_{n_1}\}$ and $K=\{k_1,k_2,\ldots,k_{n_2}\}$ be complementary subsets of $\{1,2,\ldots,n\}$ with $h_1<h_2<\ldots<h_{n_1}$ and $k_1<k_2<\ldots<k_{n_2}$. Let g_1 and g_2 be integers with $g=g_1+g_2$. The finite morphisms

$$\beta_0 : M_{g-1,n+2} \to M_{g,n}$$

and

$$\beta_{g_1,g_2,H,K} : M_{g_1,n_1+1} \times M_{g_2,n_2+1} \to M_{g,n}$$

define closed substacks $S^0_{g,n}$ and $S_{g_1,g_2,H,K}$ of $M_{g,n}$, and these are the irreducible components of $S_{g,n}$. When $g_1=0$ we write for short $S_{g_1,g_2,H,K}=S^H_{g,n}$, or simply S^H. If also $g_2=0$, then H should contain at most one of the integers 1, 2, 3.

COROLLARY 3.9.

a) *The clutching morphism* β *is finite and unramified.*
b) *When* $g_1 \neq g_2$ *or* $n \neq 0$, $\beta_{g_1g_2H,K}$ *is a closed immersion.*

PROOF. Let X be a scheme and $\pi: C \to X$ an n-pointed stable curve and let D be the curve over C obtained from $C \times_X C$ by stabilization. Since the geometric fibres of π are reduced with ordinary double points, $C_{\text{sing}} \to X$ is unramified. Define the scheme T by the fibre product

$$\begin{array}{ccc} T & \longrightarrow & X \\ \downarrow & & \downarrow \\ M & \xrightarrow{\beta} & M_{g,n} \end{array}$$

Then $T \to X$ factors as follows.

$$T \overset{\gamma}{\hookrightarrow} D_{\text{sing}} \times_C C_{\text{sing}} \to C_{\text{sing}} \to X$$

and this proves a); b) is clear.

We conclude this section with a picture of the zoo of 2-pointed stable curves of genus 2. $S_{2,2}$ and its intersections define a stratification of $M_{2,2}$ into locally closed non-singular connected strata as follows

dimension	5	4	3	2	1	0
number of components	1	4	13	24	23	10.

The 23 components of dimension 1 correspond to the curves in Figure 11.

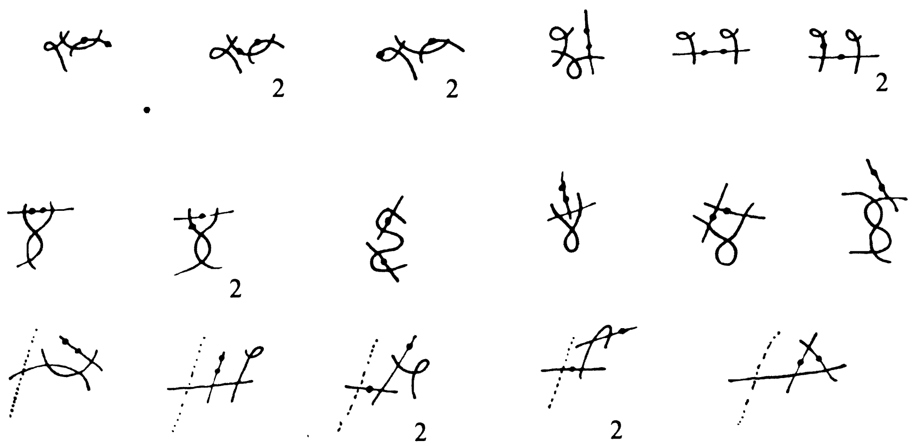


Figure 11.

The 10 components of dimension 0 correspond to the curves in Figure 12.

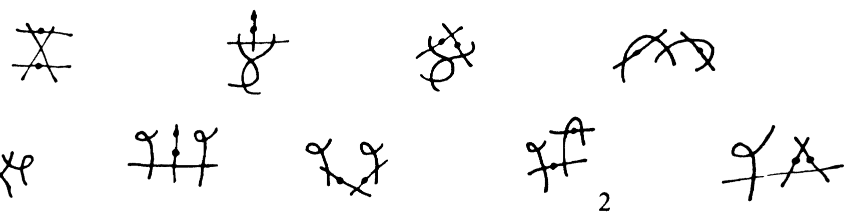


Figure 12.

The last components here are isomorphic to $M_{1,1}$ and $S_{1,1}$ respectively. (A dotted line stands for an elliptic curve. The number 2 indicates that by ordering the points we actually have 2 components of this type).

Appendix. Stably reflexive modules.

Let R and S be noetherian rings and $h: S \to R$ a ring homomorphism making R into a flat S-algebra.

DEFINITION 1. A noetherian R-module M is stably reflexive with respect to the homomorphism h, or if there can be no doubt about the homomorphism we say simply that M is stably reflexive with respect to S, if M satisfies the equivalent properties of the theorem below.

THEOREM 2. *The following are equivalent.*

1) a) *For all $i>0$ and all S-modules N*

$$\mathrm{Ext}^i_R(M, R\otimes_S N) = (0).$$

b) *The canonical map*

$$\varphi_{M,N} : M^{\vee}\otimes_S N \to \mathrm{Hom}_R(M, R\otimes_S N)$$

is an isomorphism.

a$^{\vee}$) *For all $i>0$ and all S-modules N*

$$\mathrm{Ext}^i_R(M^{\vee}, R\otimes_S N) = (0).$$

b$^{\vee}$) *The canonical map*

$$\psi_{M,N} : M\otimes_S N \to \mathrm{Hom}_R(M^{\vee}, R\otimes_S N)$$

is an isomorphism.

2) *There exists an infinite complex of finite locally free R-modules*

$$(*) \qquad \ldots \to E^{-2} \xrightarrow{d^{-2}} E^{-1} \xrightarrow{d^{-1}} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \to \ldots$$

such that for all S-modules N,

$$E^{\cdot}\otimes_S N \text{ and } E^{\vee\cdot}\otimes_S N \text{ are acyclic and } M\approx \mathrm{im}\, d^0 .$$

3. *There exists an infinite acyclic complex* $(*)$ *such that if* $B^i = \mathrm{im}\, d^i$, *then*

a) B^i *and* $B^{i\vee}$ *are S-flat,*

b) $\mathrm{Ext}^i_R(B^j, R) = \mathrm{Ext}^i_R(B^{j\vee}, R) = (0)$ *for* $i>0$,

c) $M = B^0$.

PROOF. We do a cyclic proof in the order

$$3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 3.$$

$3 \Rightarrow 2$ is immediate .

By 2) we have exact sequences

$$(*)_N \qquad (0) \to M\otimes_S N \to E^1\otimes_S N \to E^2\otimes_S N \to$$

$$(*^{\vee})_N \qquad (0) \to M^{\vee}\otimes_S N \to E^{0\vee}\otimes_S N \to E^{-1\vee}\otimes_S N \to \ldots$$

Since the canonical map

$$\psi_{F,N} : F \otimes_S N \to \mathrm{Hom}_R(F^{\vee}, R \otimes_S N)$$

is an isomorphism when F is a finite locally free R-module, 1) follows from the diagrams:

$$\begin{array}{llccccccc}
(*)_N & (0) \to & M \otimes_S N & \to & E^1 \otimes_S N & \to & E^2 \otimes_S N & \to \dots \\
& & \downarrow & & \downarrow\wr & & \downarrow\wr & \\
(**)_N & (0) \to & \mathrm{Hom}_R(M^{\vee}, R \otimes_S N) & \to & \mathrm{Hom}_R(E^{1\vee}, R \otimes_S N) & \to & \mathrm{Hom}_R(E^{2\vee}, R \otimes_S N) & \to \dots \\
(*^{\vee})_N & (0) \to & M^{\vee} \otimes_S N & \to & E^{0\vee} \otimes_S N & \to & E^{-1\vee} \otimes_S N & \to \dots \\
& & \downarrow & & \downarrow\wr & & \downarrow\wr & \\
(**^{\vee})_N & (0) \to & \mathrm{Hom}_R(M, R \otimes_S N) & \to & \mathrm{Hom}_R(E^0, R \otimes_S N) & \to & \mathrm{Hom}_R(E^{-1}, R \otimes_S N) & \to \dots
\end{array}$$

Note that $(**)_N$ and $(**^{\vee})_N$ are left exact by $(*)_S$ and $(*^{\vee})_S$ and general facts about Hom.

To show 1) $\Rightarrow$ 3) note first that b$^{\vee}$) with $N = S$ tells us that $M \approx M^{\vee\vee}$, and therefore the definition is completely symmetric with respect to M and $M^{\vee}$.

Since M is noetherian, so is $M^{\vee}$ and we can find locally free resolutions of finite R-modules of the form:

$$\dots \to E^{-2} \xrightarrow{d^{-2}} E^{-1} \xrightarrow{d^{-1}} E^0 \to M \to (0)$$
$$\dots \to E^{3\vee} \xrightarrow{d^{2\vee}} E^{2\vee} \xrightarrow{d^{1\vee}} E^{1\vee} \to M^{\vee} \to (0) .$$

By 1) $(**)_N$ and $(**^{\vee})_N$ are exact and in the two diagrams above the left vertical arrows are isomorphisms. Hence we have $(*)_N$ and $(*^{\vee})_N$ exact for all N.

Let $N \to N'$ be an injection of S-modules then by $(*)_N$ and $(*)_{N'}$

$$\begin{array}{ccccc}
(0) \to & M \otimes_S N & \to & E^1 \otimes_S N \\
& \downarrow & & \downarrow\cap \\
(0) \to & M \otimes_S N' & \to & E^1 \otimes_S N' .
\end{array}$$

Now R is S-flat so E^1 is S flat and we see from the diagram that M is S-flat too. By the remark above, $M^{\vee}$ is S-flat as well.

LEMMA. *Let* $(0) \to M'' \to M \to M' \to (0)$ *be a short exact sequence of* R-*modules such that* M *and* M' *satisfies* 1), *then* M'' *satisfies* 1).

PROOF. 1a) is clear and from the two diagrams

$$\begin{array}{ccccccccc}
\to \operatorname{Tor}_1^S(M''^{\vee}, N) \to & M'^{\vee} \otimes_S N & \to & M^{\vee} \otimes_S N & \to & M''^{\vee} \otimes_S N & \to & (0) \\
& \downarrow \sim & & \downarrow \sim & & \downarrow \varphi_{M'',N} & & \\
(0) \to & \operatorname{Hom}(M', R \otimes_S N) & \to & \operatorname{Hom}(M, R \otimes_S N) & \to & \operatorname{Hom}(M'', R \otimes_S N) & \to & (0)
\end{array}$$

$$\begin{array}{ccccccccc}
(0) \to & M'' \otimes_S N & \to & M \otimes_S N & \to & M' \otimes_S N & \to & (0) \\
& \downarrow \psi_{M'',N} & & \downarrow \sim & & \downarrow \sim & & \\
(0) \to & \operatorname{Hom}(M''^{\vee}, R \otimes_S N) & \to & \operatorname{Hom}(M^{\vee}, R \otimes_S N) & \to & \operatorname{Hom}(M'^{\vee}, R \otimes_S N) & \to & \operatorname{Ext}_R^1(M''^{\vee}, R \otimes_S N) \to
\end{array}$$

we see that the canonical maps $\varphi_{M'',N}$ and $\psi_{M'',N}$ are isomorphisms and that $\operatorname{Ext}_R^1(M''^{\vee}, R \otimes_S N) = 0$ for all N. For $i > 1$ we have exact sequences

$$(0) = \operatorname{Ext}_R^{i-1}(M'^{\vee}, R \otimes_S N) \to \operatorname{Ext}_R^i(M''^{\vee}, R \otimes_S N) \to \operatorname{Ext}_R^i(M^{\vee}, R \otimes_S N) \to$$

and the lemma is proved.

Back to proving 3). Combining the resolution $\ldots \to E^{-1} \to E^0 \to M \to (0)$ and the exact sequence $(*)_S$ we get an infinite acyclic complex (*). If B^i denotes the image of d^i it follows by the Lemma that B^i satisfies 1) for $i \leqq 0$ and $(\operatorname{im} d^{i^{\vee}})$ satisfies 1) for $i \geqq 0$. However it is clear that $(\operatorname{im} d^{i^{\vee}})^{\vee} \approx B^i$ for $i \geqq 0$ so B^i satisfies 1) for $i > 0$ too.

COROLLARY 3. *Given a short exact sequence of finite R-modules:*

$$(0) \to M'' \to M \to M' \to (0)$$

with M' stably reflexive with respect to S. Then M'' is stably reflexive with respect to S if and only if M is.

PROPOSITION 4. *If M is S-stably reflexive, then for any homomorphism $S \to T$ the $R_{(T)}$-module $M_{(T)}$ is T-stably reflexive and*

$$(*) \qquad M^{\vee} \otimes_S T \xrightarrow{\sim} \operatorname{Hom}_{R_{(T)}}(M_{(T)}, R \otimes_S T) = (M_{(T)})^{\vee} .$$

PROPOSITION 5. *Given S, R and M as before, then M is stably reflexive with respect to S if and only if for all prime ideals $\mathfrak{p} \subset R$, $M_{\mathfrak{p}}$ is stably reflexive with respect to $S_{h^{-1}(\mathfrak{p})}$.*

PROOF. For noetherian M it is well known that all the functors occuring in property 1) of Theorem 2, commute with localization.

PROPOSITION 6. *Suppose S and R are local noetherian rings and $h: S \to R$ a local homomorphism. The following properties are equivalent:*

a) *M is S-stably reflexive,*
b) *$\hat{M}$ is S-stably reflexive,*
c) *$\hat{M}$ is $\hat{S}$-stably reflexive.*

PROOF. Bourbaki [2, Chap. III, 5.4.4.]

REMARK. In view of these propositions, it is clear that stable reflexivity is a property of coherent sheaves with respect to a morphism. Moreover the property is local in the Zariski topology as well as in the étale topology.

We leave as exercise the following "local criterion of stable reflexivity".

PROPOSITION 7. *Given S, R, and M as above and let $\mathscr{J}$ be an ideal of S contained in the radical of S. Denote by S_k, R_k, and M_k $S/\mathscr{J}^k$, $R\otimes_S S/\mathscr{J}^k$, and $M\otimes_S S/\mathscr{J}^k$, respectively. Then M is stably reflexive with respect to S, if and only if, for each k, M_k is stably reflexive with respect to S_k.*

Example of a stably reflexive module.

Let S be a ring, b and c elements of S, and let R be the ring:

$$R = S[x,y]/(xy-bc)\,.$$

Note that every element $u \in R$ can be written uniquely in the form

$$u = \ldots + u_{-n}x^n + \ldots + u_{-2}x^2 + u_{-1}x + u_0 + u_1 y + u_2 y^2 + \ldots$$

with the u_n's all in S, and all but a finite number equal to zero. It will be convenient to write the elements of R as columns

$$u = \begin{bmatrix} \vdots \\ u_{-2} \\ u_{-1} \\ u_0 \\ u_1 \\ u_2 \\ \vdots \end{bmatrix}.$$

Let E be R^2, α, β, p, and q endomorphisms of E given by:

$$\alpha = \begin{pmatrix} -b & y \\ -x & c \end{pmatrix}, \quad \beta = \begin{pmatrix} -c & y \\ -x & b \end{pmatrix}, \quad p = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

LEMMA. *The diagram*

$$\begin{array}{ccccccccccc}
\to & E & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & E & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & E & \xrightarrow{\alpha} \\
 & {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow & \\
\to & E & \xrightarrow{t_\beta} & E & \xrightarrow{t_\alpha} & E & \xrightarrow{t_\beta} & E & \xrightarrow{t_\alpha} & E & \xrightarrow{t_\beta}
\end{array}$$

commutes and has exact rows.

PROOF. Commutativity is straightforward to check. Considering E as a free S-module the elements of E can be written as columns

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ \vdots \\ \vdots \\ v_{-1} \\ v_0 \\ v_1 \\ \vdots \end{bmatrix}.$$

We may then regard α and β as infinite matrixes:

$$\alpha = \left[\begin{array}{ccccccc}
-b & 0 & 0 & bc & 0 & 0 & 0 \\
0 & -b & 0 & & bc & 0 & 0 \\
0 & 0 & -b & & 0 & 1 & 0 \\
 & & & & & & 1 \\
 & & & & & & \\
-1 & & & & & & \\
0 & -1 & 0 & & c & 0 & 0 \\
0 & 0 & -bc & & 0 & c & 0 \\
0 & 0 & 0 & -bc & 0 & 0 & c
\end{array}\right]$$

$$\beta = \left[\begin{array}{ccccccc}
-c & 0 & 0 & bc & 0 & 0 & 0 \\
0 & -c & 0 & & bc & 0 & 0 \\
0 & 0 & -c & & 0 & 1 & 0 \\
 & & & & & & 1 \\
 & & & & & & \\
-1 & & & & & & \\
0 & -1 & 0 & & b & 0 & 0 \\
0 & 0 & -bc & & 0 & b & 0 \\
0 & 0 & 0 & -bc & 0 & 0 & b
\end{array}\right]$$

Let Λ be the S-homomorphism of E into E given by the matrix

$$\Lambda = \left[\begin{array}{ccc|c|ccc}
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & & -1 & 0 & 0 \\
0 & 0 & 0 & & 0 & 0 & 0 \\
\hline
0 & & & & & & \\
0 & 0 & 0 & & 0 & 0 & 0 \\
0 & 0 & 1 & & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right]$$

It is then straightforward to check that

$$\alpha\Lambda + \Lambda\beta = 1 = \beta\Lambda + \Lambda\alpha\,.$$

Define:

$$P = \left\{\begin{array}{ccc|c|ccc|ccc|c|ccc}
 & & & & & & & -1 & & & & & & \\
 & & & & & & & b & -1 & & & & & \\
 & & & & & & & & b & -1 & & & & \\
\hline
 & & & 1 & & & & & & & -1 & & & \\
\hline
 & & & & 1 & & & & & & & -1 & & \\
 & & & & & 1 & & & & & & & -1 & \\
 & & & & & & 1 & & & & & & & -1 \\
1 & & & & & & & & & & & & & \\
 & 1 & & & & & & -b & & & & & & \\
 & & 1 & & & & & & -b & & & & & \\
\hline
 & & & & -c & & & & & & 1 & & & \\
\hline
 & & & & & -c & & & & & & 1 & & \\
 & & & & & & -c & & & & & & 1 & \\
 & & & & & & & & & & & & & 1
\end{array}\right\}$$

$$Q = \left\{\begin{array}{ccc|c|ccc|ccc|c|ccc}
 & & & & & & & 1 & & & & & & \\
-b & & & & & & & & 1 & & & & & \\
-b^2 & -b & & & & & & & & 1 & & & & \\
\hline
 & & & 1 & c & c^2 & c^3 & & & & 1 & c & c^2 & c^3 \\
\hline
 & & & & 1 & c & c^2 & & & & & 1 & c & c^2 \\
 & & & & & 1 & c & & & & & & 1 & c \\
 & & & & & & 1 & & & & & & & 1 \\
-1 & & & & & & & & & & & & & \\
-b & -1 & & & & & & & & & & & & \\
-b^2 & -b & -1 & & & & & & & & & & & \\
\hline
 & & & & c & c^2 & c^3 & & & & 1 & c & c^2 & c^3 \\
\hline
 & & & & & c & c^2 & & & & & 1 & c & c^2 \\
 & & & & & & c & & & & & & 1 & c \\
 & & & & & & & & & & & & & 1
\end{array}\right\}$$

Then $PQ=QP=1$. Moreover if $\xi_1=x-b$, $\xi_2=y-c$, then

$$P\cdot\alpha = \begin{pmatrix} \xi_1 & \xi_2 \\ 0 & 0 \end{pmatrix}.$$

This shows that the ideal $\mathscr{J}$ in R generated by ξ_1 and ξ_2 is isomorphic to the image of α.

Let $t=x+y-b-c$; then t is not a zero-divisor, and if $\varepsilon_1=(x-c)/t$ *and* $\varepsilon_2=(y-b)/t$, then the fractional ideal $\mathscr{J}'$ generated by ε_1 and ε_2 is isomorphic to the image of β, which again is isomorphic to the image of ${}^t\alpha$, that is $\mathscr{J}'\approx\mathscr{J}^\vee$. One may check that this isomorphism is the right one, i.e. for $s\in\mathscr{J}'$, $t\in\mathscr{J}$, $s(t)=s\cdot t$.

Since $\varepsilon_1+\varepsilon_2=1$, it is clear that $\mathscr{J}^\vee/R$ is generated by a single element say ε_1 also $(x-b)\cdot\varepsilon_1=x\cdot 1$ and $(y-c)\cdot\varepsilon_1=-c\cdot 1$ so the map $R\to\mathscr{J}^\vee/R$ sending 1 to ε_1 factors through S, and it is easy to check that this is an isomorphism.

$$S \approx \mathscr{J}^\vee/R\,.$$

Summarizing all this we have:

1) *The ideal* $\mathscr{J}\subset R$ *is stably reflexive with respect to* S.
2) *The fractional ideal* $\mathscr{J}'$ *consisting of all elements of the total quotient ring of* R *that maps* $\mathscr{J}$ *into* R *is isomorphic to the algebraic dual of* $\mathscr{J}$.
3) $\mathscr{J}^\vee/R$ *is a free* S*-module of rank* 1.

REFERENCES

1. S. Arakelov, *Families of algebraic curves with fixed degeneracies*, Izv. Akad. Nauk. SSSR Ser. Mat. 35 (1971), 1269–1293.
2. N. Bourbaki, *Éléments de Mathématique. Algèbre Commutative* (Act. Sci. Ind. 1293), Hermann, Paris, 1961.
3. P. Deligne and D. Mumford, *The irreducibility of the space of curves of a given genus*, Inst. Hautes Études Sci. Publ. Math. 36 (1969), 75–109.
4. P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques* in *Modular functions of one variables* II, eds. W. Kuyk and P. Deligne (Proc. Internat. Summer School, Univ. Antwerp, 1972) (Lecture Notes in Math. 349), pp. 143–316. Springer-Verlag, Berlin - Heidelberg - New York, 1973.
5. A Grothendieck and J. Dieudonné, *Éléments de géométrie algèbrique*, Inst. Hautes Études Sci. Publ. Math. 11 (1961), 81–148.
6. R. Hartshorne, *Residues and duality* (Lecture Notes in Math. 20), Springer-Verlag, Berlin - Heidelberg - New York, 1966.
7. F. F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves* I, Math. Scand. 39 (1976), 19–55.
8. S. Lichtenbaum and M. Schlessinger, *The cotangent complex of a morphism*, Trans. Amer. Soc. 128 (1967), 41–70.

9. D. Mumford, *Geometric invariant theory* (Ergebnisse Math. 34) Springer-Verlag, Berlin - Heidelberg - New York, 1965.
10. D. Mumford, *Introduction to Algebraic Geometry*, Lecture Notes, Harvard University, 1967.
11. D. Mumford, *Picard groups of moduli problems*, in *Arithmetical Algebraic Geometry*, ed. O. F. G. Schilling (Proc. Conf. Purdue Univ. 1963) pp. 33–81. Harper and Row Publ., New York, 1965.
12. D. Mumford, *On the equations defining Abelian varieties* III, Invent. Math. 3 (1967), 215–244.
13. D. Mumford, *Varieties defined by quadratic equations. With an appendix by G. Kempf*, in *Questions on algebraic varieties*, (C.I.M.E., III Ciclo, Varenna, 1969) pp. 29–100, Edizioni Cremonese, Roma, 1970.
14. C. S. Seshadri, *Quotient spaces modulo reductive algebraic groups*, Ann. of Math. 95 (1972), 511–556.

MATEMATISK INSTITUTT
UNIVERSITETET I TRONDHEIM
N.T.H.
7034 TRONDHEIM
NORWAY

MATH. SCAND. 52 (1983), 200–212

THE PROJECTIVITY OF THE MODULI SPACE OF STABLE CURVES, III: THE LINE BUNDLES ON $M_{g,n}$, AND A PROOF OF THE PROJECTIVITY OF $\bar{M}_{g,n}$ IN CHARACTERISTIC 0

FINN F. KNUDSEN

Introduction.

In section 4 we construct the basic line bundles on $M_{g,n}$ and study their behaviour under pullback by contraction and clutching. This is where this paper originally was to end and then Mumford would use these results to prove projectivity as suggested by Seshadri. However Gieseker and Mumford discovered that a stable curve embedded in projective space of sufficiently high dimension is stable in the sense of [9]. Hence the projectivity can be proved in a much more natural way using the techniques of [9]. These results are published in [14].

Instead we have added to this paper sections 5, and 6, where we present the very first proof of the projectivity of the moduli space of stable curves. The result of Gieseker and Mumford is much sharper than Theorem 6.1. In fact the constant m occuring here can be chosen to equal 56/5 regardless of n, g and the characteristic.

None the less the results of this paper give considerable insight into the boundary of the moduli space and we believe this might have some general interest. Finally I want to express my gratitude to Professor Mumford for continuous support and encouragement throughout the writing of this paper.

4. Invertible sheaves on $M_{g,n}$ and their functorial properties.

Recall that a sheaf on the stack $M_{g,n}$ consists of the following data:

1) For every n-pointed stable curve $\pi\colon X \to S$ a sheaf $\mathscr{F}(\pi)$ on S.
2) For every morphism

Received August 6, 1979; in revised form November 23, 1981.

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_2}\downarrow \\ S_1 & \xrightarrow{f} & S_2 \end{array}$$

an isomorphism $\varphi_{X_1,X_2}: f^*(\mathscr{F}(\pi_2)) \xrightarrow{\sim} \mathscr{F}(\pi_1)$ which for every composition

$$\begin{array}{ccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_2}\downarrow & & {\scriptstyle \pi_3}\downarrow \\ S_1 & \xrightarrow{f} & S_2 & \xrightarrow{g} & S_3 \end{array}$$

satisfies the cocycle condition

$$\begin{array}{ccc} f^*(g^*(\mathscr{F}(\pi_3))) & \xrightarrow{f^*(\varphi_{X_2,X_3})} & f^*(\mathscr{F}(\pi_2)) \\ \approx\Big| & & \Big\downarrow \varphi_{X_1,X_2} \\ (gf)^*\mathscr{F}(\pi_3) & \xrightarrow{\varphi_{X_1,X_3}} & \mathscr{F}(\pi_1) \end{array}$$

Moreover, because $M_{g,n}$ is an algebraic stack, to define a sheaf $\mathscr{F}$ it suffices to define $\mathscr{F}(\pi)$ for every $\pi\colon X \to S$ for which $S \to M_{g,n}$ is étale, plus φ's whenever these make sense.

Then there is one and only one way, up to canonical isomorphism, to extend this to a sheaf $\mathscr{F}$ on the whole stack $M_{g,n}$. This of course because descent data are effective for étale surjective morphisms.

Let $\pi\colon X \to S$; $s_i\colon S \to X$ $(1 \leqq i \leqq n)$ be an n-pointed stable curve of genus g. By Corollary 3.3, $\Omega_{X/S}$ is a perfect complex on X. Since π is flat, $R\pi_*\Omega_{X/S}$ is a perfect complex on S. We denote by $\Lambda_{g,n}(\pi)$ the invertible sheaf on S defined by

a) $\Lambda_{g,n}(\pi) = \det(R\pi_*\Omega_{X/S})$.

The operations Ω, $R\pi_*$ and det all commute with base change, so $\Lambda_{g,n}$ is an invertible sheaf on $M_{g,n}$. Similarly we define

b) $\lambda_{g,n}(\pi) = \det(R\pi_*\omega_{X/S})$

c) $\delta_{g,n} = \lambda_{g,n} \otimes \Lambda_{g,n}^{-1}$

d) $\varkappa_{g,n}^{(i)}(\pi) = s_i^*(\omega_{X/S})$

e) $\tilde{\Lambda}_{g,n} = \Lambda_{g,n} \otimes \bigotimes_{i=1}^{n} \varkappa_{g,n}^{(i)}$

f) $\tilde{\delta}_{g,n} = \lambda_{g,n} \otimes \tilde{\Lambda}_{g,n}^{-1}$.

Besides the above invertible sheaves we also have the divisors $S_{g,n}^H$ of Definition 3.8. When no confusion is possible we drop the subscripts g, n.

Suppose $\pi\colon X \to S$ is smooth over each point of S of depth 0. Then, since $\Omega_{X/S}$ is flat over S, the associated points of $\Omega_{X/S}$ lie over the associated points of

S. Over the associated points of S, the canonical morphism $\Omega_{X/S} \to \omega_{X/S}$ is an isomorphism. Hence in this case $\Omega_{X/S} \to \omega_{X/S}$ is everywhere injective and the points in $\mathrm{Sup}(\pi_*(\omega_{X/S}/\Omega_{X/S}))$ have depth $\geqq 1$. Therefore we have

$$\delta(\pi) = \lambda \otimes \Lambda^{-1} \approx \mathcal{O}_S(\mathrm{Div}\, \pi_*(\omega_{X/S}/\Omega_{X/S}))\,.$$

$\pi\colon X \to S$ defines a morphism $S \to M_{g,n}$ and $\mathrm{Div}\,\pi_*(\omega_{X/S}/\Omega_{X/S})$ is simply the pullback to S of the divisor $S_{g,n} \subset M_{g,n}$, that is

$$\delta_{g,n} \approx \mathcal{O}_{M_{g,n}}(S_{g,n})\,.$$

Roughly speaking, $\delta_{g,n}$ is the sheaf of functions on $M_{g,n}$ regular except for simple poles at infinity.

THEOREM 4.1. *Let $\pi = \pi_{n+1}\colon M_{g,n+1} \to M_{g,n}$, and let $S^i = S^{i,n+1}_{g,n+1}$ be the divisor on $M_{g,n+1}$, which is really the image of the i-th section $s_i\colon M_{g,n} \to M_{g,n+1}$. We have*

a) $\pi^*(\lambda_{g,n}) \cong \lambda_{g,n+1}$,

b) $\pi^*(\Lambda_{g,n}) \cong \Lambda_{g,n+1}(S^1 + S^2 + \ldots + S^n)$,

c) $\pi^*(\varkappa^{(i)}_{g,n}) = \varkappa^{(i)}_{g,n+1}(-S^i)$,

d) $\varkappa^{(n+1)}_{g,n+1} \cong \omega_{M_{g,n+1}/M_{g,n}}(S^1 + S^2 + \ldots + S^n)$,

and hence

e) $\pi^*(\tilde{\Lambda}_{g,n}) \otimes \omega_{M_{g,n+1}/M_{g,n}}(S^1 + \ldots + S^n) \cong \tilde{\Lambda}_{g'n+1}\,.$

PROOF. Let $\pi\colon X \to S$, $s_i\colon S \to X$ $(1 \leqq i \leqq n)$ be an n-pointed stable curve of genus g such that the corresponding morphism $S \to M_{g,n}$ is étale. By duality we have

$$\lambda(\pi) = \det R\pi_*\omega_{X/S} \cong \det R\pi_*\mathcal{O}_X\,.$$

Consider the diagram

$$\begin{array}{ccccc} X' & \xrightarrow{q} & X \times_S X & \xrightarrow{p_2} & X \\ {\scriptstyle \pi'}\downarrow {\scriptstyle s_1'}\circlearrowright & & {\scriptstyle p_1}\downarrow {\scriptstyle \Delta}\circlearrowright & & {\scriptstyle \pi}\downarrow \\ X & = & X & \xrightarrow{\pi} & S \end{array}$$

where X' is the stabilization defined by Δ.

We have $Rq_*\mathcal{O}_{X'} \approx \mathcal{O}_{X \times_S X}$, so this proves a).

To prove b), notice that the divisor of singular curves on X is $\pi^{-1}(S_{g,n}(\pi)) + s_1 + \ldots + s_n$. Since $\Lambda \approx \lambda \otimes \delta^{-1}$ we get b).

On X' we have a short-exact sequence of sheaves

$$(0) \to q^*\mathcal{O}_{X\times_S X}(-s_i) \to \mathcal{O}_{X'}(-s_i') \to \mathcal{O}_{X'}(-s_i')|_{q^{-1}(s_i\cap\Delta)} \to (0)\,.$$

s_i' is transversal to $q^{-1}(s_i \cap \Delta)$, so taking $s_i'^*$ of this sequence leaves it exact, i.e.

$$(0) \to \pi^*(\varkappa_n^{(i)}) \to \varkappa_{n+1}^{(i)} \to \mathscr{F} \to (0)\,,$$

where $\mathscr{F}$ is a sheaf with support on the divisor s_i on X and of rank 1. Hence $\operatorname{Div}(\mathscr{F}) = s_i$ and this proves c).

To prove d) notice that the conormal bundle

$$\varkappa^{(n+1)}(\pi') \cong s_{n+1}'^*(\omega_{X'/X}(s_1' + \ldots + s_n'))\,.$$

This is because the sections never cross. By Lemma 1.4 a) we have

$$\varkappa^{(n+1)}(\pi') \approx \Delta^* p_2^*(\omega_{X/S}(s_1 + \ldots + s_n))\,.$$

But $p_2 \circ \Delta$ is the identity, so this is d).

THEOREM 4.2. *Let*

$$\alpha\colon M_{g-1,n+2} \to M_{g,n}\,,$$
$$\beta\colon M_{g_1,n_1+1} \times M_{g_2,n_2+1} \to M_{g,n}$$

be the clutching morphisms. Then

a) $\alpha^*\lambda_{g,n} \approx \lambda_{g-1,n+2}$, $\quad \beta^*\lambda_{g,n} \approx \lambda_{g_1,n_1+1} \otimes \lambda_{g_2,n_2+1}$,

b) $\alpha^*\tilde{\Lambda}_{g,n} \approx \tilde{\Lambda}_{g-1,n+2}$, $\quad \beta^*\tilde{\Lambda}_{g,n} \approx \tilde{\Lambda}_{g_1,n_1+1} \otimes \tilde{\Lambda}_{g_2,n_2+1}$,

c) $\alpha^*\tilde{\delta}_{g,n} \approx \tilde{\delta}_{g-1,n+2}$, $\quad \beta^*\tilde{\delta}_{g,n} \approx \tilde{\delta}_{g_1,n_1+1} \otimes \tilde{\delta}_{g_2,n_2+1}$.

PROOF. Consider a clutching diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & X' \\ {\scriptstyle s_1}\uparrow\uparrow{\scriptstyle s_2}\ \uparrow\downarrow{\scriptstyle \pi} & & {\scriptstyle \pi'}\downarrow\uparrow{\scriptstyle s} \\ S & = & S \end{array}$$

On X' we have a short-exact sequence

$$(0) \to \mathcal{O}_{X'} \to p_*\mathcal{O}_X \xrightarrow{s_1^* - s_2^*} \mathcal{O}_S \to (0)\,.$$

Taking $\det R\pi'_*$ of this sequence yields a). b) follows by the clutching sequence Theorem 3.5, and c) is a consequence of a) and b).

Let α be the numerical function

$$\alpha(g) = \begin{cases} 3 & \text{if } g = 0 \\ 1 & \text{if } g = 1 \\ 0 & \text{if } g \geqq 2\,. \end{cases}$$

We want to see what happens to our line bundles under pullback by the full contraction morphism

$$\pi = \pi_{\alpha+1,\alpha+2,\dots,\alpha+n} : M_{g,\alpha+n} \to M_{g,\alpha} .$$

DEFINITION. On $M_{g,n}$ we define the divisors

$$\nabla^k_{g,n} = \sum_{*H=k+1} S^H_{g,n}$$

$$D^l_{g,n} = \sum_{l\in H} S^H_{g,n} .$$

REMARK. Note that $S^H_{g,n}=\varnothing$, if $*(H\cap\{n;\ 1\leqq n\leqq\alpha(g)\})\geqq 2$.
Consider the diagram

$$\begin{array}{ccc} M_{g,\alpha+n} & \xrightarrow{\pi_{\hat{i}}} & M_{g,\alpha+1} \\ \pi_i\downarrow & & \downarrow \\ M_{g,\alpha+n-1} & \to & M_{g,\alpha} \end{array}$$

$i>\alpha$ and $\pi_{\hat{i}}$ is short for $\pi_{\alpha+1,\alpha+2,\dots,\hat{i},\dots,\alpha+n}$.

LEMMA 4.3. *With notation as above*

$$\omega_{M_{g,\alpha+n}/M_{g,\alpha+n-1}}(S^{1,i}+S^{2,i}+\dots+S^{\alpha+n,i}) \approx \pi_{\hat{i}}^*\omega_{M_{g,\alpha+1}/M_{g,\alpha}}\otimes\mathcal{O}(D^i) .$$

PROOF. We may well suppose that $i=\alpha+n$. Consider the commutative diagram

$$\begin{array}{ccccc} M_{g,\alpha+n} & \xrightarrow{\pi_{\alpha+1}} & M_{g,\alpha+n-1} & \xrightarrow{\pi_{\alpha+n-1}} & M_{g,\alpha+1} \\ \pi_{\alpha+n}\downarrow\uparrow s_i & & \pi_{\alpha+n-1}\downarrow\uparrow s_{i-1} & & \downarrow \\ M_{g,\alpha+n-1} & \xrightarrow{\pi_{\alpha+1}} & M_{g,\alpha+n-2} & \longrightarrow & M_{g,\alpha} \end{array}$$

(The diagram commutes since we renumber the sections.)

We define

$$\delta_j = \begin{cases} j & \text{if } j\leqq\alpha \\ j+1 & \text{if } j>\alpha ; \end{cases}$$

then

$$\pi^{-1}_{\alpha+1}(S^{j_1,j_2,\dots,j_k,\alpha+n-1}_{g,\alpha+n-1}) = S^{\delta_{j_1},\delta_{j_2},\dots,\delta_{j_k},\alpha+n}_{g,\alpha+n} + S^{\delta_{j_1},\dots,\delta_{j_k},\alpha+1,\alpha+n}_{g,\alpha+n} .$$

Hence

$$\pi^{-1}_{\alpha+1}\Bigg(\sum_{\substack{*H=k \\ \alpha+n-1\in H}} S^H_{g,\alpha+n-1}\Bigg) = \sum_{\substack{*H=k \\ \alpha+n\in H}} S^H_{g,\alpha+n} + \sum_{\substack{*H=k+1 \\ \alpha+n\in H \\ \alpha+1\in H}} S^H_{g,\alpha+n} - \sum_{\substack{*H=k \\ \alpha+n\in H \\ \alpha+1\in H}} S^H_{g,\alpha+n} .$$

Summing these relations over k gives

$$\pi_{\alpha+1}^{-1} D_{g,\alpha+n-1}^{\alpha+n-1} = D_{g,\alpha+n}^{\alpha+n} - S_{g,\alpha+n}^{\alpha+1,\alpha+n} .$$

By Lemma 1.6 a) we have

$$\pi_{\alpha+1}^{*} \omega_{M_{g,\alpha+n-1}/M_{g,\alpha+n-2}}(S^{1,\alpha+n-1}+\ldots+S^{\alpha+n-2,\alpha+n-1})$$
$$\approx \omega_{M_{g,\alpha+n}/M_{g,\alpha+n-1}}(S^{1,\alpha+n}+\ldots+S^{\alpha+n-1,\alpha+n}-S^{\alpha+1,\alpha+n}) ,$$

and the Lemma follows readily by induction.

The main formula of this section is

THEOREM 4.4.

$$\tilde{\Lambda}_{g,\alpha+n} \approx \pi^{*}(\tilde{\Lambda}_{g,\alpha}) \otimes \bigotimes_{i=\alpha+1}^{\alpha+n} \pi_{i}^{*}(\omega_{M_{g,\alpha+1}/M_{g,\alpha}}) \otimes \mathcal{O}(E_{g,\alpha+n})$$

where

$$E_{g,n} = \nabla_{g,n}^{1} + 2\nabla_{g,n}^{2} + \ldots + (n-1)\nabla_{g,n}^{n-1} .$$

PROOF. In order to use induction we consider the composition

$$M_{g,\alpha+n} \xrightarrow{\pi_{\alpha+n}} M_{g,\alpha+n-1} \to M_{g,\alpha} .$$

By theorem 4.1 e) and the previous lemma we have

$$\tilde{\Lambda}_{g,\alpha+n} \approx \pi_{\alpha+n}^{*}(\tilde{\Lambda}_{g,\alpha+n-1}) \otimes \omega_{M_{g,n+2}/M_{g,\alpha+n-1}}(S^{1,\alpha+n}+\ldots+S^{\alpha+n-1,\alpha+n})$$
$$\approx \pi_{\alpha+n}^{*}(\tilde{\Lambda}_{g,\alpha+n-1}) \otimes \pi_{\alpha+n}^{*}(\omega_{M_{g,\alpha+1}/M_{g,\alpha}}) \otimes \mathcal{O}(D_{g,\alpha+n}^{\alpha+n}) .$$
$$\pi_{\alpha+n}^{-1}(\nabla^{k}) = \sum_{*H=k+1} S^{H} + \sum_{\substack{*H=k+2 \\ \alpha+n\in H}} S^{H} - \sum_{\substack{*H=k+1 \\ \alpha+n\in H}} S^{H} .$$

Multiplying each of these relations by k and summing over k gives the relation

$$\pi_{\alpha+n}^{-1}(E_{g,\alpha+n-1}) = E_{g,\alpha+n} - D_{g,\alpha+n}^{\alpha+n}$$

and the theorem follows by induction.

We end this section with an inequality which is crucial in the next paragraph. Suppose $g<2$ so that $\alpha>0$ and consider the map

$$\pi_{\hat{j}} : M_{g,\alpha+n} \to M_{g,\alpha+1} .$$

On $M_{g,\alpha+1}$ we have $E=E_{g,\alpha+1}=\sum_{p=1}^{\alpha} S_{g,\alpha+1}^{p,\alpha+1}$ and

$$\sum_{j=\alpha+1}^{\alpha+n} \pi_{\hat{j}}^{-1}(E_{g,\alpha+1}) = \sum_{p=1}^{\alpha} D^{p} \cap E .$$

Since $D^p \cap D^q = \emptyset$ for $p < q \leqq \alpha$, we have

LEMMA 4.5.

$$E_{g,\alpha+n} \geqq \sum_{j=\alpha+1}^{\alpha+n} \pi_j^{-1}(E_{g,\alpha+1}) .$$

5. Ampleness of $\tilde{\Lambda}$ on the fibres of the full contraction morphism.

Let $\pi: C \to S$ be a smooth stable curve with $\alpha(g)$ sections. Then π corresponds to a morphism which we also call π

$$\pi : S \to M_{g,\alpha} \setminus S_{g,\alpha} .$$

We define the scheme $B_n(C/S)$ via the cartesian diagram

$$\begin{array}{ccc} B_n(C/S) & \to & M_{g,\alpha+n} \\ \downarrow & & \downarrow \\ S & \longrightarrow & M_{g,\alpha} \end{array}$$

$B_{n+1}(C/S)$ is an $n+\alpha$-stable curve over $B_n(C/S)$, and we have the cartesian diagram

$$\begin{array}{ccc} B_{n+1}(C/S) & \to & M_{g,\alpha+n+1} \\ {\scriptstyle \pi_{\alpha+n+1}(C/S)}\downarrow & & \downarrow \\ B_n(C/S) & \underset{\pi_{\alpha+n+1}(C/S)}{\to} & M_{g,\alpha+n} \end{array}$$

On $B_n(C/S)$ we have the line bundle

$$\tilde{\Lambda}_n(C/S) = \tilde{\Lambda}(\pi_{\alpha+n+1}(C/S)) = \pi_{\alpha+n+1}(C/S)^*(\tilde{\Lambda}_{g,\alpha+n}) .$$

PROPOSITION 5.1. *$B_n(C/S) \to S$ is a smooth and proper morphism of relative dimension n.*

PROOF. We may suppose $S = \operatorname{Spec}(k)$, k an algebraically closed field. $B_1(C/k) = C$ is smooth and proper of dimension 1 over k, so we may proceed by induction.

Let x be a point in $B_n(C/k)$. Then x corresponds to a curve E/k with $\alpha+n$ distinguished points $P_1, \ldots, P_{\alpha+n}$. One of the components of E is C.

If $\alpha+1 \leqq j \leqq \alpha+n$, the fibre of the contraction

$$\pi_j : B_n(C/k) \to B_{n-1}(C/k)$$

over the point $\pi_j(x)$ is the curve E' obtained from E by contracting the point P_j and the point $x \in E' = \pi_j^{-1}(\pi_j(x))$ is the image of P_j.

If $E = C$, clearly π_j is smooth at x. Otherwise $x \in S^{H_1} \cap S^{H_2} \cap \ldots \cap S^{H_q}$, and we may suppose that H_1 does not contain any of the other H_j's. Then if $j \in H_1$ and $j > \alpha$, the map

$$\pi_j : B_n(C/k) \to B_{n-1}(C/k)$$

is smooth at x and the proposition follows by induction. See Definition 3.8 for a definition of S^H.

In the rest of this section we fix an algebraically closed field k and consider all of our schemes to be defined over k.

Let C be a smooth stable curve of genus g and with distinguished points P_j $(1 \leqq j \leqq \alpha(g))$. We consider a morphism $\pi: S \to B_n(C)$, where S is another nonsingular irreducible complete curve. π corresponds to an $\alpha + n$-pointed stable curve $\pi: \hat{C} \to S$. Each one of the sections of π gives rise to a morphism $t_i: S \to C$. This is the composition

$$S \xrightarrow{\pi} B_n(C) \xrightarrow{\pi_i(C)} B_1(C) = C.$$

For $1 \leqq j \leqq \alpha$ we have $t_j(S) = P_j \in C$.

We wish to study the pullback to S of the line bundle $\tilde{\Lambda}_n(C)$ on $B_n(C)$. By Theorem 4.4 this is

$$\pi^*(\tilde{\Lambda}_n(\mathrm{C})) = \tilde{\Lambda}(\pi) \approx \bigotimes_{i=\alpha+1}^{\alpha+n} t_i^*(\omega_C) \otimes \mathcal{O}_S(\pi^{-1}(E)).$$

By Lemma 4.5 we have

$$\pi^{-1}(E) \geqq \pi^{-1}\left(\sum_{j=\alpha+1}^{\alpha+n} \pi_j^{-1}(E_{g,\alpha+1})\right) = \sum_{i=1}^{\alpha} \left(\sum_{j=\alpha+1}^{\alpha+n} t_j^{-1}(P_i)\right).$$

Hence we have

Proposition 5.2. *Let $\pi: S \to B_n(C)$ be as above, then*

$$\deg \pi^*(\tilde{\Lambda}_n(C)) = \deg \tilde{\Lambda}(\pi) \geqq (2g-2+\alpha) \sum_{i=\alpha+1}^{\alpha+n} \deg(t_i).$$

In particular,

$$\deg \pi^*(\tilde{\Lambda}_n(C)) \geqq \sum_{i=1}^{\alpha+n} \deg(t_i).$$

Definition 5.3. Let X be a complete scheme and S an integral curve in X. We denote by $m_P(S)$ the multiplicity of a point P on the curve, and by $m(S)$ the number

$$m(S) = \sup_{P \in S} \{m_P(S)\} .$$

LEMMA 5.4. *For all stable nonsingular C and for all integral curves $S \subset B_n(C)$*

$$\deg \tilde{\Lambda}_n(C)|_S \geqq m(S) .$$

PROOF. We prove the lemma by induction with respect to n. For $n=1$ $S=C$ $=B_1(C)$, and by Theorem 4.4

$$\tilde{\Lambda}_1(C) \approx \omega_C \otimes \mathcal{O}_C(E) .$$

In this case $E=\sum_{j=1}^{\alpha} P_j$, so $\deg \mathcal{O}_C(E)=\alpha$. Hence

$$\deg \tilde{\Lambda}_1(C) = 2g-2+\alpha \geqq 1 = m(S) .$$

Assume that the lemma is true for all $k<n$ and fix notation

$$\begin{aligned} &S \subset B_n(C) \text{ an integral curve ,} \\ &\hat{S} \text{ the normalization of } S , \\ &\pi\colon S \to B_n(C) \\ &t_i\colon S \to C \quad (1 \leqq i \leqq \alpha+n) , \\ &\hat{t}_i\colon \hat{S} \to C \quad (1 \leqq i \leqq \alpha+n) . \end{aligned}$$

If all the t_i's are constant maps, at least three of them are equal. If r is the maximum number of equal maps we get a factorization via the clutching morphism

$$S \hookrightarrow B_{n-r+1}(C) \times B_{r-2}(\boldsymbol{P}_k^1) \hookrightarrow B_n(C)$$

and the lemma follows from Theorem 4.2 b).

Suppose then that t_i is not a constant map. Then on the one hand we have

$$\deg \tilde{\Lambda}_n(C)|_S \geqq \sum_{j=\alpha+1}^{\alpha+n} \deg \hat{t}_j \geqq \deg \hat{t}_i .$$

On the other hand, if P is a point of S, P' its image in C^n, and S' the image of S in C^n, we have

$$\deg \hat{t}_i \geqq m_{P'}(S') \geqq m_P(S) .$$

The second inequality is obvious. To prove the first inequality, let H be the divisor $p_i^{-1}(t_i(P))$. Then by the very definition of multiplicity we have

$$m_{P'}(S') \leqq (H,S')_{P'} \leqq (H,S') = \deg \hat{t}_i ,$$

and the lemma follows by induction.

THEOREM 5.5. *For all n and all nonsingular stable curves C, $\tilde{\Lambda}_n(C)$ is ample.*

PROOF. For this it is enough to state *Seshadri's ampleness criterion*:

Let L be an invertible sheaf on a complete scheme X, then L is ample if and only if there is an $\varepsilon > 0$ such that $\deg L|_S \geqq \varepsilon \cdot m(S)$ for every integral curve S in X.

6. Proof of projectivity in characteristic 0.

In this section let k be the field of complex numbers. All schemes and morphisms will be defined over k. In particular by $M_{g,n}$ in this section we will mean

$$M_{g,n} \times_{\mathrm{Spec}(\mathbf{Z})} \mathrm{Spec}(k) .$$

THEOREM 6.1. *For all pairs g,n with $2g-2+n>0$, the stack $M_{g,n}$ is coarsely represented by a normal projective variety $\bar{M}_{g,n}$. More precisely, there is a morphism*

$$\Phi : M_{g,n} \to \bar{M}_{g,n}$$

such that

1) *Φ induces an isomorphism*

$$\Phi(k)\colon \left\{\begin{matrix}\text{isomorphism of classes} \\ \text{of objects in } M_{g,n}(k)\end{matrix}\right\} \xrightarrow{\sim} \bar{M}_{g,n}(k) .$$

2) *$\bar{M}_{g,n}$ is normal and proper over* Spec (k).
3) *There exists an integer N such that $\lambda^{\otimes N}$ and $\tilde{\delta}^{\otimes N}$ are pullbacks of invertible sheaves on $\bar{M}_{g,n}$, which we also write $\lambda^{\otimes N}$ and $\tilde{\delta}^{\otimes N}$. Finally there exists a number $m>0$ such that if $N \mid a$, $N \mid b$ and $a \geqq mb > 0$, then,*

$$\lambda^{\otimes a} \otimes (\tilde{\delta}^{-1})^{\otimes b} \quad \textit{is ample on } \bar{M}_{g,n} .$$

REMARK. It is not hard to see that 1) and 2) imply that Φ is universal for all morphisms from $M_{g,n}$ to schemes. To prove the theorem we introduce some auxiliary schemes. Fix an integer $e \geqq 3$ and let $\{C, x_1, x_2, \ldots, x_n)$ be an n-pointed stable curve of genus g. We define

$$d = e(2g-2+n) = \deg(\omega_C(x_1 + \dots + x_n)^{\otimes e}),$$
$$P(t) = dt-(g-1) = \text{Hilbert polynomial of } \omega_C(x_1 + \dots + x_n)^{\otimes e},$$
$$\nu = P(1) = h(\omega_C(x_1 + \dots + x_n)^{\otimes e}).$$

Consider the Hilbert scheme $\mathrm{Hilb}^P_{\boldsymbol{P}^{\nu-1}}$ of subschemes of $\boldsymbol{P}^{\nu-1}$ with Hilbert polynomial $P(t)$ and let

$$H_{g,n} \subset \mathrm{Hilb}^P_{\boldsymbol{P}^{\nu-1}} \times (\boldsymbol{P}^{\nu-1})^n$$

be the locally closed subscheme representing $n+1$ tuples $(C, x_1, \dots, x_n)$, where the x_i's are distinct smooth points on C, C with these x_i's is an n-pointed stable curve and C is embedded in $\boldsymbol{P}^{\nu-1}$ in such a way that $\mathcal{O}(1)|_C \approx \omega_C(x_1 + \dots + x_n)^{\otimes e}$. Thus $H_{g,n}$ represents the functor

$$H_{g,n}(S) = \left\{ \begin{array}{l} \text{pairs } (\{\pi\colon X \to S,\ s_i\colon S \to X\}, \alpha) \\ \text{consisting of an } n\text{-pointed stable} \\ \text{curve over } S \text{ and an } S\text{-isomorphism} \\ \alpha\colon \boldsymbol{P}(\pi_* \omega_{X/S}(s_1 + \dots + s_n)^{\otimes e} \xrightarrow{\sim} \boldsymbol{P}_S^{\nu-1} \end{array} \right\} \Big/ \begin{array}{l} \text{modulo} \\ \text{isomorphisms}. \end{array}$$

PGL$(\nu-1)$ acts on $H_{g,n}$ and $M_{g,n}$ is the stack theoretic quotient of $H_{g,n}$ by PGL$(\nu-1)$. According to ([15, Theorem 6.1]), $H_{g,n}$ has a finite normal Galois covering $H^*_{g,n}$ (not at all unique) on which PGL$(\nu-1)$ acts freely, commuting with the Galois group Γ and with local cross sections in the Zariski topology, and $H^*_{g,n} \to H_{g,n}$ is a PGL$(\nu-1)$ morphism. $H^*_{g,n}$ is a principal fibre bundle over a normal variety $X_{g,n}$ with group PGL$(\nu-1)$, and the action of Γ descends to $X_{g,n}$. On $H_{g,n}$ we have the universal family of n-pointed stable curves in $\boldsymbol{P}^{\nu-1}$. Pulling this back to $H^*_{g,n}$ and dividing by PGL$(\nu-1)$ we see that $X_{g,n}$ has a family of n-pointed stable curves on it. Then we have a commutative diagram

$$\begin{array}{ccc} H^*_{g,n} & \longrightarrow & H_{g,n} \\ \downarrow & & \downarrow \\ X_{g,n} & \xrightarrow{q} & M_{g,n} \end{array}$$

where $q\colon X_{g,n} \to M_{g,n}$ is finite and surjective; hence $X_{g,n}$ is proper over k, since $M_{g,n}$ is. Therefore by [15, Remark 6.1], it suffices to prove that $\lambda^{P_m} \otimes \delta^{-1}$ is ample on $X_{g,n}$ for some m, and then $\bar{M}_{g,n} = X_{g,n}/\Gamma$ will have all the required properties.

By contraction, $X_{g,n}$ has a family of stable curves over it, so by ([1, Theorem 1.1, and Lemma 1.4]) there is a morphism

$$t\colon X_{g,n} \to S_g = \left\{\begin{array}{l}\text{Satake compactification of}\\ \text{the moduli space of}\\ \text{polarized abelian}\\ \text{varieties of dimension } g\end{array}\right.$$

such that:

1) If x is a point in $X_{g,n}$ corresponding to an n-pointed stable curve C, then $t(x)$ corresponds to the abelian part of the generalized jacobian of C.
2) If S_g is embedded in projective space by modular forms of weight m, then

$$t^*(\mathcal{O}(1)) \approx \lambda^{\otimes m}.$$

Let x be a point of $X_{g,n}$ corresponding to a stable curve C and let $C_1, \dots, C_k$ be the non-rational components of the normalization of C. Let

$$m = n+2g-2-2\sum_{i=1}^{k}(g_i-1)-(\# \text{ of elliptic } C_i)$$

where $g_i = \text{genus } C_i$.

Let X be the variety:

$$X = \coprod_{\substack{n_1+\dots+n_k+n'_1+\dots+n'_l+1=m\\ n_i\geq 1 \text{ (0 if } C_i \text{ ellipic)}\\ n'_i\geq 0}} \left(\coprod B_{n_1}(C_1)\times\dots\times B_{n_k}(C_k)\times B_{n'_1}(\mathbf{P}^1)\times\dots\times B_{n'_l}(\mathbf{P}^1)\right)$$

in which the inner sum contains one copy for each choice of pairing all but n of the base points, leading to a stable curve by clutching.

Note that each point of X defines k different n_i-pointed stable curves of genus g_i (n_i+1-pointed when $g_i=1$) and l different n'_i+3 pointed stable curves of genus 0. Identifying all but n of these in pairs in such a way that the result is connected and taking into account the definition of m, one sees that this gives an n-pointed stable curve of genus g.

Note that we get in this way all stable curves C' such that the normalizations of C and C' minus their rational components are isomorphic. By Torelli's theorem, we get all C' such that the abelian part of the jacobians of C and C' are isomorphic.

Clutching the points in the various configurations corresponding to each component of X, we get a morphism $\beta\colon X \to M_{g,n}$. Let $q_x\colon t^{-1}(t(x)) \to M_{g,n}$ be the restriction of q and let $X' = \text{Isom}(\beta, q_x)$.

Then we get a diagram

$$\begin{array}{ccc} X' & \xrightarrow{p_1} & t^{-1}(t(x)) \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle q_x} \\ X & \xrightarrow{\beta} & M_{g,n}, \end{array}$$

where p_2 is finite and p_1 is finite and surjective.

Let δ be the invertible sheaf on X defined on each component as

$$p_1^*\tilde{\delta}_{g_1,n_1}\otimes\ldots\otimes p_k^*\tilde{\delta}_{g_{n_k},n_k}\otimes p_{k+1}^*\tilde{\delta}_{0,n_1'}\otimes\ldots\otimes p_{k+1}^*\tilde{\delta}_{0,n_l'}\,.$$

Then by Theorem 4.2 we have

$$p_1^*(\tilde{\delta}_{g,n}|_{t^{-1}t(x)}) = p_2^*(\tilde{\delta})\,.$$

Since p_2 is finite and p_1 is finite surjective, it follows from Theorem 5.5 that $\tilde{\delta}_{g,n}^{-1}$ is ample on $t^{-1}(t(x)$ hence there is an $m>0$ such that $\lambda^{\otimes m}\otimes\tilde{\delta}^{-1}$ is ample on $X_{g,n}$.

This proves Theorem 6.1.

REFERENCES

1. S. Arakelov, *Families of algebraic curves with fixed degeneracies*, Izv. Akad. Nauk. SSSR Ser. Mat. 35 (1971), 1269–1293.
2. N. Bourbaki, *Éléments de Mathématique. Algèbre Commutative* (Act. Sci. Ind. 1293), Hermann, Paris, 1961.
3. P. Deligne and D. Mumford, *The irreducibility of the space of curves of a given genus*, Inst. Hautes Études Sci. Publ. Math. 36 (1969), 75–109.
4. P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques* in *Modular functions of one variables* II, eds. W. Kuyk and P. Deligne (Proc. Internat. Summer School, Univ. Antwerp, 1972) (Lecture Notes in Math. 349), pp. 143–316. Springer-Verlag, Berlin - Heidelberg - New York, 1973.
5. A Grothendieck and J. Dieudonné, *Éléments de géométrie algèbrique*, Inst. Hautes Études Sci. Publ. Math. 11 (1961), 81–148.
6. R. Hartshorne, *Residues and duality* (Lecture Notes in Math. 20), Springer-Verlag, Berlin - Heidelberg - New York, 1966.
7. F. F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves* I Math. Scand. Scand. 39 (1976), 19–55.
8. S. Lichtenbaum and M. Schlessinger, *The cotangent complex of a morphism*, Trans. Amer. Soc. 128 (1967), 41–70.
9. D. Mumford, *Geometric invariant theory* (Ergebnisse Math. 34) Springer-Verlag, Berlin - Heidelberg - New York, 1965.
10. D. Mumford, *Introduction to Algebraic Geometry*, Lecture Notes, Harvard University.
11. D. Mumford, *Picard groups of moduli problems*, in *Arithmetical Algebraic Geometry*, ed. O. F. G. Schilling (Proc. Conf. Purdue Univ. 1963) pp. 33–81. Harper and Row Publ., New York, 1965.
12. D. Mumford, *On the equations defining Abelian varieties* III, Invent. Math. 3 (1967), 215–244.
13. D. Mumford, *Varieties defined by quadratic equations. With an appendix by G. Kempf*, in *Questions on algebraic varieties*, (C.I.M.E., III Ciclo, Varenna, 1969) pp. 29–100, Edizioni Cremonese, Roma, 1970.
14. D. Mumford, *Stability of projective varieties*, Enseign. Math. 23 (1977), 39–110.
15. C. S. Seshadri, *Quotient spaces modulo reductive algebraic groups*, Ann. of Math. 95 (1972), 511–556.

MATEMATISK INSTITUTT
UNIVERSITETET I TRONDHEIM
N.T.H.
7034 TRONDHEIM
NORWAY