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HILBERT'S FOURTEENTH PROBLEM - THE FINITE GENERATION
OF SUBRINGS SUCH AS RINGS OF INVARIANTS

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1. INTRODUCTION

The precise statement of the problem is this:

Let k be a field

Let K be a subfield of the rational functions in n -variables over k :

$$k \subset K \subset k(x_1, \dots, x_n).$$

(n.b. all such K are automatically finitely generated over k as fields)

Is the ring:

$$K \cap k[x_1, \dots, x_n]$$

finitely generated over k ?

The motivation for this question came from its affirmative answer by Hilbert and others in certain very interesting cases: e.g., say $\text{char}(k) = 0$, suppose $G = \text{SL}(m)$ is acting linearly on k^n , and suppose K is defined as the field of G -invariant rational functions. Then $K \cap k[x_1, \dots, x_n]$ is just the ring of G -invariant polynomials and Hilbert had proven that this was finitely generated. Unfortunately, it turns out that the answer is, in general, NO: $K \cap k[x_1, \dots, x_n]$ may require an infinite number of generators. A beautiful counter-example was discovered by M. Nagata [13] in 1959. It would appear that after Hilbert's discovery of the extremely general finiteness principle on which his proof in the $\text{SL}(m)$ -invariant case was based, namely "Hilbert's basis theorem" on the finite generation of all ideals in $k[x_1, \dots, x_n]$, Hilbert was overly optimistic about finiteness results in other algebraic contexts. However my belief is that it was not at all a blind alley: that on the one hand its failure reveals some very significant and far-reaching subtleties in the category of varieties; and that the search for cases where it and

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related geometric questions are correct is a very important area of research in algebraic geometry. In fact, my guess is that it was Hilbert's idea to take a question that heretofore had been considered only in the narrow context of invariant theory and thrust it out into a much broader context where it invited geometric analysis and where its success or failure had to have far-reaching algebro-geometric significance. We will discuss the problem in 3 sections - first in the case of invariant theory where K is the field of G -invariant functions for some G , second in its geometric form involving linear systems formulated and analyzed first by Zariski [23], and thirdly as a special case of the general problem of forming quotient spaces of varieties by algebraic equivalence relations.

2. INVARIANT THEORY

Hilbert's proof of the finiteness when K is the field of G -invariant functions, $G = SL(m)$, $\text{char}(k) = 0$ is so very elegant and simple that it should really be part of every mathematician's bag of tricks. So I would like to begin by running through this marvelous proof: to begin with, it is known that if V is any finite-dimensional polynomial representation of $SL(m)$ in $\text{char. } 0$, then V is completely reducible. In particular, there is a unique decomposition:

$$V = V^G \oplus V_1$$

where V^G is the subspace of invariant vectors and V_1 is a G -stable subspace containing no invariants. Let ρ_V be the projection of V onto V^G with kernel V_1 . Next, let $R = k[X_1, \dots, X_n]$, and let $R^G \subset R$ be the ring of invariants. R and R^G are graded rings, i.e.,

$$R = \bigoplus R_k, \quad R_k = \text{vector space of homogeneous degree } k \text{ polynomials}$$

$$\text{and } R^G = \bigoplus R_k^G, \quad R_k^G = G\text{-invariants in } R_k.$$

Thus the operators

$$\rho_{R_k}: R_k \longrightarrow R_k^G$$

patch together into a projection

$$\rho_R: R \longrightarrow R^G.$$

A simple argument using the uniqueness of ρ shows that ρ_R satisfies the identity:

$$\rho_R(fg) = f\rho_R(g), \quad f \in R^G, g \in R.$$

Now we let

$$R_+^G = \bigoplus_{k>0} R_k^G$$

and let $I = R_+^G \cdot R$ be the ideal in R generated by all invariants of positive degree. Hilbert's Basis Theorem asserts that

$$I = \sum_{i=1}^N f_i \cdot R$$

for some $f_1, \dots, f_N \in I$; we can assume if we like that each f_i is in fact in R^G and homogeneous of some degree d_i . Then Hilbert asserts that these f_i generate R^G as ring! He proves this by induction on degree: choose $g \in R_n^G$ and assume all $h \in R_{n'}^G$, for $n' < n$ are polynomials in the f_i 's. Then $g \in I$, hence there is an expression:

$$g = \sum_{i=1}^N a_i f_i, \quad a_i \in R_{n-d_i}.$$

Apply ρ_R :

$$g = \rho_R g = \sum \rho_R(a_i f_i) = \sum (\rho_R a_i) f_i.$$

Then $\rho_R a_i \in R_{n-d_i}^G$ which is a polynomial in the f_i 's by induction, hence so is g !

What was the history of invariant theory after Hilbert? First of all, Hilbert did not give the above abstract description of ρ , but rather an explicit construction of ρ , called "Cayley's Ω -process" in which ρ appears in the Universal enveloping algebra of $\mathfrak{sl}(m)$. As mentioned in Hilbert's problem itself, A. Hurwitz [7] had already observed and H. Weyl was later to use effectively the fact that if $k = \mathbb{C}$, (and we can reduce easily any $\text{char. } 0$ case to the case $k = \mathbb{C}$), then

$$\rho x = \int_{g \in SU(m)} g^*(x) \cdot dg$$

$SU(m)$ = special unitary group

dg = Haar measure

Via the fact that any reductive algebraic group over \mathbb{C} has a Zariski-dense compact subgroup, this gives us an explicit construction for the projection ρ for any such groups, hence a proof of finiteness. The final step - to observe that no explicit formula for ρ is needed but one merely must know the complete

reducibility of all finite-dimensional representations to construct ρ abstractly - was taken by M. Schiffer in 1933 (unpublished; it appeared in H. Weyl's "Classical Groups" [22], Supplement C).

In char. p , no semi-simple group has the property that all its representations are completely reducible. For instance, think of $SL(2)$ acting on the 3-dimensional space of quadratic forms

$$V = k \cdot x^2 + k \cdot xy + k \cdot y^2.$$

In char. 2, $k \cdot x^2 + k \cdot y^2$ is an invariant subspace with no complement. Therefore the Schiffer-Hilbert method breaks down. However, very recently, W. Haboush [25] has succeeded in proving the following Theorem which I conjectured in [9]:

THEOREM: If a semi-simple (or even reductive) algebraic group G acts on a vector space V and leaves fixed a vector $v \in V$, there is a polynomial function f on V such that:

- i) $f(v) \neq 0$
- ii) f is G -invariant.

In char. 0, f exists and may be taken linear by complete reducibility. Seshadri [17] had previously proven that such f 's exist when $G = SL(2)$. Nagata [14] has proven that if G has the property of the Theorem (this is sometimes stated as " G is semi-reductive"), then the ring of G -invariants* is finitely generated, i.e., whenever G acts linearly on $kx_1 + \dots + kx_n$, then $k[x_1, \dots, x_n]^G$ is finitely generated. Therefore, it follows that the ring of invariants is finitely generated for G reductive.

* We have not made precise before whether by G -invariants we meant polynomials $f(x_1, \dots, x_n)$ which were identically invariant, i.e.,

$$f(g(x)) - f(x) \equiv 0 \text{ as function of } g,$$

or f 's which were invariant separately under every $g \in G(k)$ (the k -rational points of G). If k is infinite, $G(k)$ is Zariski-dense in G and there is no difference between these 2 concepts. But if k is finite there is a difference: in this case $G(k)$ is finite and I wish G -invariant to mean identically invariant.

For groups G which are not semi-simple or reductive (i.e., which have a "unipotent radical"), very little is known even in char. 0 about finiteness of the ring of invariants. I know of only 2 results --

- a) Weitzenbock [21] (cf. Also [16]) proved $k[x_1, \dots, x_n]^G$ finitely generated if $G = \mathbb{G}_a$ (i.e., $\mathbb{G}_a =$ the additive group of the ground field),
- b) Nagata's counter-example [13] is a non-finitely generated ring $k[x_1, \dots, x_n]^G$ where G is commutative, but G is a product of many groups \mathbb{G}_a and many groups \mathbb{G}_m (here $\mathbb{G}_m =$ the multiplicative group of the ground field*).

3. ZARISKI'S FORMULATION WITH LINEAR SYSTEMS

We recall that if X is a non-singular projective variety (or more generally if X is normal) and D is a positive divisor on X (i.e., $D = \sum n_i E_i$, $E_i \subset X$ a subvariety of codimension 1 and $n_i \geq 0$), then we define:

$$\mathcal{L}(D) = \left\{ \begin{array}{l} \text{vector space of rational functions } f \text{ on } X \\ \text{with poles bounded by } D, \text{ i.e., } \forall E \subset X \text{ of} \\ \text{codimension 1,} \\ \text{ord}_E f \geq -(\text{mult. of } E \text{ in } D). \end{array} \right\}$$

(Either $\mathcal{L}(D)$ or the family of divisors that occurs as the zeroes of the functions $f \in \mathcal{L}(D)$ is called a linear system on X .) Zariski introduced the 2 rings:

$$R(D) = \bigcup_{n=0}^{\infty} \mathcal{L}(nD) = \left\{ \begin{array}{l} \text{ring of rational functions } f \text{ with} \\ \text{poles of any order but only on } D \end{array} \right\}$$

$$R^*(D) = \bigoplus_{n=0}^{\infty} \mathcal{L}(nD).$$

* In concrete terms, a representation of \mathbb{G}_a^n is a commutative group of matrices all of the form

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

in a suitable basis of $kx_1 + \dots + kx_n$. A representation of \mathbb{G}_m^n is a commutative group of diagonal matrices

$$\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

in a suitable basis of $kx_1 + \dots + kx_n$.

The ring $R^*(D)$, though apparently much bigger than $R(D)$, is easily shown to be isomorphic to $R(D_1)$ for a suitable divisor D_1 on a variety X_1 which is a \mathbb{P}^1 -bundle over the variety X you start with. So the class of rings $R^*(D)$ is really a subset of the class of rings $R(D)$. More generally, for any divisors* D_1, \dots, D_k , we can define a k -times graded ring:

$$R^*(D_1, \dots, D_k) = \bigoplus_{n_1=0}^{\infty} \cdots \bigoplus_{n_k=0}^{\infty} \mathcal{L}(\sum_1 D_i)$$

and this is also isomorphic to $R(D_1)$ for a suitable D_1 on an X_1 (which is now a \mathbb{P}^k -bundle over X .) In his penetrating article [23], Zariski showed that Hilbert's rings $K \cap k[x_1, \dots, x_n]$ were isomorphic to rings of the form $R(D)$ for a suitable X and D ; asked more generally whether all the rings $R(D)$ might not be finitely generated; and proved $R(D)$ finitely generated if $\dim X = 1$ or 2 . I want to outline the procedure for finding X and D such that:

$$K \cap k[x_1, \dots, x_n] \cong R(D).$$

First of all, X is to be a suitable projective variety with function field K^{**} . For any such X , the inclusion of fields

$$K \subset k(X_1, \dots, X_n)$$

defines a "rational map"

$$\pi: \mathbb{P}^n \longrightarrow X$$

i.e., π is a many-valued map whose graph in $\mathbb{P}^n \times X$ is a subvariety and which is single-valued on a Zariski-open subset $U \subset \mathbb{P}^n$. Let $r = \dim X = \text{tr.d.}_K K$. Then if X is chosen "sufficiently blown up", one can make π^{-1} nice in the sense:

$$\forall x \in X, \text{ the full inverse image } W_x = \pi^{-1}[x] \\ \text{has dimension } n-r.$$

Roughly speaking, we have a fibration of \mathbb{P}^n by $(n-r)$ -dimensional algebraic sets W_x such that K is the field of rational functions constant on each

* If D has some negative coefficients, an $f \in \mathcal{L}(D)$ should have corresponding zeroes of order at least that coefficient.

** If resolution is known for this dimension and characteristic one would take X non-singular; if not, one takes X to be normal and $R(D)$ is defined as before.

W_x , i.e., invariant generically under the equivalence relation defined by belonging to the same W_x . Of course, these W_x 's may become singular and in general will meet at certain "bad" points of \mathbb{P}^n , namely where the map π is not single-valued. Now let D_1, \dots, D_k be the subvarieties of X of codimension 1 such that $\pi^{-1}[D_i] \subset (\text{the hyperplane at } \omega, \mathbb{P}^n - \mathbb{A}^n)$. Then for all rational functions f on X , f has poles only on $\bigcup D_i$ if and only if $f \cdot \pi$ has poles only at ω , hence

$$K \cap k[x_1, \dots, x_n] = R(\sum D_i).$$

Unfortunately, it was precisely by focusing so clearly the divisor-theoretic content of Hilbert's 14th problem that Zariski cleared the path to counter-examples. The history is this -

- i) Rees [15] in 1958 found a 3-dimensional X and a D with $R(D)$ infinitely generated. His X was birational to $\mathbb{P}^2 \times E$ (E an elliptic curve).
- ii) Nagata [13] in 1959 found that for suitable points $P_1, \dots, P_r \in \mathbb{P}^2$, if X is the surface obtained by blowing up each P_i into a rational curve E_i , then

$$R^*(\mathcal{L}, -\sum_1^r E_i) \quad (\mathcal{L} \text{ a line not through any } P_i)$$

is infinitely generated; and that this ring was a ring of invariants $k[x_1, \dots, x_{2r}]^G$ as mentioned in §1.

- iii) Zariski [24] in 1962 returned to the problem and pursuing some constructions which had been considered in different contexts by Grauert [3] and Nagata [12], found that it was not at all uncommon for $R^*(D)$ to be infinitely generated when $\dim X = 2$ (hence for $R(D)$ to be infinitely generated when $\dim X = 3$).

I would like to describe the situation Zariski looked at because it is a very useful source of counter-examples to several problems and illustrates some basic facts about the category of algebraic varieties. Suppose you have

- a) a non-singular surface X ,
- b) a curve $E \subset X$ of genus $g > 0$ such that
 - i) $(E^2) < 0$ (i.e., the normal bundle to E in X has negative curvature)
 - ii) $\text{Pic } X \longrightarrow \text{Pic } E$ is injective (i.e., if a line bundle L on X is trivial on E , then it is trivial on X).

Such a situation is not hard to obtain: start with any sufficiently general hypersurface section H_0 on X_0 and blow up enough generic points on H_0 to make its normal bundle negative. First of all, here is what Grauert observed about this situation: analytically, E can be blown down, i.e., there is a normal analytic surface X_1 and $\pi: X \rightarrow X_1$ mapping E to a point x but bijective elsewhere. But X_1 is not a variety: if it were, x would have an affine neighborhood $U \subset X$, hence $C = X_1 - U$ would be a curve not containing x , hence $\pi^{-1}(C)$ would be a curve on X disjoint from E , hence "twisting by $\pi^{-1}(C)$ " we get a line bundle $\mathcal{O}_X(\pi^{-1}C)$ trivial on E but not trivial on X : contradiction.

Zariski did this: let $H \subset X$ be a hyperplane section, let $a = (H.E)$, (the intersection number of H and E), let $(E^2) = -b$. Then he showed

$$R^*(bH + aE)$$

is not finitely generated. The reason is this — look for functions f on X with poles $kbH + kaE$, some $k \geq 1$. If at some $P \in E$, $x = 0$ is the local equation of E , expand f :

$$f = \frac{g_0}{x^k} + \frac{g_1}{x^{k-1}} + \dots$$

and consider the function g_0 on E . Suitably interpreting what g_0 means, g_0 comes out as a section of a line bundle on E ; in fact the line bundle $\mathcal{O}_X(kbH+kaE)$ on X restricted to E . This has degree 0 but by assumption (bii) is not trivial. So it has no sections and $g_0 \equiv 0$, i.e., f can have at most poles of type $kbH + (ka-1)E$. On the other hand, Zariski showed that there is a fixed k_0 such that for all k , there are functions f with poles of type $kbH + \max(0, ka-k_0)E$. To see the implications of this, say for instance that $k_0 = 1$: then for all k , let

$$f_k \in \mathcal{L}(kbH + kaE)$$

have a pole $kbH + (ka-1)E$. Then for all k ,

$$f_k \notin \left(\text{subring of } R^*(bH + aE) \text{ generated by } 1, f_1, \dots, f_{k-1} \right)$$

since every function in the degree k piece of the subring has a pole of at most $kbH + (ka-2)E$: Taking into account that $R^*(bH+aE)$ is graded, it requires at least one generator in each degree, hence is not finitely generated!

Are there any positive results asserting that $R(D)$ and $R^*(D)$ are finitely generated in some cases? When $\dim X = 2$, Zariski's paper [24] gives a thorough analysis of when $R^*(D)$ is finitely generated. In

higher dimensions, at the moment, the best results are numerical criteria on D implying that D is ample, which in turn implies very quickly that both $R(D)$ and $R^*(D)$ are finitely generated. Here "D ample" means that for some $n \geq 1$, nD is a hyperplane section of X in a suitable projective embedding. These criteria use intersection numbers and are as follows:

- 1.) Nakai's Criterion: if for every subvariety $Y \subset X$, $(Y, D^r) > 0$ where $r = \dim Y$, then D is ample.
- 2.) Seshadri's Criterion: if there is an $\epsilon > 0$ such that for every curve $C \subset X$, $(C, D) > \epsilon \cdot \left[\max_{P \in C} (\text{mult. of } P \text{ on } C) \right]$, then D is ample.

For proofs, see Hartshorne's book [4], Chapter I.

4. QUOTIENT SPACES BY ALGEBRAIC EQUIVALENCE RELATIONS

Another way of generalizing Hilbert's problem is to ask: given a variety X , and

- (1) a finite union of subvarieties of $X \times X$
- (2) set-theoretically, an equivalence relation on X

when is there another variety Y and a surjective morphism $f: X \rightarrow Y$ such that

$$R = \{ (x_1, x_2) \mid f(x_1) = f(x_2) \}?$$

For short, we speak* of Y as X/R . Two cases of particular interest are i) a group G acts on X and $R = \{ (x, gx) \mid x \in X, g \in G \}$, and ii) E is a subvariety of X to be "blown down" and $R = (\text{diagonal}) \cup (E \times E)$. In Hilbert's case, $X = \mathbb{A}^n$ (affine space) but one is given R only generically by specifying the subfield K (i.e., $R = \{ (x_1, x_2) \mid x_1, x_2 \text{ belong to some } W_x \}$ in the notation of §2); Hilbert's problem can be broken up into 2 steps — first extend this equivalence relation nicely to one on all of \mathbb{A}^n , second prove \mathbb{A}^n/R exists and is an affine variety, in which case Hilbert's ring $k[X_1, \dots, X_n] \cap K$ is just the affine coordinate ring of \mathbb{A}^n/R .

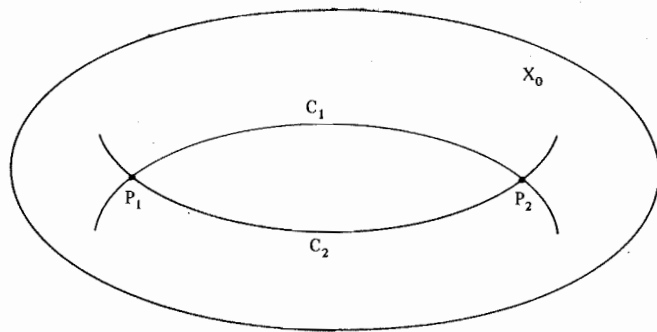
Returning to the general case, it is always possible to find a Zariski open subset $U \subset X$ stable under R such that U/R exists (this may be proven for instance using Chow coordinates of the equivalence classes).

*The requirements do not determine Y uniquely, but in all cases that arise, there are natural extra conditions one imposes that make Y unique if it exists at all.

Equivalently the field of rational functions K on X/R is easy to construct and then any model Y of K realizes X/R on some sufficiently small Zariski-open U in X . The real problem is a birational one of finding a Y which works everywhere. However, as in Zariski's divisor formulation of the problem, one is confronted straightway by a raft of counter-examples:

- 1.) Grauert's example [3] described in §3 of an $E \subset X$, where $\dim X = 2$, $(E^2) < 0^*$ and E can be blown down analytically but not algebraically,
- 2.) Hironaka [6] found a beautiful example of a complete (though non-projective) variety X on which $\mathbb{Z}/2\mathbb{Z}$ acts freely, but $X/(\mathbb{Z}/2\mathbb{Z})$ is not a variety at all,
- 3.) Nagata and I found ([9], p. 83) examples of $\mathrm{PGL}(n)$ acting freely on quasi-projective varieties X such that the orbit space $X/\mathrm{PGL}(n)$ is not a variety.

In rough outline, here is the idea of Hironaka: take a 3-dimensional projective variety X_0 with 2 curves C_1, C_2 in it crossing transversely at 2 points P_1, P_2 and with $\mathbb{Z}/2\mathbb{Z}$ acting on X_0 interchanging the C 's and the P 's:



We then blow up C_1 and C_2 in X_0 to obtain X . However, where the C 's cross, we must specify the order in which the C 's are blown up — so at P_1 , we blow up C_1 first, then in the resulting variety we blow up C_2 ; at P_2 ,

* If $(E^2) \geq 0$, then E cannot be blown down even analytically so of course one cannot construct X/R , $R = (\mathrm{diag})U(E \times E)$, algebraically. For general equivalence relations R one asks first that R have some reasonable properties ensuring that X/R exists in the analytic context.

we blow up C_2 first, then in the result, we blow up C_1 . Then $\mathbb{Z}/2\mathbb{Z}$ still acts on X . However, say $Y = X \bmod (\mathbb{Z}/2\mathbb{Z})$ were a variety. Since X_0 is projective, it can be shown (cf. e.g. [10], p. 111) that $X_0 \bmod (\mathbb{Z}/2\mathbb{Z})$ is a variety Y_0 . In Y_0 , C_1 and C_2 have the same image D and P_1 and P_2 have the same image Q . Then Y would be obtained from Y_0 by blowing up D ; but at Q , the 2 branches of D must be blown up in a definite order. As D is an irreducible curve, these 2 branches cannot be distinguished by rational functions! This turns out to mean that Y in fact does not exist in the category of algebraic varieties.

Confronted with these counter-examples, people have had 2 reactions: a) find criteria for X/R to exist as a variety, or b) instead enlarge the category you are working in. The ploy (b) was most notably successful in Weil's hands in his 2nd proof of the Riemann hypothesis for curves over finite fields [20]. His idea here required the construction of the Jacobian variety of such a curve. At that time, only affine and projective varieties had been considered. Weil invented the category of what he called abstract varieties — now called simply varieties — and constructed the Jacobian as one of these. Subsequently he and Chow independently showed that the Jacobian was actually a projective variety; however, at the time, Weil instead developed the theory of "abstract" varieties far enough to by-pass the question of projectivity and prove the Riemann Hypothesis using these Jacobians. Matsusaka [8] made an initial attempt at enlarging the category even further. However it was M. Artin who found, I believe, the most natural enlargement: he calls these new objects algebraic spaces (cf. [1] and [2]). One way to define these is simply to introduce them as formal quotients X/R , where X is a scheme and R is an étale equivalence relation, i.e., $R \subset X \times X$ is a subscheme such that the projection

$$p: R \longrightarrow X$$

is étale — essentially makes R into an unramified covering over X . Artin then went on to show that the category of algebraic spaces is closed under apparently all "reasonable" further quotient operations $X \longrightarrow X/R$. For details we refer the reader to his papers, which make algebraic spaces into a very effective and powerful tool.

Still you may have a sentimental attachment to familiar old varieties. It would appear especially that projective varieties play such a central technical role in algebraic geometry that it may be virtually impossible to eliminate their use even if you wanted to. In any case, it is very interesting to prove, when possible, that X/R is actually a projective variety. I would like to state one such result concerning orbit spaces:

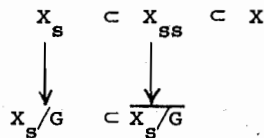
Suppose:

- X = a projective variety over k
- G = a semi-simple (or more generally reductive) algebraic group over k , acting on X
- $X \subset \mathbb{P}^n$: an embedding such that the action of G on X extends to an action on \mathbb{P}^n .

Then there are canonical open subsets

- $X_s \subset X$: the set of "stable" points
- $X_{ss} \subset X$: the set of "semi-stable" points

such that X_s, X_{ss} are G -invariants, and there is a diagram:



where $\overline{X_s/G}$ is a projective variety, X_s/G is an open subset of $\overline{X_s/G}$, $X_s \rightarrow X_s/G$ makes X_s/G into an orbit space by G , and $X_{ss} \rightarrow X_{ss}/G$ makes $\overline{X_{ss}/G}$ into the quotient of X_{ss} by a cruder equivalence relation \sim defined by:

$$x \sim y \text{ if } \overline{O^G(x)} \cap \overline{O^G(y)} \cap X_{ss} \neq \emptyset$$

(here $O^G(x)$ = G -orbit of x). This theorem is proven in my book [9] when $\text{char.} = 0$, and the part about X_s is proven by Seshadri [18] when $\text{char.} = p$. This X_{ss} -part in $\text{char.} p$ follows from the recent results of Haboush[25] discussed in §1. See [11] for examples and a discussion of this result. This result has proven very useful for proving that various moduli spaces are quasi-projective varieties (and not "just" algebraic spaces).

The above theorem is in fact a natural extension of Hilbert's own ideas about the ring of invariants, especially as developed in his last big paper on the subject, "Über die vollen Invariantensystemen" [5]. To indicate this, let me define X_{ss} . Assume for simplicity that there is actually a representation of G on k^{n+1} which induces the action of G on the \mathbb{P}^n ambient to X . We then make the definition:

If $x \in X$, then

$$x \in X_{ss} \iff \left\{ \begin{array}{l} \forall \text{ homomorphisms } \lambda: \mathbb{G}_m \rightarrow G, \text{ let } x(\lambda) = \lim_{t \rightarrow 0} \lambda(t)(x). \\ \text{Let } x(\lambda)^* \in k^{n+1} \text{ be homogeneous coordinates for } x(\lambda), \\ \text{so that } \lambda(t)[x(\lambda)^*] = t^r \cdot [x(\lambda)^*] \text{ for some } r \in \mathbb{Z}. \\ \text{We ask } r < 0 \text{ for all } \lambda. \end{array} \right.$$

Now let R be the homogeneous coordinate ring of X , and let R_+^G be the invariants with no constant term. Then we have Hilbert's result:

$$X - X_{ss} = V(R_+^G).$$

Contrary to the usual credo that Hilbert eliminated the interest in studying special cases in invariant theory, my belief is that some of the most challenging problems still open in invariant theory concern special cases. I would like to raise two rather broad questions:

PROBLEM Let S be the parameter space for a family $\{X_s \mid s \in S\}$ of non-singular projectively normal subvarieties $X_s \subset \mathbb{P}^n$. Assume $\text{PGL}(n+1)$ acts on S so that for all $g \in \text{PGL}(n+1)$, $X_{g(s)} = g(X_s)$. Assume this action is proper. Then is the quotient $S/\text{PGL}(n+1)$ always a variety?

PROBLEM Now that we have computers, is there a practical way to actually find generators of such classical rings of invariants as those of a binary or ternary n -ic (i.e., $\text{SL}(2)$ or $\text{SL}(3)$ acting on the space of homogeneous degree n polynomials in 2 or 3 variables)? After an extraordinary effort, Shioda [19] only recently found these for binary octics.

Added in proof: Independent of W. Haboush's work, E. Formanek and C. Procesi have recently in a preprint entitled "Mumford's Conjecture for the general linear group" given another very beautiful proof of the semi-reductivity of $\text{GL}(n)$ and $\text{SL}(n)$.

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PROBLEM 15. RIGOROUS FOUNDATION OF
SCHUBERT'S ENUMERATIVE CALCULUS

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ABSTRACT

Schubert's calculus was first interpreted and rigorously justified by van der Waerden (1929) by means of the calculus of algebraic cohomology classes developed by Lefschetz. Entirely algebraic treatments of the foundations of Schubert's calculus have become possible through the jumbled efforts of a great many mathematicians, who have contributed to the constructions of algebraic intersection rings to replace the topological cohomology ring. However, this work does not constitute a complete solution to Hilbert's fifteenth problem; for, in the statement and explanation of the problem, Hilbert makes clear his interest in the effective computability and actual verification of the geometrical numbers of classical enumerative geometry. Due primarily to Schubert (1886), the classical method of obtaining certain numbers, like the number $(1! 2! \dots d! h!)/(n-d)! \dots n!$ of d -planes in n -space meeting $h = (d+1)(n-d)$ general $(n-d-1)$ -planes, was vindicated topologically by Ehresmann (1934) and algebraically by Hodge (1941, 1942) by means of an explicit determination of the cohomology ring, and respectively, of an equivalent algebraic intersection ring, on the Grassmann manifold. In the offing, there is the exciting hope of the development in algebraic geometry of a general enumerative theory of singularities of mappings, a theory of Thom polynomials, which will, among other things, unify and justify the classical work dealing with prescribed conditions of intersection and contact imposed on linear spaces. Classically, conditions of intersection and contact were imposed on other figures as well. For example, Schubert (1879), in his book, obtains the number 666, 841, 048 of quadric surfaces tangent to 9 given quadric surfaces in space, and the number 5, 819, 539, 783, 680 of twisted cubic space curves tangent to 12 given quadric surfaces. Today, we cannot vouch for the accuracy of these two spectacular numbers, nor do we even know whether Schubert's method is basically sound.

AMS (MOS) subject classifications (1970). Primary 14-03, 14-01, 14N10; Secondary 14C15, 14M15, 14E99, 14M99.

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