



## Pathologies IV

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## PATHOLOGIES IV.

By DAVID MUMFORD.

In this note I would like to use the beautifully simple method introduced by Tony Iarrobino [1]—when he proved that there are 0-dimensional subschemes of  $\mathbf{P}^3$  which are not specializations of reduced subschemes—to prove here that there are also reduced and irreducible complete curves which are not specializations of non-singular curves. Since there are no global obstructions in deforming reduced curves, this also shows that there are complete reduced 1-dimensional local rings with no flat deformation which is generically smooth.

Start with a complete non-singular curve  $C$  of genus  $g$  with no automorphisms over an algebraically closed ground field  $k$ . Choose a point  $x \in C$  and a large even integer  $\nu$ . Note that if  $V$  is any  $k$ -vector space where

$$m_{x,C}^{2\nu} \subset V \subset m_{x,C}^{\nu}$$

then  $k + V$  is a subring of  $\mathcal{O}_{x,C}$ . For each such  $V$ , define a new curve:

$$\pi: C \rightarrow C(V)$$

by:

- (a)  $\pi$  is a bijection, and an isomorphism

$$\text{res } \pi: C - \{x\} \xrightarrow{\approx} C(V) - \{\pi x\}.$$

- (b)  $\mathcal{O}_{\pi x, C(V)} = k + V$ .

Note that if  $V_1, V_2$  are two such vector spaces, then

$$C(V_1) \approx C(V_2) \Rightarrow V_1 = V_2.$$

(In fact,  $C$  is the normalization of each  $C(V)$ ; hence any  $o: C(V_1) \xrightarrow{\approx} C(V_2)$  lifts to  $o': C \rightarrow C$  which must be the identity, hence  $k + V_1 = \mathcal{O}_{\pi(x), C(V_1)} = \mathcal{O}_{\pi(x), C(V_2)} = k + V_2$ .) Moreover, the curves  $C(V)$  can all be fitted together into

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a family if we fix the integer  $\dim_k V/m_{x,C}^{2\nu}$ : for all  $k, 0 \leq k \leq \nu$ ,

let  $G =$  Grassmanian of  $k$ -dimensional subspaces of  $m_{x,C}^\nu/m_{x,C}^{2\nu}$ ,

let  $\mathcal{V} \subset (m_{x,C}^\nu/m_{x,C}^{2\nu}) \otimes_k \mathcal{O}_G$  be the universal family,

let  $C(\mathcal{V})$  be the scheme equal to  $C \times G$  as topological space, with structure sheaf defined by:

$$\begin{array}{ccc}
 \mathcal{O}_{C \times G} & \xrightarrow{\alpha} & (\mathcal{O}_{x,C}/m_{x,C}^{2\nu}) \otimes_k \mathcal{O}_G \longrightarrow 0 \\
 & & \cup \\
 \cup & & [k + (m_{x,C}^\nu/m_{x,C}^{2\nu})] \otimes_k \mathcal{O}_G \\
 & & \cup \\
 \alpha^{-1}(\mathcal{V} + \mathcal{O}_G) & \dashrightarrow & \mathcal{V} + \mathcal{O}_G \\
 \parallel \text{def.} & & \\
 \mathcal{O}_{C(\mathcal{V})} & & 
 \end{array}$$

Since  $[(\mathcal{O}_{x,C}/m_{x,C}^{2\nu}) \otimes_k \mathcal{O}_G] / \mathcal{V}$  is a locally free  $\mathcal{O}_G$ -sheaf,  $\mathcal{O}_{C(\mathcal{V})}$  is flat over  $\mathcal{O}_G$ , i.e.,  $C(\mathcal{V})$  is flat over  $G$ .

Now choose  $k = \nu/2$  and calculate:

- (i)  $\dim G = k(\nu - k) = \nu^2/4$
- (ii)  $p_a(C(V)) = g + \dim_k [\mathcal{O}_{x,C} / \mathcal{O}_{\pi(x), C(V)}]$   
 $= g + \left(\frac{3\nu}{2} - 1\right)$

Therefore if  $\nu \gg 0, \dim G \geq 3p_a(C(V)) - 3!$  I claim that this implies that almost all the curves  $C(V)$  are not specializations of non-singular curves, because of:

LEMMA . Let  $p: \mathcal{C} \rightarrow S$  be a flat and proper family of reduced and irreducible singular curves  $C_s = p^{-1}(s)$  such that

- (a)  $\forall s \in S, \{s' | C_{s'} \approx C_s\}$  is finite
- (b)  $p_a(C_s) \geq 2, S$  is irreducible and  $\dim S \geq 3p_a(C_s) - 3,$

then almost all curves  $C_s$  are not specializations of non-singular curves.

Proof. If the conclusion is false, then after replacing  $S$  by a Zariski open subset we can extend the family  $\mathcal{C}/S$  like this:

$$\begin{array}{ccc}
 \mathcal{C} & \longrightarrow & \mathcal{C}^* \\
 \downarrow & & \downarrow \\
 S & \longrightarrow & S^*
 \end{array}
 \quad ; \quad \begin{array}{l} S^* \text{ irreducible,} \\ \dim S^* = \dim S + 1 \end{array}$$

so that  $\mathcal{C}^*$  is generically smooth over  $S^*$ . In fact  $\mathcal{C}$  will carry a relatively ample  $L$ , so we may use  $p_*L^{\otimes n}$  ( $n \gg 0$ ) to embed  $\mathcal{C}$  in some  $\mathbf{P}^N$ -bundle  $\mathcal{P}$  over  $S$ . Moreover, if a  $C_s$  ( $s \in S_0$ ) is abstractly a specialization of a non-singular curve, so is the embedded curve  $C_s \subset \mathbf{P}^N$ . So take  $S^*$  to be a suitable subvariety of the Hilbert scheme of  $\mathcal{P}$  over  $S$ . Once we have  $\mathcal{C}^*/S^*$ , consider the two induced families:

$$\mathcal{C}_i^* = \mathcal{C}^* \times_{S^*} (S^* \times S^*) \quad (\text{formed via } p_i : S^* \times S^* \rightarrow S^*, i = 1, 2)$$

and the scheme

$$I = \text{Isom}_{S^* \times S^*}(\mathcal{C}_1^*, \mathcal{C}_2^*)$$

whose points over  $(s_1, s_2) \in S^* \times S^*$  are isomorphisms  $o : C_{s_1} \rightarrow C_{s_2}$ . Look at the morphisms:

$$q \begin{array}{c} \downarrow \\ I \\ \downarrow \\ S^* \end{array} \delta \quad \begin{array}{l} q(o : C_{s_1} \rightarrow C_{s_2}) = s_1 \\ \delta(s) = [\text{id.} : C_s \rightarrow C_s] \end{array}$$

Since  $\dim S^* = \dim S + 1 > 3p_a(C_s) - 3$ , whenever  $C_s$  is non-singular, the same non-singular curve must occur in the family  $\mathcal{C}^*/S^*$  infinitely often; thus when  $C_s$  is non-singular, some component of  $q^{-1}(s)$  through  $\delta(s)$  is positive-dimensional. Now by upper semi-continuity of dimensions of fibres of a morphism, it follows that for every  $s$ ,  $q^{-1}(s)$  has a positive-dimensional component through  $\delta(s)$ . Now let  $D_1 = \text{Im}(S \rightarrow S^*)$ ,  $D_2 = \{s \mid C_s \text{ is singular}\}$ ; then  $\overline{D_1}$  is a component of  $D_2$  and let  $D_1^0 = D_1 - (\text{closure of } D_2 - D_1)$ . Choose  $s \in D_1^0$  and consider how  $q^{-1}(s)$  can have a positive-dimensional component  $\gamma$  through  $\delta(s)$ . By (b),  $\text{Aut}(C_s)$  is finite; by (a), there are only finitely many  $s' \in D_1$  with  $C_s \approx C_{s'}$ ; certainly  $C_s \not\approx C_{s'}$  if  $s' \in S^* - D_2$  because  $C_s$  is singular while  $C_{s'}$  is non-singular; and since  $s \notin (\text{closure } D_2 - D_1)$ ,  $\gamma$  cannot lie over  $(s) \times \overline{D_2 - D_1}$  in  $S^* \times S^*$ . Thus there is nowhere for  $\gamma$  to go! Contradiction.

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REFERENCES.

- [1] A. Iarrobino, "Reducibility of the families of 0-dimensional schemes on a variety," *Inv. Math.* 15 (1972), pp. 72-77.