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The self-intersection formula and the 'formule-clef'

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Introduction. We shall consider exclusively algebraic non-singular quasi-projective irreducible varieties over an algebraically closed field. If V is such a variety $\mathscr{A}(V)$ will be the Chow ring of rational equivalence classes of cycles of V and

$$\phi_*: \mathscr{A}(V_1) \to \mathscr{A}(V_2)$$

the group homomorphism defined by any proper morphism $\phi: V_1 \to V_2$. Also

$$\phi^*: \mathscr{A}(V_2) \to \mathscr{A}(V_1)$$

denotes the ring homomorphism defined by ϕ.

Let X, Y be two varieties. Assume that Y is a subvariety of X (by subvariety, we will always mean *closed* subvariety), let $i: Y \to X$ be the inclusion and E the normal bundle of Y in X. Then Grothendieck conjectured the *self-intersection formula*

$$i^* i_*(y) = y c_r(E), \tag{1}$$

for any $y \in \mathscr{A}(Y)$, where $r = \operatorname{codim}_X(Y)$ and $c_r(E)$ is the rth Chern class of E.

Consider the blowing-up diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\ j\ } & X' \\ {\scriptstyle \rho_E}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\ i\ } & X \end{array}$$

where $Y' = P(E)$ is the projective bundle associated to E. The normal bundle of Y' in X' is the tautological bundle $\check{L}_E$ of E and one has the exact sequence

$$0 \to \check{L}_E \to \rho_E^* E \to E^{(1)} \to 0$$

which defines $E^{(1)}$ (4). Then Grothendieck also conjectured:

$$f^* i_*(y) = j_*(\rho^*(y)\, c_{r-1}(E^{(1)})) \tag{2}$$

for any $y \in \mathscr{A}(Y)$, which he called the 'formule-clef' for calculating $\mathscr{A}(X')$.

Our aim is to prove both formulae. Especially (1) looks so innocuous that it is hard to believe it is not false or trivial; for instance they are both well known in singular cohomology when the ground field is $\mathbb{C}$. They were first conjectured by Grothendieck in 1957 ((2), exposé 0), and were proven *modulo torsion* by a very roundabout method in (2), cf. esp. exposé XIV (4·4). They were subsequently used in (6). Their analogues in étale cohomology are apparently to be published in SGA5. Our proof is completely elementary, but requires a good deal of manipulation.

1. *Preliminaries.* The notation will be that of (7) as far as possible. If E is a vector bundle

$$C(t, E) = \sum_{i=0}^{r} c_i(E)t^i,$$

where $r = \operatorname{rank}(E)$, will be its Chern polynomial and $[\lambda C](t, E)$ the polynomial in which an operator λ has been applied to its coefficients. The point of this notation is that if a suitable element x is substituted for t then in $[\lambda C](x, E)$ the operator λ does not apply to the powers of x. The reversed Chern polynomial is defined as

$$\bar{C}(t, E) = t^r C(t^{-1}, E).$$

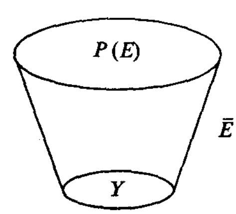
One has

$$[\rho_E^* C](t, E) = C(t, \rho_E^* E) = C(t, E^{(1)}) \,.\, C(t, \check{L}_E)$$

by applying the additivity formula to the exact sequence of the introduction. Let $\xi_E = c_1(L_E)$ so that $C(t, \check{L}_E) = 1 - t\xi_E$. Multiplying by $(1-t\xi_E)^{-1}$ and comparing constant terms, we get:

$$c_{r-1}(E^{(1)}) = \sum_{i=0}^{r-1} \rho_E^* c_i(E) \,.\, \xi_E^{r-1-i}. \tag{1·1}$$

Let $G = E \oplus \mathbf{1}_Y$ where $\mathbf{1}_Y$ is the trivial line bundle over Y. Then $\bar{E} = P(G)$ is the projective closure of E. The canonical inclusions of E and $\mathbf{1}_Y$ in G give two subvarieties of $\bar{E}$ isomorphic to $P(E)$ and Y respectively, which can be called the 'roof' and the 'floor' of $\bar{E}$. We shall identify the roof with $P(E)$ and the floor with Y.



If $\bar{\imath}$: $Y \to \bar{E}$ and $\bar{\jmath}$: $P(E) \to \bar{E}$ are the inclusion maps then

$$\rho_G \bar{\imath} = Id_Y, \quad \rho_G \bar{\jmath} = \rho_E, \tag{1·2}$$

where ρ_G is the projection of $P(G)$. One has

$$\bar{\jmath}_*(1) = \xi_G. \tag{1·3}$$

and

$$\bar{\imath}_*(1) = [\rho_G^* \bar{C}](\xi_G, E) \tag{1·4}$$

according to Scott's formula ((5) and (7)). The first equality implies

$$\bar{\imath}^*(\xi_G) = 0 \tag{1·5}$$

because $P(E) \cap Y = \varnothing$ in $\bar{E}$. One can now prove the self intersection formula for the special case where Y is regarded as a subvariety of the variety $\bar{E}$ as follows.† For any $y \in \mathscr{A}(Y)$, $\bar{\imath}_*(y) = \bar{\imath}_* \bar{\imath}^* \rho_G^*(y)$ by (1·2). By projection formula $\bar{\imath}_*(y) = \rho_G^*(y) . \bar{\imath}_*(1)$ hence

$$\bar{\imath}_*(y) = \rho_G^*(y) . [\rho_G^* \bar{C}] (\xi_G, E), \tag{1·6}$$

using also (1·4). Applying $\bar{\imath}^*$ on both sides and taking into account (1·5),

$$\bar{\imath}^* \bar{\imath}_*(y) = \bar{\imath}^* \rho_G^*(y) . \bar{\imath}^* \rho_G^* c_r(E) = \bar{\imath}^* \rho_G^*(y c_r(E)) = y c_r(E),$$

since $\bar{\imath}^* \rho_G^* = Id$. Hence

$$\bar{\imath}^* \bar{\imath}_*(y) = y . c_r(E) \tag{1·7}$$

for any $y \in \mathscr{A}(Y)$.

A useful remark in this situation is that $\bar{\jmath}_*$ is injective. In fact, if α is any element of $\mathscr{A}(P(E))$ then $\alpha = \sum_0^{r-1} \rho_E^* a_i . \xi_E^i$, so $\alpha = \bar{\jmath}^* \sum_0^{r-1} \rho_G^* a_i . \xi_G^i = \bar{\jmath}^* \beta$, say. Then

$$\bar{\jmath}_* \alpha = \bar{\jmath}_* \bar{\jmath}^* \beta = \beta . \bar{\jmath}_*(1) = \beta \xi_G$$

by (1·3). So $\bar{\jmath}_* \sum_0^{r-1} \rho_E^* a_i . \xi_E^i = \sum_0^{r-1} \rho_G^* a_i . \xi_G^{i+1}$. As the minimal equations of ξ_E and ξ_G are respectively of degrees $r-1$ and r it follows that $\bar{\jmath}_*$ is injective.

LEMMA 1·1. ('Excision lemma' cf. (3), 4–30, Lemma 4). *Let U, V be two varieties, V a subvariety of U, $W = U \setminus V$ and Z a cycle of U such that its restriction Z_W to W is rationally equivalent to zero. Then there exists a cycle Γ of V which is rationally equivalent to Z on U.*

COROLLARY 1·2. *Consider the blowing-up diagram*

$$\begin{array}{ccc} V' & \longrightarrow & U' \\ \downarrow & & \downarrow \phi \\ V & \longrightarrow & U \end{array}$$

and let Z be a cycle of U such that no irreducible component of Z is contained in V. Let Z' be the proper transform of Z by ϕ. Assume that the restriction Z_W of Z to $W = U \setminus V$ is rationally equivalent to zero. Then there is a cycle Γ of V' such that Z' is rationally equivalent to Γ.

Proof. $W' = U' \setminus V'$ is isomorphic to W by ϕ and the restriction $Z'_{W'}$ of Z' to W' corresponds to Z_W. Hence $Z'_{W'}$ is rationally equivalent to zero. Apply (1·1) to Z', U' and V'.

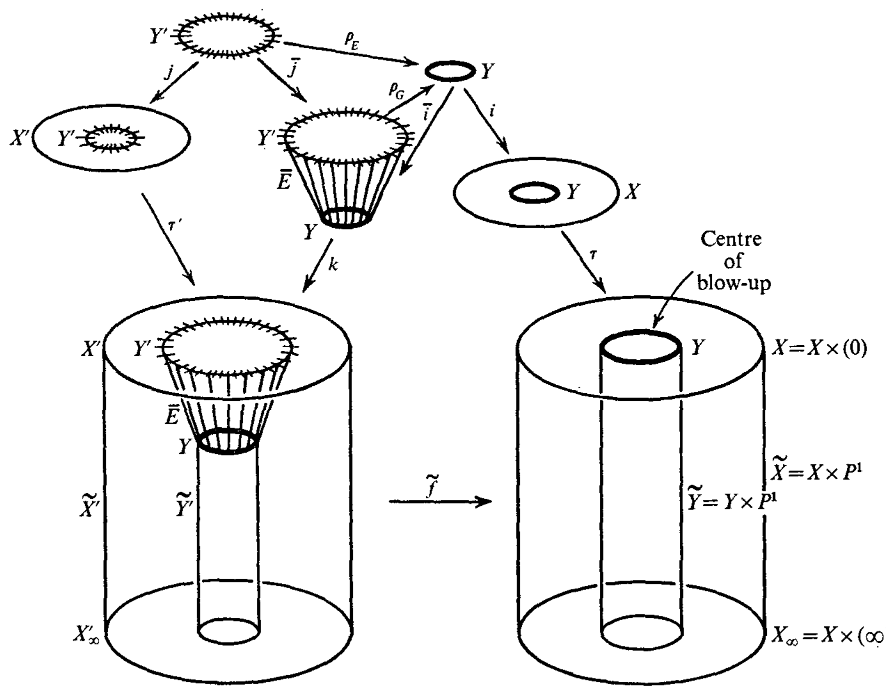
2. *The construction.* Let $\tilde{X} = X \times \mathbb{P}^1$ and blow it up along $Y \times 0$ to get a morphism $\tilde{f}: \tilde{X}' \to \tilde{X}$. One can identify X with $X \times 0$ and its proper transform by $\tilde{f}^{-1}$ with X'.

† A no more difficult argument enable us to establish the self intersection formula in the case where X and Y are both projective bundles over the same base space.

Then $\tilde{f}|X' = f$ and the total transform of X is $\tilde{f}^{-1}(X) = X' + \bar{E}$, because the normal bundle of $Y \times 0$ in $\tilde{X}$ is $E \oplus 1_Y$, $Y \times 0$ is a simple subvariety of $X \times 0$ and

$$\operatorname{codim}_{\tilde{X}}(X \times 0) = 1.$$

Also $\bar{E}$ is attached to X' along Y' its roof, $\tilde{Y}'$ is the proper transform of $Y\ (= Y \times \mathbb{P}^1)$ and $\tilde{f}$ induces an isomorphism $\tilde{Y}' \xrightarrow{\sim} \tilde{Y}$.



For convenience we shall list the maps needed in the sequel: The projection

$$\pi' : \tilde{Y}' \to Y.$$

The inclusions†

$$i : Y \to X, \quad \bar{\imath} : Y \to \bar{E}, \quad \tilde{\imath}' : \tilde{Y}' \to \tilde{X}',$$
$$j : Y' \to X', \quad \bar{\jmath} : Y' = P(E) \to \bar{E},$$
$$k : \bar{E} \to \tilde{X},$$
$$\beta : X_\infty \to \tilde{X}, \quad \beta' : X'_\infty \to \tilde{X}',$$
$$\tau : X \to \tilde{X}, \quad \tau' : X' \to \tilde{X}'.$$

Lemma 2·1. $k_*(1) = \beta'_*(1) - \tau'_*(1)$.

Proof. $\tau_*(1) = \beta_*(1)$ and $\tilde{f}^*\tau_*(1) = k_*(1) + \tau'_*(1)$, $\tilde{f}^*\beta_*(1) = \beta'_*(1)$.

Lemma 2·2. (i) $\bar{\imath}^* = \pi'_* \tilde{\imath}'^* k_*$, (ii) $\tau'^* k_* = j_* \bar{\jmath}^*$.

Proof. (i). Any subvariety Z of $\bar{E}$ properly intersecting Y on $\bar{E}$, also intersects $\tilde{Y}'$ properly on $\tilde{X}'$ because $\operatorname{codim}_{\bar{E}}(Y) = \operatorname{codim}_{\tilde{X}'}(\tilde{Y}')$ and $Z \cap Y = Z \cap \tilde{Y}'$. The proof of (ii) is similar taking into account that $\operatorname{codim}_{\bar{E}}(Y') = \operatorname{codim}_{\tilde{X}'}(X') = 1$.

† β and τ are just abbreviations for 'bottom' and 'top'.

LEMMA 2·3. $\bar{\imath}^* k^* k_* = 0$.

Proof. By Lemma 2·2, (i) for any $u \in \mathscr{A}(\bar{E})$, $\bar{\imath}^* k^* k_* u = \pi'_* \bar{\imath}'^* k_* k^* k_* u$. Then

$$k_* k^* k_* u = k_*(u) k_*(1)$$

by projection formula, $k_*(u).k_*(1) = k_*(u)(\beta'_*(1) - \tau'_*(1))$ by Lemma 2·1 and $k_*(u).\beta'_*(1) = 0$, because $\bar{E} \cap X'_\infty = \varnothing$. It follows that $k_* k^* k_* u = -k_*(u)\tau'_*(1)$. Finally $\pi'_* \bar{\imath}'^* k_* k^* k_* u = -\pi'_* \bar{\imath}'^* (k_*(u).\tau'_*(1)) = 0$ because $k_*(u).\tau'_*(1)$ is represented by a cycle on X' which will be disjoint from $\tilde{Y}' = \mathrm{Im}(\bar{\imath}')$.

3. *The self intersection formula.* For any cycle A of X denote $A \times \mathbb{P}^1$ by $\tilde{A}$ and let $\tilde{A}'$ be the proper transform of $\tilde{A}$ by $\tilde{f}$. The following lemma is straightforward to verify:

LEMMA 3·1. (i) *If A is a subvariety of Y, then $\tilde{A}'$ is isomorphic with $\tilde{A}$ and it intersects the floor of $\bar{E}$ along A; moreover $\tilde{A}'.\bar{E}$ is defined on $\tilde{X}'$, $\tilde{A}'.\bar{E} = A$ and $\tilde{A}'.X' = 0$.*

(ii) *If A intersects Y properly on X then the cycle $\tilde{f}^{-1}(\tilde{A})$ is defined and $\tilde{f}^{-1}(\tilde{A}) = \tilde{A}'$; the cycle $\tilde{A}'.\bar{E}$ is defined on $\tilde{X}'$ and $\tilde{A}'.\bar{E} = \rho_G^{-1}(A.Y)$.*

(iii) *Under the assumption of* (ii) *the cycle $X'.\tilde{A}'$ is defined on $\tilde{X}'$ and $X'.\tilde{A}' = f^{-1}(A)$.*

THEOREM 1. *For any $y \in \mathscr{A}(Y)$*

$$i^* i_*(y) = y c_r(E),$$

where E is the normal bundle of Y in X.

Proof. One can assume that y is the class of a subvariety B of Y. Let B_1 be a cycle of X properly intersecting Y and rationally equivalent to B. According to Lemma 3·1, (i)

$$\tilde{B}'.\bar{E} = B \tag{3·1}$$

and

$$\tilde{B}'.X' = 0. \tag{3·2}$$

By (ii) and (iii) of the same lemma

$$\tilde{f}^{-1}(\tilde{B}_1) = \tilde{B}'_1, \tag{3·3}$$

$$\tilde{B}'_1.\bar{E} = \rho_G^{-1}(B_1.Y), \tag{3·4}$$

$$\tilde{B}'_1.X' = f^{-1}(B_1). \tag{3·5}$$

Also the class of $B_1.Y$ in $\mathscr{A}(Y)$ is

$$cl_Y(B_1.Y) = i^* i_*(y). \tag{3·6}$$

The cycle $\tilde{B} - \tilde{B}_1$ is rationally equivalent to zero on $\tilde{X}$. By Corollary 1·2, with $\phi = \tilde{f}$ and $Z = \tilde{B} - \tilde{B}_1$, there exists a cycle Γ of $\bar{E}$ rationally equivalent to $\tilde{B}' - \tilde{B}'_1$ on $\tilde{X}'$, i.e. in rational equivalence classes

$$cl_{\tilde{X}'}(\tilde{B}') - cl_{\tilde{X}'}(\tilde{B}'_1) = k_*(\gamma), \tag{3·7}$$

where $\gamma \in \mathscr{A}(\bar{E})$ is the class of Γ. According to (3·1) $k^* cl_{\tilde{X}'}(\tilde{B}') = \bar{\imath}_*(y)$. Also from (3·4) and (3·6) one deduces $k^* cl_{\tilde{X}'}(\tilde{B}'_1) = \rho_G^* i^* i_*(y)$. These give, by applying k^* to (3·7),

$$\bar{\imath}_*(y) - \rho_G^* i^* i_*(y) = k^* k_*(\gamma). \tag{3·8}$$

Applying $\bar{\imath}^*$ on both sides and taking into account that $\bar{\imath}^* k^*k_* = 0$ by Lemma 2·3, $\bar{\imath}^*\bar{\imath}_*(y) - \bar{\imath}^*\rho_G^* i^*i_*(y) = 0$, hence $\bar{\imath}^*\bar{\imath}_*(y) = i^*i_*(y)$ because $\rho_G \bar{\imath} = Id_Y$. To conclude apply (1·7).

Note. Since $i^*i_*(y) = yc_r(E)$ one can write (3·8) as follows

$$\bar{\imath}_*(y) = \rho_G^*(yc_r(E)) + k^*k_*(\gamma). \tag{3·9}$$

4. *The 'formule-clef' for $\mathscr{A}(X')$.*

LEMMA 4·1. *With notations of section* 3

$$-\bar{\jmath}^*(\gamma) = \rho_E^*(y) . c_{r-1}(E^{(1)}).$$

Proof. By (3·9) $k^*k_*\gamma = \bar{\imath}_*(y) - \rho_G^*(y . c_r(E))$. Also

$$k^*k_*(\gamma) = -\gamma . \xi_G = -\gamma . \bar{\jmath}_*(1) = -\bar{\jmath}_*\bar{\jmath}^*(\gamma)$$

using the self-intersection formula, the fact that $\check{L}_G$ is the normal bundle of $\bar{E}$ in $\tilde{X}'$ and the formula $c_1(\check{L}_G) = -\xi_G$ for the first equality; and using (1·3) for the second formula. By (1·6) and (1·3) again:

$$\begin{aligned}\bar{\imath}_*(y) - \rho_G^*(y . c_r(E)) &= \rho_G^*(y) . ([\rho_G^* \bar{C}](\xi_G, E) - \rho_G^* c_r(E)) \\ &= \rho_G^*(y) . \xi_G . \left(\sum_{i=0}^{r-1} \xi_G^{r-1-i} . \rho_G^* c_i(E)\right) \\ &= \rho_G^*(y) . \bar{\jmath}_* \bar{\jmath}^* \left(\sum_{i=0}^{r-1} \xi_G^{r-1-i} . \rho_G^* c_i(E)\right).\end{aligned}$$

But $\bar{\jmath}^*\xi_G = \xi_E$ and $\rho_E = \rho_G \bar{\jmath}$, so putting everything together and using (1·1), we get:

$$\begin{aligned}-\bar{\jmath}_*\bar{\jmath}^*(\gamma) &= \bar{\imath}_*(y) - \rho_G^*(y . c_r(E)) \\ &= \bar{\jmath}_*\left(\rho_E^*(y) . \sum_{i=0}^{r-1} \xi_E^{r-1-i} . \rho_E^* c_i(E)\right) \\ &= \bar{\jmath}_*(\rho_E^*(y) . c_{r-1}(E^{(1)}).\end{aligned} \tag{4·1}$$

But $\bar{\jmath}_*$ is injective as remarked in section 1, so we may cancel it in (4·1).

THEOREM 2. *For any* $y \in \mathscr{A}(Y)$

$$j_*(\rho^*(y) . c_{r-1}(E^{(1)})) = f^*i_*(y).$$

Proof. By applying τ'^* to (3·7), $-\tau'^* cl_{\tilde{X}'}(\tilde{B}_1') = \tau'^* k_*(\gamma)$ since $\tilde{B}'$ is a cycle of $\tilde{Y}'$ and $\tilde{Y}' \cap X' = \varnothing$. Also $\tau'^* cl_{\tilde{X}'}(\tilde{B}_1') = f^*i_*(y)$ according to (3·5). Hence

$$-f^*i_*(y) = \tau'^*k_*(\gamma).$$

By (ii) of Lemma 2·2, $\tau'^*k_*(\gamma) = j_*\bar{\jmath}^*(\gamma)$. But in Lemma (4·1) we have shown that $-\bar{\jmath}^*(\gamma) = \rho_E^*(y)\, c_{r-1}(E^{(1)})$. Putting this together, we get the 'formule-clef'.

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