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MATSUSAKA'S BIG THEOREM

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§1. INTRODUCTION

The goal of this note is to present an outline of Matsusaka's proof [4],[5] of the following Theorem:

THEOREM 1. Let $P(k)$ be a rational polynomial with integral values for all integers k. Then there is a k_o such that for every non-singular complex projective variety V, and every ample line bundle L on V with

$$\chi(V, L^{\otimes k}) = P(k),$$

then $L^{\otimes k}$ is very ample if $k \geq k_o$.

The proof can be divided into 2 parts. The more difficult part consists in the proof of Theorem 2 below, and a somewhat easier but still subtle part consists in checking that Theorem 2 implies Theorem 1.

THEOREM 2. Given constants $\epsilon > 0$, $\gamma, k_o, n \in \mathbb{Z}$ and $t \in \mathbb{Q}$, there is a $k_1 = k_1(\epsilon, \gamma, k_o, t, n)$ for which the following holds: Let V be any normal projective variety of dimension n over any algebraically closed field k; let C be an ample divisor on V and let D be a codimension 1 cycle on V; assume $\gamma = (C^n)$, and

$$t = \frac{(D \cdot C^{n-1})}{(C^n)} .$$

Assume

$$\dim H^o(\mathcal{O}_V(kD)) \geq \frac{(\frac{1}{2}+\epsilon)}{n!} t^n (kC)^n), \text{ for all } k \geq k_o.$$

Then for every $k \geq k_1$, one can find a subspace

$$\Lambda \subset H^o(\mathcal{O}_V(kD))$$

such that the induced rational map

$$\phi_\Lambda: V \longrightarrow \mathbb{P}^N$$

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is birational and does not blow down any codimension one subvarieties. Moreover:

$$\deg \phi_\Lambda(V) \leq \gamma k^n t^n.$$

Many special cases of Theorem 1 are well known. If V is a curve of genus g, then L is ample if and only if $\deg L \geq 1$, and it is well known that in this case $L^{\otimes k}$ is very ample if $k \geq (2g+1)/\deg L$. If V is an abelian variety, L ample, then $L^{\otimes k}$ is very ample if $k \geq 3$ (Lefschetz: cf. Mumford [8], §17). If V is a K3-surface, L ample, then again $L^{\otimes k}$ is very ample if $k \geq 3$ (Mayer [7], Saint-Donat [9]). If V is a normal surface of general type with its rational curves E with $(E^2) = -2$ blown down, and $L = \mathcal{O}_V(K)$, then $L^{\otimes k}$ is very ample if $k \geq 5$ (Enriques, Kodaira [3], Bombieri [2]). For arbitrary surfaces and ample L's in any characteristic, the Theorem was proven in Matsusaka-Mumford [6].

Once Theorem 1 is established, one may apply the theory of Hilbert schemes or of Chow varieties to conclude that the set of polarized varieties (V,L) with given Hilbert polynomial $P(k)$ may all be parametrized by a quasi-projective scheme. (In particular this family contains all deformations of the polarized variety (V,L) because the Hilbert polynomial is invariant under deformation.)

Matsusaka's proof of Theorem 1 is non-cohomological, unlike for instance Bombieri's approach to canonically polarized surfaces. Theorem 1 would follow immediately if, for instance, one could solve directly the following:

PROBLEM: Given $P(k)$, find k_o such that for every (V,L) with Hilbert polynomial $P(k)$ and every $x,y \in V$,

$$H^i(V, \mathfrak{m}_x \cdot \mathfrak{m}_y \cdot L^{\otimes k}) = (0), \quad \text{all } i \geq 1,\ k \geq k_o.$$

Conversely, Matsusaka's result implies that such a k_o exists because it implies that the quadruples (V,L,x,y) form a "bounded family".

We want to add a word about the completeness of our presentation of Matsusaka's proof. We believe the careful reader can reconstruct the whole proof from what we say. However in some places we have not written out fully various details. In particular, a more complete version would include a whole section working out the elementary properties of Matsusaka's operation $\Lambda^{[j]}$ (cf. §2 below): instead we simply introduce these without proof where they are needed.

§ 2. PROOF OF THEOREM 2

(I.): The first step is to find a k_2 depending only on γ, k_o, t such that for all $k \geq k_2$, the rational map

$$\phi_{kD}: V \longrightarrow \mathbb{P}(H^o(\mathcal{O}_V(kD)))$$

satisfies $\dim \phi_{kD}(V) = n$. We shall in fact prove:

LEMMA 2.1. If $\Lambda \subset H^o(\mathcal{O}_V(kD))$ and

$$\dim \Lambda > \max_{1 \leq i \leq n-1} (i + \gamma t^i k^i)$$

then $\dim \phi_\Lambda(V) = n$.

Because of our assumed lower bound on $h^o(kD)$ one gets immediately a $k_2(\gamma, k_o, t)$ such that $\Lambda = H^o(\mathcal{O}_V(kD))$ satisfies this for $k \geq k_2$.

To prove the lemma, let $W = \phi_\Lambda(V)$. We show in fact that if $\dim W = i$, then

$$\dim \Lambda \leq i + \gamma t^i k^i.$$

Firstly, recall the well known fact (cf. [6], Th. 3) that for any projective variety $X \subset \mathbb{P}^n$, X not in any hyperplane:

$$n+1 \leq \deg X + \dim X.$$

In particular, if $X = W$, $\mathbb{P}^n = \mathbb{P}(\Lambda)$, then

$$\dim \Lambda \leq \deg W + i,$$

so it suffices to prove

$$\deg W \leq \gamma t^i k^i.$$

To transform the inequality on deg W into an estimate on V itself, Matsusaka introduces an interesting new concept of the <u>variable j-fold intersection cycle</u> $\Lambda^{[j]}$ of a linear system $\Lambda \subset \Gamma(V, M)$. This is the codimension j cycle, defined only up to rational equivalence on V, obtained in either of the following ways: let $B \subset V$ be the base points of the linear system Λ so that Λ defines a morphism

$$\phi_\Lambda: \ V-B \longrightarrow \mathbb{P}(\Lambda).$$

Take the closure in V of $\phi_\Lambda^{-1}(L)$, $L \subset \mathbb{P}(\Lambda)$ a general codimension j linear space; or take the closure in V of the intersection cycle $V(s_1)\cdot \cdots \cdot V(s_j)$ in V-B, where the s_i are j general element of

Λ. If $\Lambda = \Gamma(V, \mathcal{O}_V(E))$ for some divisor E, write $(E)^{[j]}$ for $\Lambda^{[j]}$. Note that if $\Lambda_1 \subset \Lambda_2$ are 2 linear systems, then

$$\Lambda_2^{[i]} \underset{\text{rat.eq.}}{\sim} \Lambda_1^{[i]} + \text{eff. cycle.}$$

With this concept, we find:

$$\begin{aligned} \deg W &\leq \# \text{ of components of } \Lambda^{[i]} \\ &\leq (C^{n-i}.\Lambda^{[i]}) \\ &\leq (C^{n-i}.(kD)^{[i]}) , \end{aligned}$$

hence Lemma 2.1 follows by taking $E = kD$ in the following key result:

PROPOSITION 2.2: Let V be a normal projective variety of dimension n, C an ample divisor and E a codimension one cycle on V such that $\dim \phi_{kE}(V) \geq i$ for $k \gg 0$. Then

$$(E^{[i]}.C^{n-i}) \leq (C^n)\cdot\left(\frac{(C^{n-1}\cdot E)}{(C^n)}\right)^i$$

<u>Proof</u>: Replacing C by ℓC multiplies both sides by ℓ^{n-i} so we may assume C very ample. Let V' be a general intersection of n-i divisors $C_1,\cdots,C_{k-i} \in |C|$, let $C' = V'.C$ and let $E' = V'.E$. Then one sees easily that

$$(E^{[i]}.C^{n-i})_V = (E')^{[i]}_{V'}$$

$$(C^n)_V = (C'^i)_{V'}$$

$$(C^{n-1}.E)_V = (C'^{i-1}.E')_{V'} ,$$

hence replacing V by V', we may assume $i = n$. Now $\dim \phi_{kE}(V) = n$ for $k \gg 0$, hence in fact ϕ_{kE} is birational for $k \gg 0$ [i.e., if $W_k = \phi_{kE}(V) \subset \mathbb{P}_{N_k}$ and if $\pi: W_k' \longrightarrow W_k$ is the normalization of W_k in the field $k(V)$, then $\pi^*(\mathcal{O}_{W_k}(1))$ is ample on W_k' and

$$\Gamma(W_k', \pi^*(\mathcal{O}_{W_k}(\ell)) \subset \Gamma(V, \mathcal{O}(k\ell E)).]$$

We may also replace E by kE to prove the Proposition because:

$$(kE)^{[n]} \geq k^n(E)^{[n]}$$

$$(C^{n-1}.kE)^n = k^n(C^{n-1}.E)^n$$

(because the base locus of $|kE|$ is contained in the base locus of $|E|$, and the variable intersection of n general divisors in $|kE|$ specializes to k^n times the variable intersection of n general divisors in $|E|$ plus some components in the base locus of $|E|$.) So we may assume ϕ_E is birational.

Now let $W = \phi_E(V)$. Since ϕ_E is birational, $\deg W = (E^{[n]})$. Moreover, if $k \gg 0$:

$$\begin{aligned} h^o(V,\mathcal{O}_V(kE)) &\geq h^o(W,\mathcal{O}_W(k)) \\ &= \deg W \cdot \frac{k^n}{n!} + \text{lower terms} \\ &= E^{[n]} \cdot \frac{k^n}{n!} + \text{lower terms.} \end{aligned}$$

The Proposition now follows from considering the upper bound on $h^o(\mathcal{O}_V(kE))$ as $k \longrightarrow \infty$, which is given by

PROPOSITION 2.3 ("Q-estimate"): Let V be a projective variety of dimension n, let C be a hyperplane section of V and let $\mathfrak{I}$ be a torsion-free rank 1 sheaf on V. Then

$$h^o(\mathfrak{I}) \leq \binom{[t]+n}{n}\gamma + \binom{[t]+n-1}{n-1}$$

where

$$t = \deg \mathfrak{I}/\deg V \qquad \gamma = \deg V \;. \qquad \left(\begin{matrix}\text{degree measured via C}\\ \text{as in Kleiman, Annals, 1966}\end{matrix}\right)$$

<u>Proof</u>: For $n = 1$, the inequality reads

$$h^o(\mathfrak{I}) \leq ([t]+1)\gamma + 1$$

which follows from the Riemann-Roch estimate:

$$h^o(\mathfrak{I}) \leq \deg \mathfrak{I} + 1 = t\gamma + 1.$$

We proceed by induction, assuming the result true on a general hyperplane section C. First we need to find a hyperplane C such that C is again a variety and $\mathfrak{I} \otimes \mathcal{O}_C$ is still torsion-free. Indeed almost all C's are varieties (Seidenberg's Theorem) and for $\mathfrak{I}\otimes\mathcal{O}_C$ to be torsion-free, it suffices to make sure $\operatorname{depth}_{\mathcal{O}_x}(\mathfrak{I}_x) \geq 2$ for all $x \in C$ except the generic point of C. Since there are only finitely many $x \in V$ with $\operatorname{codim}_V\{\overline{x}\} \geq 2$ and $\operatorname{depth}_{\mathcal{O}_x}(\mathfrak{I}_x) = 1$ (cf. for instance EGA, Ch. 4, §10.8), this is possible. Then one has the exact sequences:

$$0 \longrightarrow \mathfrak{I}(-k-1) \longrightarrow \mathfrak{I}(-k) \longrightarrow \mathfrak{I}\otimes\mathcal{O}_C(-k) \longrightarrow 0,$$

hence

$$h^0(\mathfrak{I}(-k)) \leq h^0(\mathfrak{I}(-k-1)) + h^0(\mathfrak{I}\otimes\mathcal{O}_C(-k)).$$

But $h^0(\mathfrak{I}(-k)) \neq 0$ implies there is a homomorphism

$$0 \longrightarrow \mathcal{O}_V(kC) \longrightarrow \mathfrak{I}$$

hence $\deg \mathfrak{I} \geq k \deg V$, hence $[t] \geq k$. Thus

$$h^0(\mathfrak{I}) \leq \sum_{k=0}^{[t]} h^0(\mathfrak{I}\otimes\mathcal{O}_C(-k)).$$

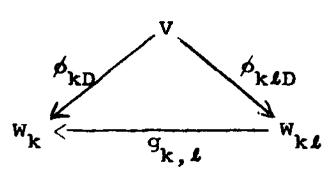
Using the estimate on C, we get:

$$h^0(\mathfrak{I}) \leq \sum_{k=0}^{[t]} \binom{[t]-k+n-1}{n-1}\gamma + \binom{[t]-k+n-2}{n-2}$$

$$= \binom{[t]+n}{n}\gamma + \binom{[t]+n-1}{n-1} \qquad \text{QED}$$

(II.): This completes the first step: if $W_k = \phi_{kD}(V)$, we have a k_2 such that if $k \geq k_2$, $\dim W_k = n$. The second step is to find a k_3 also depending only on ϵ, γ, k_0, t such that if $k \geq k_3$, then ϕ_{kD} is birational. We will in fact produce an ℓ_0 such that $\phi_{\ell_0 k_2 D}$ is birational. Note that since $k \geq k_0$ implies kD is effective, then for $k \geq k_3 = \ell_0 k_2 + k_0$, $\Gamma(\mathcal{O}(kD)) \supseteq \Gamma(\mathcal{O}(\ell_0 k_2 D))$, hence ϕ_{kD} is also birational.

To produce ℓ_0, consider for each $k \geq k_2$, $\ell > 1$, the diagram of rational maps:



Note that $\deg(\phi_{k\,D}) = \deg(\phi_{k\ell D})\cdot\deg(g_{k,\ell})$.

LEMMA 2.4: There is an integer ℓ (depending on ϵ, n and t) such that if $\deg(\phi_{kD}) > 1$ and $\deg(g_{k,\ell}) = 1$, then one must have

$$\frac{(k\ell D)^{[n]}}{k^n \ell^n} > (1+\epsilon)^{1/n} \cdot \frac{(kD)^{[n]}}{k^n} .$$

<u>Proof</u>: Choose ℓ such that

$$\binom{\ell(1+\epsilon)^{1/n}+n+1}{n} + \frac{2}{t^n}\binom{\ell(1+\epsilon)^{1/n}+n}{n-1} < \frac{1+2\epsilon}{n!}\,\ell^n .$$

This is possible because for $\ell >> 0$, the left hand side grows like $(1+\epsilon)\ell^n/n!$. Now W_k and $W_{k\ell}$ both have explicit projective embeddings:

$$W_k \subset \mathbb{P}(H^0(\mathcal{O}_V(kD)))$$

$$W_{k\ell} \subset \mathbb{P}(H^0(\mathcal{O}_V(k\ell D))).$$

Since $g_{k,\ell}$ is birational by assumption, let $U \subset W_k$ be the domain of definition of $g_{k,\ell}^{-1}$. Then the morphism

$$U \longrightarrow W_{k\ell} \subset \mathbb{P}(H^0(\mathcal{O}_V(k\ell\ D)))$$

is defined by an invertible sheaf $\mathfrak{J}'$ on U and $h^0(k\ell D)$ sections s_i' of $\mathfrak{J}'$ generating $\mathfrak{J}'$. There is a unique torsion free sheaf $\mathfrak{J}$ on W_k plus sections s_i generating $\mathfrak{J}$, which restrict on U to $\{\mathfrak{J}', s_i'\}$. Thus $h^0(\mathfrak{J}) \geq h^0(k\ell D)$. On the other hand, for the given projective embedding of W_k, we calculate:

$\deg \mathfrak{J}$ = # of intersections on V outside base locus of $|kD|$ of (n-1) general sections of $\mathcal{O}(kD)$, one section of $\mathcal{O}(k\ell D)$,

hence:

$$\ell^{n-1}.\deg \mathfrak{J} \leq (k\ell D)^{[n]} .$$

Now combine the assumed lower bound on $h^0(k\ell D)$, and the upper bound on $h^0(\mathfrak{J})$ given by the Q-estimate to get:

$$\frac{\frac{1}{2}+\epsilon}{n!} t^n (k\ell)^n \gamma \le h^o(k\ell D)$$
$$\le h^o(\mathfrak{J})$$
$$\le \binom{[\frac{\deg \mathfrak{J}}{\deg W_k}]+n}{n} \deg W_k + \binom{[\frac{\deg \mathfrak{J}}{\deg W_k}]+n-1}{n-1} .$$

Moreover:

$$\deg W_k = (kF)^{[n]}/\deg(\phi_{kD})$$
$$\le \frac{t^n k^n \gamma}{2} \text{ by Prop. 2.2}$$

and

$$[\frac{\deg \mathfrak{J}}{\deg W_k}] \le \frac{(k\ell D)^{[n]}}{\ell^{n-1}(kD)^{[n]}} + 1$$
$$= \ell R + 1$$

if

$$R = \frac{(k\ell D)^{[n]}}{\ell^n (kD)^{[n]}} .$$

Hence

$$\frac{\frac{1}{2}+\epsilon}{n!} t^n k^n \ell^n \gamma \le \binom{\ell R+n+1}{n} \frac{t^n k^n \gamma}{2} + \binom{\ell R+n}{n-1}$$

hence

$$\frac{1+2\epsilon}{n!} \ell^n \le \binom{\ell R+n+1}{n} + \frac{2}{t^n} \binom{\ell R+n}{n-1} .$$

If $R \le (1+\epsilon)^{1/n}$, this contradicts the inequality that ℓ was chosen to satisfy. Thus $R > (1+\epsilon)^{1/n}$. <u>QED</u>

However, for all k,

$$\frac{(kD)^{[n]}}{k^n} \le \gamma t^n$$

by Prop. (2.2). Hence starting at any ϕ_{kD}, we see that:

$$\deg \phi_{(k\ell^e D)} < \deg \phi_{kD} \quad \text{if} \quad e > \frac{n \log(\gamma t^n k^n)}{\log(1+\epsilon)} .$$

Since we know that ϕ_{k_2D} is finite-to-one and

$$\deg \phi_{k_2D} \leq (k_2D)^{[n]} \leq \gamma t^n k_2^n,$$

it follows that $\phi_{(k_2 \ell e)D}$ is birational if

$$e > n\gamma t^n k_2^n \frac{\log(\gamma t^n k^n)}{\log(1+\epsilon)} .$$

(III): This completes the second step: we have a k_3 such that if $k \geq k_3$, ϕ_{kD} is birational. The third step is to find a k_4 also depending only on ϵ, γ, k_o, t such that if $k \geq k_4$ then there is a $\Lambda \subset |kD|$ such that ϕ_Λ is birational and does not blow down any codimension 1 subvarieties of V. We will in fact only produce a k_5 such that if $k \geq k_5$, then there is a $\Lambda' \subset |kD|$ such that $\dim \phi_{\Lambda'}(V) = n$ and $\phi_{\Lambda'}$ does not blow down any codimension 1 subvarieties. Setting $k_4 = k_3+k_5$ and Λ = minimal sum of $|k_3D|$ and Λ', we get the Λ with all the properties. The proof is very similar to the beautifully simple Method of Albanese by which for any projective n-dimensional variety X one constructs a $\Lambda \subset H^o(\mathcal{O}_X(k))$ such that $\phi_\Lambda(X)$ is birational to X and has no points of multiplicity $> n!$ (cf. [1],(12.4.4)) We show in fact:

LEMMA (2.5). Choose k such that

$$tk \geq 2n\cdot n!(1+\frac{1}{\gamma}) \quad \text{and} \quad k \geq k_o.$$

Then there is a $\Lambda \subset H^o(\mathcal{O}_V(kD))$ such that $\dim \phi_\Lambda(V) = n$ and ϕ_Λ does not blow down any codimension 1 subvarieties.

Proof: Note by the assumption on k,

$$n\gamma(tk)^{n-1}+n \leq \gamma(tk)^{n-1}(n+\frac{n}{\gamma}) \leq \frac{\gamma(tk)^n}{2n!} < h^o(kD).$$

Also $tk \geq 1$, so by Lemma (2.1) any $\Lambda \subset H^o(kD)$ for which

$$\dim \Lambda \geq n + \gamma(tk)^{n-1}$$

has the property $\dim \phi_\Lambda(V) = n$. So any Λ such that

$$\operatorname{codim} \Lambda \leq (n-1)\gamma(tk)^{n-1}$$

has this property too. What we will do is this: starting

* See also Lecture 1, §5 of Lipman's article, " Introduction to resolution of singularities", in this volume.

with $\Lambda_o = H^o(\mathcal{O}_V(kD))$, choose a sequence of subspaces

$$\Lambda_o \supset \Lambda_1 \supset \Lambda_2 \supset \cdots\cdot$$

with $\dim \Lambda_i/\Lambda_{i+1} = 1$, until we reach the desired Λ. In fact, say Λ_r is chosen but there is still an $E \subset V$, $\dim E = n-1$, such that

$$\dim \phi_{\Lambda_r}(E) = i-1, \qquad n-1 \geq i \geq 1.$$

If s is the multiplicity to which E occurs as a fixed component of Λ_r, let $\Lambda_r' \subset H^o(\mathcal{O}_V(kD-sE))$ be the linear system such that $\Lambda_r = (e^{\otimes s})\otimes\Lambda_r'$, e = canonical section of $\mathcal{O}_V(E)$. Let x be a general point of E. Define

$$\Lambda_{r+1}' = \{s \in \Lambda_r' \mid s(x) = 0\},$$

$$\Lambda_{r+1} = (e^{\otimes s}) \otimes \Lambda_{r+1}' .$$

Note that ϕ_{Λ_r} is defined at x, so define

$$Z = \text{closure of } \phi_{\Lambda_r}^{-1}(\phi_{\Lambda_r}(x)).$$

Then $\dim Z = n-i$ if x is sufficiently general, and if $s \in \Lambda_r'$ vanishes at x, it vanishes on all of Z. Thus

$$(\Lambda_r')^{[i]} \underset{\text{rat.eq.}}{\sim} (\Lambda_{r+1}')^{[i]} + Z + \text{eff. cycle}.$$

But $\Lambda_r^{[i]} = (\Lambda_r')^{[i]}$ and $\Lambda_{r+1}^{[i]} = (\Lambda_{r+1}')^{[i]}$, so it follows

$$(\Lambda_r^{[i]}.c^{n-i}) > (\Lambda_{r+1}^{[i]}.c^{n-i}).$$

Since for each j,

$$(\Lambda_r^{[j]}.c^{n-j}) \geq (\Lambda_{r+1}^{[j]}.c^{n-j}),$$

it follows that the invariant

$$\delta(\Lambda) = \sum_{j=1}^{n-1} (\Lambda^{[j]}.c^{n-j})$$

decreases when you pass from Λ_r to Λ_{r+1}. But by Prop. (2.2)

$$\delta(\Lambda) \leq \delta(kD) \leq \sum_{j=1}^{n-1} \gamma(kt)^j \leq (n-1)\gamma(kt)^{n-1}.$$

Since we have "this much room" in $H^0(\mathcal{O}(kD))$, we can find a Λ for which no E is blown down. <u>QED</u>

§3. TH. 2 $\Longrightarrow$ TH. 1

This is the part of the proof that involves char. 0 because we want to apply Kodaira's Vanishing Theorem. The idea is to apply Theorem 2 to V with C,D chosen so that

$$L = \mathcal{O}_V(C)$$

$$L^{m_o} \otimes \Omega_V^n = \mathcal{O}_V(D).$$

Here m_o will be chosen below depending only on P so as to make Theorem 2 apply. Note that for $m > 0$ by Kodaira Vanishing and Serre duality:

$$(*) \qquad \begin{aligned} \dim H^0(L^m \otimes \Omega_V^n) &= \chi(L^m \otimes \Omega_V^n) \\ &= (-1)^n \chi(L^{-m}) \\ &= (-1)^n P(-m): \text{ call this } P'(m). \end{aligned}$$

Moreover, by Riemann-Roch,

$$P(k) = (C^n)\frac{k^n}{n!} - \frac{(K_V \cdot C^{n-1})}{2(n-1)!}k^{n-1} + \text{lower degree terms}$$

hence P determines the integer $\gamma = (C^n)$ and, once m_o is chosen, P determines $(D.C^{n-1})$ and hence $(D.C^{n-1})/(C^n)$ too. Finally, we need a lower bound for $\dim H^0(\mathcal{O}_V(kD))$ of the type used in Th. 2. This is obtained as follows -

a) Say $P'(m_1) > 0$, so that by (*) in divisor notation m_1C+K is an effective divisor. Then:

$$\begin{aligned} kD &= k(m_oC + K) \\ &= (k-1)(m_1C+K) + ((k(m_o-m_1)+m_1)C + K). \end{aligned}$$

The first term is an effective divisor, so

$$\begin{aligned}\dim H^o(\mathcal{O}_V(kD)) &\geq \dim H^o(\mathcal{O}_V((k(m_o-m_1)+m_1)C + K)) \\ &= P'(k(m_o-m_1)+m_1) \\ &= \frac{(C^n)}{n!}(k(m_o-m_1)+m_1)^n + \text{lower degree terms in } k \\ &= \frac{((kC)^n)}{n!}(m_o-m_1)^n + \text{lower degree terms in } k.\end{aligned}$$

b) But

$$\begin{aligned}t \underset{\text{def}}{=} \frac{(D.C^{n-1})}{(C^n)} &= \frac{((m_oC+K).C^{n-1})}{(C^n)} \\ &= m_o + \frac{(K.C^{n-1})}{(C^n)},\end{aligned}$$

i.e.,

$$\dim H^o(\mathcal{O}_V(kD)) \geq \frac{((kC)^n)}{n!}\left[t-m_1-\frac{(K.C^{n-1})}{(C^n)}\right]^n + \text{lower degree terms in } k.$$

If m_o and hence t is large enough, the term $[\ \ ]^n$ is at least $\frac{3}{4}t^n$ and then for k_o large enough, we certainly obtain:

$$\dim \Gamma(\mathcal{O}_V(kD)) \geq \frac{((kC)^n)}{n!}\cdot\frac{5}{8}t^n, \qquad \text{if } k \geq k_o.$$

Thus Theorem 2 applies for some m_o and k_o readily computed in terms of the polynomial P alone. Thus we can find k, so that for every (V,L)

$$\begin{aligned}\exists\ \Lambda \subset \Gamma(V,\mathcal{O}_V(k_1D)) &= \Gamma(V,\mathcal{O}_V(k_1(m_oC+K))) \\ &= \Gamma(V,L^{k_1m_o}\otimes(\Omega_V^n)^{k_1})\end{aligned}$$

for which ϕ_Λ is birational and does not blow down any divisors - we abbreviate this to "Λ is quasi-ample".

Now let's analyze the projective variety $U = \phi_\Lambda(V)$. By **Prop.** (2.2) we know:

$$(**) \qquad \deg U \leq \gamma k_1^n t^n.$$

Automatically then, the ambient space $\mathbb{P}(\Lambda)$ has its dimension bounded as follows:

$$\dim \mathbb{P}(\Lambda) \leq \deg U + n - 1 \underset{\text{def}}{=} N.$$

It follows that the set of varieties U lies in a bounded family when

(V,L) varies over all pairs with Hilbert polynomial P! This is the key point, from which we want to argue backwards, obtaining eventually the boundedness of the set of pairs (V,L). From this point on, we leave the area in which we can make explicit estimates, and rely on general results asserting that various numbers are bounded when calculated for some set of varieties and divisors in a bounded family. The first point is that if U_{nor} is the normalization of U, then there is a k_2 such that for all U with degree bounded by (**), the pullback of $\mathcal{O}_U(k_2)$ to U_{nor} is very ample. It follows that if we choose a suitable $\Lambda' \subset \Gamma(V,\mathcal{O}_V(k_1k_2D))$, then $\phi_{\Lambda'}(V) \cong U_{nor}$. Replacing k_1 by k_1k_2 and Λ by Λ', this means we may assume that U is always normal. Call these Λ "normally quasi-ample". In that case, working with "Weil"-divisors on U, i.e., cycles of codimension 1, we may define the total transform $\phi_\Lambda(E)$ for every divisor E on V; and because ϕ_Λ does not contract any divisors, this defines an injection of the groups of Weil-divisors

$$\mathrm{Div}(V) \xhookrightarrow[\phi_\Lambda]{} \mathrm{Div}(U)$$

such that

a) $\phi_\Lambda(E)$ eff. $\Longleftrightarrow$ E eff.

b) $\phi_\Lambda((f)_V) = (f)_U$.

Thus ϕ^*_Λ sets up an isomorphism between

$$\Gamma(V,\mathcal{O}_V(E)) \xrightarrow{\approx} \Gamma(U,\mathcal{O}_U(\phi_\Lambda E)).$$

Moreover, if $U_o \subset U$ is the maximal open set such that

$$\phi_\Lambda^{-1}: \ U_o \longrightarrow V$$

is a morphism, then codim $U-U_o \geq 2$. ϕ_Λ^{-1} then defines an injection:

$$(\phi_\Lambda^{-1})^*: \ \Omega^n_V \longrightarrow \Omega^n_U|_{U_o} .$$

This implies that

$$\phi_\Lambda(K_V) = K_U + (\text{eff. divisor}).$$

It follows that if $E \in |\ell C+K_V|$, then

$$\begin{aligned}\deg \phi_\Lambda(E) &= \ell \deg \phi_\Lambda(C) + \deg \phi_\Lambda(K_V)\\ &\leq \ell(C.\Lambda^{[n-1]}) + \deg K_U\\ &\leq \gamma k_1^{n-1}t^{n-1}\ell + \deg K_U.\end{aligned}$$

Of course, $\deg K_U$ is bounded when U varies over all U's with degree bounded by (**): call this bound κ.

We can now reveal the diagram on which the rest of the proof is based. We consider 3 sets, related by 2 maps, as follows:

$$B = \left\{\begin{array}{l}\text{Set of all } (V,L,s_0,\cdots,s_N,E_0,E_1)\\ V,L \text{ as in } A;\ E_0 \in |m_1C+K_V|;\\ E_1 \in |(m_1+1)C+K_V|;\text{ and } s_0,\cdots,s_N \in \Gamma(V,\mathcal{O}_V(k_1D))\\ \text{spanning a normally quasi-ample } \Lambda\end{array}\right\}$$

$$\alpha: B \to A \quad (\text{ignore}(s_i),(E_i)) \qquad\qquad \beta: B \to C$$

$$A = \left\{\begin{array}{l}\text{Set of all } (V,L),\\ V \text{ non-sing., proj.,}\\ L \text{ ample with}\\ \text{Hilbert Polyn. } P\end{array}\right\} \qquad C = \left\{\begin{array}{l}\text{Set of } (U,F_0,F_1),\text{ where}\\ U \subset \mathbb{P}^N \text{ is an n-dimensional}\\ \text{normal subvariety, } F_i \text{ divisors on } U\\ \deg F_i \leq \gamma k_1^{n-1}t^{n-1}(m_1+i)+\kappa\\ \deg U \leq \gamma k_1^n t^n\\ h^0(F_0+\ell(F_1-F_0)) \geq P'(\ell+m_1)\\ \text{all } \ell \geq 0\end{array}\right\}$$

Here m_1,k_1,N,γ, and t are chosen as above, in particular so that α is surjective. β is defined by

$$U = \phi_\Lambda(V)$$
$$F_i = \phi_\Lambda(E_i).$$

Note that this is OK because

$$\begin{aligned}H^0(U,\mathcal{O}_U(F_0+\ell(F_1-F_0))) &\cong H^0(U,\mathcal{O}_U(\phi_\Lambda(E_0+\ell(E_1-E_0))))\\ &\cong H^0(V,\mathcal{O}_V((m_1+\ell)C+K_V))\end{aligned}$$

which has dimension exactly $P'(m_1+\ell)$. Also, set C is isomorphic to the set of points on a locally closed subset of a union of 3-way products of Chow Varieties, i.e., each U,F_1,F_2 has a Chow form, normal is an open condition and $h^0(\cdots) \geq c$ are all closed conditions. Thus C has a natural structure of a (reducible) variety.

LEMMA 3.1: β is injective.

Proof: In fact, to recover V from (U,F_1,F_2), let $\Phi_\ell: U \longrightarrow \mathbb{P}^M$ be the rational map defined by $H^0(U,\mathcal{O}_U(\ell(F_1-F_0)))$. Then if $\ell >> 0$, $V = \Phi_\ell(U)$ (using the fact that

$$H^0(U,\mathcal{O}_U(\ell(F_1-F_0))) \cong H^0(V,L^{\otimes \ell})$$

and that L is ample on V). Moreover $\phi_\Lambda = \Phi_\ell^{-1}$, L is the line bundle associated to $\Phi_\ell(F_1-F_0)$, $E_i = \Phi_\ell(F_i)$, and $(s_0,\cdots,s_N)$ are the sections of $\mathcal{O}_V(k_1D)$ corresponding to the canonical sections $(X_0,\cdots,X_N)$ of $\mathcal{O}_U(1)$ via

$$\phi_\Lambda^*: \quad H^0(\mathcal{O}_U(1)) \longrightarrow H^0(\mathcal{O}_V(k_1D)). \qquad \text{QED}$$

LEMMA 3.2: The image of β is Zariski-open.

Proof: It is elementary to see that the image of β is a countable union of locally closed subsets of the set C. Therefore it is enough to show that for any valuation ring R and morphism ϕ: Spec R $\longrightarrow$ C, if the closed point is in the image of β, then so is the generic point. Then over R, we get a flat family of normal varieties (by Hironaka's lemma) $\mathcal{U} \longrightarrow$ Spec R, plus divisors $\mathfrak{F}_1,\mathfrak{F}_2$ on $\mathcal{U}$. For every ℓ,m, let

$$M_{\ell,m} = H^0(\mathcal{U},\mathcal{O}(m\mathfrak{F}_0+\ell m(\mathfrak{F}_1-\mathfrak{F}_0))).$$

$M_{\ell,m}$ is a finitely generated torsion-free and hence free R-module, and $\sum_m M_{\ell,m} = R_\ell$ is an R-algebra. If $k = R/M$ is the residue field, $\overline{U},\overline{F}_0,\overline{F}_1$ is the induced triple over k, we get an injection:

$$\sigma_{\ell,m}: \quad M_{\ell,m} \otimes_R k \hookrightarrow H^0(\overline{U},\mathcal{O}_{\overline{U}}(m\overline{F}_0+\ell m(\overline{F}_1-\overline{F}_0))).$$

Let $(\overline{U},\overline{F}_0,\overline{F}_1) = \beta(\overline{V},\overline{L},\overline{s}_0,\cdots,\overline{s}_N;\overline{E}_1,\overline{E}_2)$. If K is the fraction field of R, U^*,F_0^*,F_1^* is the induced triple over K, we get an isomorphism:

$$M_{\ell,m}\otimes_R K \xrightarrow{\approx} H^0(U^*,\mathcal{O}_{U^*}(mF_0^*+\ell m(F_1^*-F_0^*))).$$

But then it follows that

$$\begin{aligned}\dim_k M_{\ell,1}\otimes_R k &\leq \dim_k H^0(\overline{U},\mathcal{O}_{\overline{U}}(\overline{F}_0+\ell(\overline{F}_1-\overline{F}_0)))\\ &= \dim_k(H^0(\overline{V},\mathcal{O}_{\overline{V}}((m_1+\ell)\overline{C}+K_{\overline{V}}))\\ &= P'(m_1+\ell)\\ &\leq \dim_K H^0(U^*,\mathcal{O}_{U^*}(F_0^*+\ell(F_1^*-F_0^*)))\\ &= \dim_K M_{\ell,1}\otimes_R K.\end{aligned}$$

Since $M_{\ell,1}$ is free, the 2 extremes are equal, so equality holds everywhere. In particular, $\sigma_{\ell,1}$ is an isomorphism

$$\sigma_{\ell,1}\colon\ M_{\ell,1}\otimes_R k \xrightarrow{\approx} H^0(\overline{U},\mathcal{O}_{\overline{U}}(\overline{F}_0+\ell(\overline{F}_1-\overline{F}_0))).$$

Now on $\overline{V}$, since $\overline{C}$ is ample, for $\ell \gg 0$ it follows that the ring

$$\sum_{m=0}^{\infty} H^0(\overline{V},\mathcal{O}_{\overline{V}}((m_1+\ell)\overline{C}+K_{\overline{V}})^{\otimes m})$$

is generated by its elements of degree 1 and that $\overline{V}$ is its Proj. This implies that the ring

$$\sum_{m=0}^{\infty} H^0(\overline{U},\mathcal{O}_{\overline{U}}(m\overline{F}_0+m\ell(\overline{F}_1-\overline{F}_0)))$$

is generated by its elements of degree 1. But since $\sigma_{\ell,1}$ is surjective, this implies that $\sigma_{\ell,m}$ is surjective too: i.e., if $\ell \gg 0$, there is an isomorphism of rings:

$$\sigma_\ell\colon\ R_\ell\otimes_R k \xrightarrow{\approx} \sum_{m=0}^{\infty} H^0(\overline{U},\mathcal{O}_{\overline{U}}(m\overline{F}_0+m\ell(\overline{F}_1-\overline{F}_0))).$$

Therefore $\mathrm{Proj}(R_\ell\otimes_R k)\cong\overline{V}$. So $\mathcal{V}=\mathrm{Proj}(R_\ell)$ itself is a flat family of schemes of Spec R with special fibre $\overline{V}$. Moreover since $R_\ell\otimes_R k$ is generated by its elements of degree 1, R_ℓ is also generated by its elements of degree 1. Therefore $\mathrm{Proj}(R_\ell)$ comes equipped with a line bundle $\mathcal{O}_{\mathcal{V}}(1)$, which on the closed fibre $\overline{V}$ is just $\mathcal{O}_{\overline{V}}((m_1+\ell)\overline{C}+K_{\overline{V}})$, i.e., $\overline{L}^{-m_1+\ell}\otimes\Omega^n_{\overline{V}}$. Since $\overline{V}$ is non-singular, $\mathcal{V}$ is smooth over R. Moreover by deformation theory $\overline{L}$ lifts to a unique invertible sheaf $\mathcal{L}$ on $\mathcal{V}$ such that

$$\mathcal{O}_{\mathcal{V}}(1)\cong\mathcal{L}^{m_1+\ell}\otimes\Omega^n_{\mathcal{V}/R}.$$

Let (V^*,L^*) be the generic fibre of $(\mathcal{V},\mathcal{L})$. It is now easy to see that the rational map

$$\mathcal{V} \longrightarrow \mathcal{U}$$

defines $s_0,\cdots,s_N,\mathcal{E}_1,\mathcal{E}_2$ on $\mathcal{V}$, hence $s_0^*,\cdots,s_N^*,E_0^*,E_1^*$ on V^* such that $(U^*,F_0^*,F_1^*) = \beta(V^*,L^*,s_0^*,\cdots,s_N^*,E_0^*,E_1^*)$. QED

Heuristically, this shows that $\beta(B)$ is a "limited family", hence so is B, hence so is A. To be precise, note that all elements of B can be parametrized a suitable countably infinite set of families each defined over a base space B_α which is an algebraic variety. Then $\beta(B_\alpha)$ is at least a constructible subset of $C_0 = \mathrm{Im}\ \beta$. But assuming the ground field k is uncountable*, then a (reducible) variety C_0 which is a countable union of constructible subsets $\beta(B_\alpha)$ is also a finite union of them: hence B is a finite union of B_α's.

*The other way of arguing is to look at 2 countable algebraically closed ground field $\bar{\mathbb{Q}} \subset k$, where $\bar{\mathbb{Q}}$ = field of algebraic numbers and k has infinite transcendence degree over $\mathbb{Q}$. Considering k-rational points, we get a bijection

$$\beta:\ B(k) \longrightarrow C_0(k)$$

but each B_α may be assumed to be defined over $\bar{\mathbb{Q}}$. Apply the elementary compactness assertion: if any set of $\bar{\mathbb{Q}}$-rational constructible sets covers $C_0(k)$, a finite subset already covers $C_0(k)$.

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