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A REMARK ON THE PAPER OF M. SCHLESSINGER

by David Mumford

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In the conference itself, I spoke on a theorem asserting the existence of "semi-stable" reductions for analytic families of varieties over a disc, smooth outside the origin. This talk turned out to be difficult to transcribe into a paper of moderate size and instead will be incorporated into the notes of a seminar which I am running together with G. Kempf, B. Saint-Donat, and Tai, which we will publish in the Springer Lecture Notes.

Here I would like to add a footnote to Schlessinger's calculations of versal deformations.[1] He studied the situation: $V =$ complex $n+1$-dimensional vector space; $\mathbf{P}(V) = n$-dimensional projective space of 1-dimensional subspaces of V; $Y \subset \mathbf{P}(V)$ a smooth r-dimensional variety, $r \geqq 1$; $C \subset V$ the cone over Y.

Let $L = \mathcal{O}_Y(1)$. Assume:

$$H^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(k)) \to H^0(Y, L^k) \text{ is surjective, } k \geqq 1$$

(We may also assume by replacing $\mathbf{P}(V)$ by a linear space that it is an isomorphism for $k = 1$). Then he proved:

a) There is a natural injection of functors:

$$\bar{H} = \left\{\begin{matrix}\text{Deformations}\\ \text{of } Y \text{ in } \mathbf{P}(V)\end{matrix}\right\} \Big/ \begin{matrix}\text{projective}\\ \text{automorphisms}\end{matrix} \to \left\{\begin{matrix}\text{Deformations}\\ \text{of } C\end{matrix}\right\}$$

b) T_C^1 has a natural graded structure

$$T_C^1 = \bigoplus_{k=-\infty}^{+\infty} (T_C^1)_k$$

such that $(T_C^1)_0 \cong$ image of Zariski tangent space to $\bar{H}$,

c) If $(T_C^1)_k = (0)$ for $k \neq 0$, then $\bar{H}$ is *isomorphic* to the functor of deformations of C, i.e., all deformations of C remain conical.

d) If $r \geqq 2$ and L is sufficiently ample on Y, then the condition in (c) is satisfied.

What I would like to show here is:

d′) If $r = 1$, L is sufficiently ample on Y and Y has genus $\geqq 2$ and is not hyperelliptic, then again the condition in (c) is satisfied.

This gives:

Corollary. There exist normal singularities of surfaces with no non-singular deformation!

To prove (d′), we let $U = C - (0)$ and use the exact sequences:

$$\begin{array}{ccccc} \Gamma(V, \theta_V) & \xrightarrow{\alpha} & \Gamma(C, N_C) & \rightarrow & T_C^1 \rightarrow 0 \\ \| & & \| & & \\ \Gamma(V-(0), \theta_V) & & \Gamma(U, N_C). & & \end{array}$$

Now $\mathbf{C}^*$ acts in a natural way on both θ_V and N_C, and if $\pi\colon V-(0) \rightarrow \mathbf{P}(V)$ is the projection, then both $\pi_*\theta_V$ and π_*N_C decompose into direct sums of their eigenspaces for the various characters of $\mathbf{C}^*$. Moreover, the $\mathbf{C}^*$ invariant sections are:

$$(\pi_*\,\theta_V)^{\mathbf{C}^*} \cong \mathcal{O}_{\mathbf{P}(V)}(1) \otimes_{\mathbf{C}} V$$

$$(\pi_* N_C)^{\mathbf{C}^*} \cong N_Y$$

and α induces the map $\alpha' = \gamma \circ \beta$

$$\alpha'\colon \mathcal{O}_{\mathbf{P}(V)}(1) \otimes_C V \xrightarrow{\beta} \theta_{\mathbf{P}(V)} \xrightarrow{\gamma} N_Y$$

(β = standard map).

Thus we get:

$$\begin{array}{ccc} \Gamma(V-(0), \theta_V) & \xrightarrow{\alpha} & \Gamma(U, N_C) \\ \| & & \| \\ \bigoplus_{\nu=-\infty}^{+\infty} \Gamma(\mathbf{P}(V), \mathcal{O}(\nu+1) \otimes_C V) & \longrightarrow & \bigoplus_{\nu=-\infty}^{+\infty} \Gamma(Y, N_Y(\nu)). \end{array}$$

So if

$$(T_C^1)_\nu = \operatorname{coker}[\Gamma(\mathbf{P}^n, \mathcal{O}(\nu+1) \otimes_{\mathbf{C}} V) \xrightarrow{\alpha'_\nu} \Gamma(Y, N_Y(\nu))],$$

then $T_C^1 = \bigoplus_{\nu=-\infty}^{+\infty} (T_C^1)_\nu$. We must compute these groups.

The idea is to determine N_Y explicitly on Y without actually using the embedding of Y defined by L. Consider in fact $N_Y^*(1)$ via the dual of α' as a subbundle of $\mathcal{O}_Y \otimes_{\mathbf{C}} V^*$

$$N_Y^*(1) \subset \theta^*_{\mathbf{P}(V)}(1)\big|_Y \subset \mathcal{O}_Y \otimes_C V^*$$

hence for every $x \in Y$:

$$[N_Y^*(1) \otimes \mathcal{O}_x/m_x] \subset [\overset{*}{\mathcal{O}}_{\mathbf{P}(V)}(1) \otimes \mathcal{O}_x/m_x] \subset V^* .$$

It is easy to see that under these inclusions, if $x' \in C$ lies over x:

$$\mathcal{O}^*_{\mathbf{P}(V)}(1) \otimes \mathcal{O}_x/m_x \quad = \quad \begin{Bmatrix} \text{space of linear forms } l \text{ on } V \\ \text{such that } l(x') = 0 \end{Bmatrix}$$

$$N_Y^*(1) \otimes \mathcal{O}_x/m_x \quad = \quad \begin{Bmatrix} \text{space of linear forms } l \text{ on } V \\ \text{such that } l(x') = 0 \text{ and} \\ l = 0 \text{ is tangent to } Y \text{ at } x \end{Bmatrix} .$$

But now by assumption:

$$V^* \cong \Gamma(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(1)) \xrightarrow{\approx} \Gamma(Y, L)$$

and under this isomorphism, the linear forms l such that $l=0$ and is tangent to Y at x go over to the sections of L vanishing at x to 2nd order, i.e. $\Gamma(Y, m_x^2 \cdot L)$. Now consider

$$\begin{array}{c} \Delta \subset Y \times Y \text{ with } p_1^* L(-2\Delta) \\ \Big\downarrow p_2 \\ Y \quad \text{with } p_{2,*}[p_1^* L(-2\Delta)] . \end{array}$$

Then it is easily seen that $p_{2,*}[p_1^* L(-2\Delta)]$ is a locally free sheaf on Y and that

$$\begin{aligned} p_{2,*}[p_1^* L(-2\Delta)] \otimes \mathcal{O}_x/m_x &\cong \Gamma(Y \otimes \{y\}, p_1^* L(-2\Delta) \otimes_{\mathcal{O}_Y} \mathcal{O}_x/m_x) \\ &\cong \Gamma(Y, m_x^2 \cdot L) . \end{aligned}$$

Thus the two sub-bundles:

a) $p_{2,*}[p_1^* L(-2\Delta)] \subset p_{2,*}[p_1^* L] = \Gamma(Y, L) \otimes_{\mathbf{C}} \mathcal{O}_Y$

b) $N_Y^*(1) \subset V^* \otimes_{\mathbf{C}} \mathcal{O}_Y \cong \Gamma(Y, L) \otimes_{\mathbf{C}} \mathcal{O}_Y$

are equal. Now assume $r = 1$, so that Y is a curve and $\mathcal{O}(-2\Delta)$ is an invertible sheaf on $Y \times Y$. Then by Serre duality for the morphism p_2, we can identify $N_Y(-1)$ as a quotient of $V \otimes_{\mathbf{C}} \mathcal{O}_Y$ or $\Gamma(Y, L)^* \otimes_{\mathbf{C}} \mathcal{O}_Y$:

$$\begin{array}{ccccc} V \otimes_{\mathbf{C}} \mathcal{O}_Y & \xrightarrow{\alpha'(-1)} & N_Y(-1) & \to 0 \\ \wr\| & & \wr\| & \\ \Gamma(Y, L)^* \otimes_{\mathbf{C}} \mathcal{O}_Y & \longrightarrow & \mathrm{Hom}(p_{2,*}[p_1^* L(-2\Delta)], \mathcal{O}_Y) & \to 0 \\ \wr\| & & \wr\| & \\ R^1 p_{2,*}(\mathrm{Hom}(p_1^* L, \Omega_{Y \times Y/Y})) & \to & R^1 p_{2,*}(\mathrm{Hom}(p_1^* L(-2\Delta), \Omega_{Y \times Y/Y})) & \to 0 \\ \wr\| & & \wr\| & \\ R^1 p_{2,*}(p_1^*(\Omega_Y \otimes L^{-1})) & \longrightarrow & R^1 p_{2,*}(p_1^*(\Omega_Y \otimes L^{-1})(2\Delta)) & \to 0 . \end{array}$$

We want to show that $(T_C^1)_\nu = (0)$ if $\nu \neq 0$, i.e.,

$$\Gamma(Y, R^1 p_{2,*}[p_1^*(\Omega_Y \otimes L^{-1})] \otimes L^\nu) \to \Gamma(Y, R^1 p_{2,*}[p_1^*(\Omega_Y \otimes L^{-1})(2\Delta)] \otimes L^\nu)$$

is surjective if $\nu \neq 1$. If $\deg L > 2g$, then $p_{2,*}$ of the two sheaves in square brackets is zero, hence by the Leray spectral sequence for p_2, the above map is the same as:

$$\begin{aligned} &H^1(Y \times Y, p_1^*(\Omega_Y \otimes L^{-1}) \otimes p_2^* L^\nu) \\ &\qquad \to H^1(Y \times Y, p_1^*(\Omega_Y \otimes L^{-1}) \otimes p_2^* L^\nu \otimes \mathcal{O}(2\Delta)). \end{aligned}$$

We treat the surjectivity in three cases:

Case I. $\nu \geqq 2$: Consider the sheaf cokernel

$$p_1^*(\Omega_Y \otimes L^{-1}) \otimes p_2^* L^\nu \to p_1^*(\Omega_Y \otimes L^{-1}) \otimes p_2^* L^\nu \otimes \mathcal{O}(2\Delta) \to K_\nu \to 0.$$

It is a sheaf of $\mathcal{O}_{2\Delta}$-modules so it lies in an exact sequence between $\mathcal{O}_\Delta \cong \mathcal{O}_Y$-modules

$$\begin{array}{ccccccc} 0 \to & (\mathcal{O}(\Delta) \otimes \mathcal{O}_\Delta) \otimes L^{\nu-1} \otimes \Omega_Y & \to & K_\nu & \to & (\mathcal{O}(2\Delta) \otimes \mathcal{O}_\Delta) \otimes L^{\nu-1} \otimes \Omega_Y & \to 0 \\ & \wr\Vert & & & & \wr\Vert & \\ & L^{\nu-1} & & & & L^{\nu-1} \otimes (\Omega_Y)^{-1}. & \end{array}$$

So if $\deg L > 4g-4$, $H^1(K_\nu) = (0)$ when $\nu \geqq 2$.

Case II. $\nu = 0$: Consider the Leray spectral sequence for p_1. Since we have assumed Y is not hyperelliptic

a) $p_{1,*}\mathcal{O}_{Y \times Y}(2\Delta) \cong p_{1,*}\mathcal{O}_{Y \times Y}$

and

b) $R^1 p_{1,*}\mathcal{O}_{Y \times Y}(2\Delta)$ is a locally free sheaf $\mathcal{E}$ of rank $g-2$. Now we have:

$$\begin{array}{ccccccc} 0 \to & H^1(Y, \Omega_Y \otimes L^{-1}) & \to & H^1(Y \times Y, p_1^* \Omega_Y \otimes L^{-1}) & \to & H^0(Y, \Omega_Y \otimes L^{-1} \otimes R^1 p_{1,*}\mathcal{O}_{Y \times Y}) & \to \\ & \downarrow \wr \text{ by (a)} & & \downarrow & & \downarrow & \\ 0 \to & H^1(Y, \Omega_Y \otimes L^{-1} \otimes p_{1,*}\mathcal{O}(2\Delta)) & \to & H^1(Y \times Y, p_1^* \Omega_Y \otimes L^{-1}(2\Delta)) & \to & H^0(Y, \Omega_Y \otimes L^{-1} \otimes \mathcal{E}) & - \end{array}$$

Note that $\mathcal{E}$ does not depend on L. So by (b) there is an integer n_0 depending only on Y such that if $\deg L > n_0$, then $(\Omega_Y \otimes \mathcal{E}) \otimes L^{-1}$ has no sections.

Case III: $\nu \leqq -1$: Surjectivity in this case always follows from surjectivity when $\nu = 0$. In fact, if we know that

$$V \to \Gamma(Y, N_Y \otimes L^{-1}) \to 0$$

is surjective, I claim $\Gamma(Y, N_Y \otimes L^{-\nu}) = (0)$, $\nu \geqq 2$. If not, $N_Y \otimes L^{-2}$ has a non-zero section s. Then for all $t \in \Gamma(Y, L) \cong V^*$, $t \otimes s$ is a non-zero section of $N_Y \otimes L^{-1}$. Thus we must get *all* sections of $N_Y \otimes L^{-1}$ in this way. But this means that all these sections are proportional, hence do not generate $N_Y \otimes L^{-1}$. But since

$$V \otimes \mathcal{O}_Y \to N_Y \otimes L^{-1}$$

is surjective and $V \otimes \mathcal{O}_Y$ is generated by its sections, so is $N_Y \otimes L^{-1}$. This is a contradiction, so $s = 0$.

This completes the proof of (d'). Finally two remarks:

(A) If you look at the case $Y = \mathbf{P}^1$, $L = \mathcal{O}_{\mathbf{P}^1}(k)$, then C = cone over the rational curve of degree n in $\mathbf{P}^n$ and the sequences we have used enable us to compute T_C^1 easily. In fact it turns out that if $k \geqq 3$,

$$\begin{aligned}(T_C^1)_l &= (0), \quad \text{if } l \neq -1 \\ \dim (T_C^1)_{-1} &= 2k - 4.\end{aligned}$$

It seems most reasonable to conjecture that the versal deformation space of this C is a non-singular $k - 1$-dimensional space but with a 0-dimensional embedded component at the origin if $k \geqq 4$.[2]

(B) What happens in the hyperelliptic case? If, for instance, $\pi: Y \to \mathbf{P}^1$ is the double covering and $L = \pi^* \mathcal{O}_{\mathbf{P}^1}(k)$, then C is itself a double covering of the rational cone considered in (A) which is known to have non-singular deformations. Do these lift to deformations of this C?

NOTES

1. M. Schlessinger, "On Rigid Singularities," in this volume, pp. 147–162.

2. H. Pinkham has recently proved that this is true if $k \geqq 5$, but if $k = 4$, the versal deformation space has two components, a smooth 3-dimensional one and a smooth 1-dimensional one crossing normally at the origin! (Cf. "Deformations of cones with negative grading," *J. of Algebra*, to appear.)

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