BI-EXTENSIONS OF FORMAL GROUPS

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In the Colloquium itself, I announced that all abelian varieties can be lifted to characteristic zero. The proof of this, as sketched there, is roughly as follows.

(i) It suffices to prove that every char p abelian variety is a specialization of a char p abelian variety with multiplicative formal group (an "ordinary" abelian variety), since Serre (unpublished) has shown that these admit liftings.

(ii) A preliminary reduction of the problem was made to abelian varieties X such that the invariant

$$\alpha(X) = \dim_k \text{Hom}\,(\alpha_p, X)$$

is 1.

(iii) A method was found to construct deformations of a polarized abelian variety from deformations of its polarized Dieudonné module.

(iv) Finally, some simple deformations of polarized Dieudonné modules were constructed to establish the result.

However, it seems premature to give this proof here, since the basic method used in (iii) promises to give much fuller information on the local structure of the formal moduli space of a polarized abelian variety, and this would make my *ad hoc* method obsolete. I want instead to give some basic information on the main new technical tool which is used in (iii).

1. Cartier's result. In the note [1], Cartier has announced a module-theoretic classification of formal groups over arbitrary ground-rings R. We require only the special case where $p = 0$ in R, which is foreshadowed in Dieudonné's original paper [2], before the category men got a hold of it, modifying the technique until the restriction "R = perfect field" came to seem essential.

DEFINITION. *Let R be a ring of characteristic p. Let $W(R)$ be the ring of Witt vectors over R, and let*

$$(a_0, a_1, a_2, \ldots)^{\sigma} = (a_0^p, a_1^p, a_2^p, \ldots),$$
$$(a_0, a_1, a_2, \ldots)^{t} = (0, a_0, a_1, \ldots).$$

Then A_R will denote the ring

$$W(R)[[V]][F]$$

modulo the relations:

(a) $FV = p$,

(b) $VaF = a^t$,

(c) $Fa = a^{\sigma}F$,

(d) $aV = Va^{\sigma}$,

for all $a \in W(R)$.

THEOREM (Dieudonné-Cartier). *There is a covariant equivalence of categories between*

(A) *the category of commutative formal groups Φ over R, and*

(B) *the category of left A_R-modules M such that*

(a) $\bigcap_i V^i M = (0)$,

(b) $Vm = 0 \Rightarrow m = 0$, *all $m \in M$,*

(c) *M/VM is a free R-module of finite rank.*

The correspondence between these 2 categories can be set up as follows. Recall first that a formal group Φ/R (by which we mean a set of n power series $\phi_i(x_1, \ldots, x_n; y_1, \ldots, y_n)$, $1 \leqslant i \leqslant n$, satisfying the usual identities, c.f. Lazard [3]) defines a covariant functor F_Φ from R-algebras S to groups : i.e. $\forall\, S/R$,

$$F_\Phi(S) = \{ (a_1, \ldots, a_n) \mid a_i \in S, a_i \text{ nilpotent} \}$$

where

$$(a_1, \ldots, a_n).(b_1, \ldots, b_n) = (\phi_1(a_1, \ldots, a_n; b_1, \ldots, b_n), \ldots, \phi_n(a_1, \ldots, a_n; b_1, \ldots, b_n)).$$

N. B. In what follows, we will often call the functor F_Φ instead of the power series Φ the formal group, for simplicity.

Let $\widehat{\mathbf{W}}$ be the functor

$$\begin{cases} \widehat{\mathbf{W}}(S) = \{ (a_0, a_1, \ldots) \mid a_i \in S, a_i \text{ nilpotent, almost all } a_i = 0\}, \\ \text{gp law} = \text{Witt vector addition.} \end{cases}$$

Then we attach to the commutative formal group Φ the set

$$M = \mathrm{Hom}_{\text{gp. functors}/R}(\widehat{\mathbf{W}}, F_\Phi),$$

and since $A_R \cong \mathrm{Hom}(\widehat{\mathbf{W}}, \widehat{\mathbf{W}})^0$, we can endow M with the structure of left A_R-module. Conversely, to go in the other direction, first note that any A_R-module M as in the theorem can be resolved:

$$(*) \qquad 0 \longrightarrow A_R^n \xrightarrow{\beta} A_R^n \xrightarrow{\alpha} M \longrightarrow 0.$$

In fact, choose $m_1, \ldots, m_n \in M$ whose images mod VM are a basis of M/VM as R-module. Define

$$\alpha(P_1, \ldots, P_n) = \sum_{i=1}^{n} P_i\, m_i.$$

It is easy to check that Fm_i can be expanded in the form $\sum_{j=1}^{n} Q_{ij}(V)\, m_j$, Q_{ij} a power series in V with coefficients in $W(R)$. Define

$$\beta(P_1, \ldots, P_n) = \left(\sum_{i=1}^{n} P_i \cdot Q_{i1} - \delta_{i1} F, \ldots, \sum_{i=1}^{n} P_i \cdot Q_{in} - \delta_{in} F \right).$$

It is not hard to check that (*) is exact. Then β defines a monomorphism of group functors β^*: $(\widehat{\mathbf{W}})^n \to (\widehat{\mathbf{W}})^n$, and let F be the quotient functor $(\widehat{\mathbf{W}})^n / \beta^*(\widehat{\mathbf{W}})^n$. Then F is isomorphic to F_Φ for one and—up to canonical isomorphism—only one formal group Φ.

Moreover, we get a resolution of the functor F_Φ :

$$0 \longrightarrow (\widehat{\mathbf{W}})^n \xrightarrow{\beta^*} (\widehat{\mathbf{W}})^n \longrightarrow F_\Phi \longrightarrow 0.$$

When R is a perfect field, the above correspondence can be extended to an analogous correspondence between p-divisible groups over R and $W(R)[F, V]$-modules of suitable type (c.f. [4], [5]).

However, it does not seem likely at present that such an extension exists for non-perfect R's. This is a key point.

2. Bi-extensions of abelian groups. Let A,B,C be 3 abelian groups. A bi-extension of $B \times C$ by A will denote a set G on which A acts freely, together with a map

$$G \xrightarrow{\pi} B \times C$$

making $B \times C$ into the quotient G/A, together with 2 laws of composition:

$$+_1 : G \underset{B}{\times} G \to G \quad ; \quad +_2 : G \underset{C}{\times} G \to G$$

$$\text{def}\Big\| \qquad\qquad \text{def}\Big\|$$

$\{(g_1, g_2) \mid \pi(g_1), \pi(g_2)$ have same B-component$\}$ $\quad$ $\{(g_1, g_2) \mid \pi(g_1), \pi(g_2)$ have some C-component$\}$

These are subject to the requirements:

(i) for all $b \in B$, $G'_b = \pi^{-1}(b \times C)$ is an abelian group under $+_1$, π is a surjective homomorphism of G'_b onto C, and via the action of A on G'_b, A is isomorphic to the kernel of π;

(ii) for all $c \in C$, $G^2_c = \pi^{-1}(B \times c)$ is an abelian group under $+_2$, π is a surjective homomorphism of G^2_c onto B, and via the action of A on G^2_c, A is isomorphic to the kernel of π;

(iii) given $x, y, u, v \in G$ such that

$$\pi(x) = (b_1, c_1)$$
$$\pi(y) = (b_1, c_2)$$
$$\pi(u) = (b_2, c_1)$$
$$\pi(v) = (b_2, c_2),$$

then

$$(x +_1 y) +_2 (u +_1 v) = (x +_2 u) +_1 (y +_2 v).$$

This may seem like rather a mess, but please consider the motivating example: let X be an abelian variety over an algebraically closed field k, let $\hat{X}$ be its dual, and let P be the universal, or Poincaré, line bundle on over $X \times \hat{X}$. Then P_k, the underlying set of closed points of P, is a bi-extension of $X_k \times \hat{X}_k$ by k^* !

Notice that if G is a bi-extension of $B \times C$ by A, then $\pi^{-1}(B \times 0)$ splits canonically into $A \times B$, and $\pi^{-1}(0 \times C)$ splits canonically into $A \times C$. In fact, we can lift B to $\pi^{-1}(B \times 0)$ by mapping $b \in B$ to the element of G which is the identity in $\pi^{-1}(b \times C)$; and we can lift C to $\pi^{-1}(0 \times C)$ by mapping $c \in C$ to the element of G which is the identity in $\pi^{-1}(B \times c)$.

Bi-extensions can be conveniently described by co-cycles: choose a (set-theoretic) section

$$G \underset{\pi}{\overset{s}{\rightleftarrows}} B \times C$$

Via s and the action of A on G, we construct an isomorphism

$$G \cong A \times B \times C$$

such that the action of A on G corresponds to the action of A on $A \times B \times C$ which is just addition of A-components, leaving the B- and C-components fixed. Then $+_1$ and $+_2$ go over into laws of composition on $A \times B \times C$ given by:

$$(a, b, c) +_1 (a', b, c') = (a + a' + \phi(b; c, c'), b, c + c')$$

$$(a, b, c) +_2 (a', b', c) = (a + a' + \psi(b, b'; c), b + b', c).$$

For $+_1$, $+_2$ to be abelian group laws, we need:

(a) $$\phi(b; c + c', c'') + \phi(b; c, c') = \phi(b; c, c' + c'') + \phi(b; c', c'')$$
$$\phi(b; c, c') = \phi(b; c', c);$$

(b) $$\psi(b + b', b''; c) + \psi(b, b'; c) = \psi(b, b' + b''; c) + \psi(b', b''; c)$$
$$\psi(b, b'; c) = \psi(b', b; c).$$

The final restriction comes out as:

(c) $$\phi(b+b';c,c') - \phi(b;c,c') - \phi(b';c,c')$$
$$= \psi(b,b';c+c') - \psi(b,b';c) - \psi(b,b';c').$$

What are the co-boundaries? If you alter s by adding to it a map $\rho : B \times C \to A$, then you check that the new ϕ', ψ' are related to the old ones by

$$\phi'(b;c,c') - \phi(b;c,c') = \rho(b,c+c') - \rho(b,c) - \rho(b,c')$$
$$\psi'(b,b';c) - \psi(b,b';c) = \rho(b+b',c) - \rho(b,c) - \rho(b',c).$$

Using this explicit expression by co-cycles and co-boundaries, it is clear that the set of all bi-extensions of $B \times C$ by A forms itself an abelian group, which we will denote

$$\text{Bi-ext } (B \times C, \ A).$$

It is also clear, either from the definition or via co-cycles, that Bi-ext is a covariant functor in A, and a contravariant functor in B and C.

3. Bi-extensions of group-functors.

DEFINITION. *If F,G,H are 3 covariant functors from the category of R-algebras to the category of abelian groups, a bi-extension of $G \times H$ by F is a fourth functor K such that for every R-algebra S, $K(S)$ is a bi-extension of $G(S) \times H(S)$ by $F(S)$ and for every R-homomorphism $S_1 \to S_2$, the map $K(S_1) \to K(S_2)$ is a homomorphism of bi-extensions (in the obvious sense). In particular, if F,G,H are formal groups, this gives us a bi-extension of formal groups.*

If F,G,H are formal groups, it is easy again to compute the bi-extensions K by power series co-cycles. In fact, one merely has to check that:

(i) there is a functorial section

$$K \underset{\pi}{\overset{s}{\rightleftarrows}} G \times H$$

(this follows using the "smoothness" of the functor F, i.e. $F(S) \to F(S/I)$ is surjective if I is a nilpotent ideal);

(ii) any morphism of functors from one product of formal groups to another such product is given explicitly by a set of power series over R in the appropriate variables.

In fact, we will be exclusively interested in the case where $F = \widehat{\mathbf{G}}_m$ is the formal multiplicative group; that is

$$\widehat{\mathbf{G}}_m(S) = \left\{ \begin{array}{l} \text{Units in } S \text{ of form } 1+x,\ x \text{ nilpotent,} \\ \text{composed via multiplication.} \end{array} \right\}$$

Then if G and H are formal groups in variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$, a bi-extension of $G \times H$ by $\widehat{\mathbf{G}}_m$ is given by 2 power series $\sigma(x_1, \ldots, x_n; y_1, \ldots, y_m, y_1', \ldots, y_m')$, $\tau(x_1, \ldots, x_n, x_1', \ldots, x_n'; y_1, \ldots, y_m)$ with constant terms 1 such that—abbreviating n-tuples and m-tuples:

$$\sigma(x; \Phi(y, y'), y'').\sigma(x; y, y') = \sigma(x; y, \Phi(y', y'')).\sigma(x; y', y'')$$

$$\sigma(x; y, y') = \sigma(x; y', y)$$

$$\tau(\Psi(x, x'), x'', y).\tau(x, x'; y) = \tau(x, \Psi(x', x''); y).\tau(x', x''; y)$$

$$\tau(x, x'; y) = \tau(x', x; y)$$

$$\sigma(\Psi(x, x'); y, y').\sigma(x; y, y')^{-1}.\sigma(x'; y, y')^{-1} = \tau(x, x'; \Phi(y, y')). \tau(x, x'; y)^{-1}. \tau(x, x'; y')^{-1},$$

if Φ, Ψ are the group laws of G and H respectively.

We want one slightly non-trivial fact about general bi-extensions. This result gives essentially the method for computing Bi-ext's via resolutions.

PROPOSITION 1. *Let E, G, G' be abelian group functors as above. Suppose*

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow G \longrightarrow 0$$

$$0 \longrightarrow F_1' \longrightarrow F_0' \longrightarrow G' \longrightarrow 0$$

are 2 exact sequences of such functors. Then

$$\mathrm{Ker}\{\text{Bi-ext}\,(G \times G', E) \longrightarrow \text{Bi-ext}\,(F_0 \times F_0', E)\}$$

$$\cong \frac{\{(f,g) \mid f\colon F_0 \times F_1' \to E \text{ and } g : F_1 \times F_0' \to E \text{ bi-homomorphisms } \operatorname{res} f = \operatorname{res} g \text{ on } F_1 \times F_1'\}}{\{(f,g) \mid \exists h\colon F_0 \times F_0' \to E \text{ bi-homomorphism, } f \text{ and } g \text{ restrictions of } h\}}$$

The proof goes along these lines: let H be a bi-extension of $G \times G'$ by E. If it lies in the above kernel, then the induced bi-extension of $F_0 \times F_0'$ is trivial:

$$H \underset{(G \times G')}{\times} (F_0 \times F_0') \cong E \times F_0 \times F_0'.$$

Consider the equivalence relation on the functor $E \times F_0 \times F_0'$ induced by the mapping of it onto H. It comes out that there are maps $f\colon F_0 \times F_1' \to E$, $g\colon F_1 \times F_0' \to E$ such that this equivalence relation is generated by

$$(a,b,c) \sim (a + f(b,\bar{c}),\, b,\, c + \bar{c}),\ a \in E(S),\ b \in F_0(S)$$
$$c \in F_0'(S),\ \bar{c} \in F_1'(S). \quad (1)$$

and

$$(a,b,c) \sim (a + g(\bar{b},c),\, b + \bar{b},\, c),\quad a \in E(S),\ b \in F_0(S)$$
$$\bar{b} \in F_1(S),\ c \in F_0'(S). \quad (2)$$

Moreover, f and g have to be bi-homomorphisms with $\operatorname{res} f = \operatorname{res} g$ on $F_1 \times F_1'$. Conversely, given such f and g, define the functor H to be the quotient of $E \times F_0 \times F_0'$ by the above equivalence relation. H turns out to be a bi-extension. Finally, the triviality of H can be seen to be equivalent to f and g being the restrictions of a bi-homomorphism $h\colon F_0 \times F_0' \to E$.

4. Bi-extensions of $\hat{\mathbf{W}}$.

PROPOSITION 2. Bi-ext $(\hat{\mathbf{W}} \times \hat{\mathbf{W}}, \hat{\mathbf{G}}_m) = (0)$.

PROOF. Consider functors F from (R-algebras) to (abelian groups) which are isomorphic as set functors to D^I, where

$$D^I(S) = \{(a_i) \mid a_i \in S, \text{ all } i \in I,\ a_i \text{ nilpotent, almost all } a_i = 0\}$$

and where I is an indexing set which is either finite or countably infinite. Note that all our functors are of this type. Then I claim that for all R of char p, all such F, there is a canonical retraction p_F:

$$\operatorname*{Hom}_{\text{set-functors}}(\hat{\mathbf{W}}, F) \underset{\text{inclusion}}{\overset{p_F}{\rightleftarrows}} \operatorname*{Hom}_{\text{gp-functors}}(\hat{\mathbf{W}}, F)$$

which is functorial both with respect to (1) any homomorphism $F \to G$, and (2) base changes $R_1 \to R_2$.

The construction of p_F is based on Theorem 1 of Cartier's note [1]. Let $\hat{\mathbf{W}}^*$ be the full Witt group functor (i.e. based on all positive integers, rather than powers of p), and let i: $D \to \hat{\mathbf{W}}^*$ be the canonical inclusion used in [1]. Then Theorem 1 asserts that for all formal groups F, every morphism $\phi: D \to F$ extends uniquely to a homomorphism u: $\hat{\mathbf{W}}^* \to F$.

$$\begin{array}{ccc} D & \xrightarrow{\ \phi\ } & F \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle u} & \\ \hat{\mathbf{W}}^* & & \end{array}$$

Cartier informs me that this theorem extends to all F's of our type. On the other hand, $\hat{\mathbf{W}}$, over a ring of char p, is a direct summand of $\hat{\mathbf{W}}^*$:

$$\hat{\mathbf{W}}^* \underset{\pi}{\overset{j}{\rightleftarrows}} \hat{\mathbf{W}}.$$

Construct p_F as follows: given $f: \hat{\mathbf{W}} \to F$, let $\phi =$ res to D of $f \circ \pi$; let $u =$ extension of ϕ to a homomorphism u; let $p_F(f) = u \circ j$.

Now let F be a bi-extension of $\hat{\mathbf{W}} \times \hat{\mathbf{W}}$ by $\hat{\mathbf{G}}_m$. For every R-algebra S and every $a \in \hat{\mathbf{W}}(S)$, let F'_a (resp F''_a) denote the fibre functor of F over $\{a\} \times \hat{\mathbf{W}}$ (resp $\hat{\mathbf{W}} \times \{a\}$) (i.e. $F_a(T) = \{b \in F(T) \mid 1^{\text{st}}$ (resp 2^{nd}) component of $\pi(b)$ is induced by a via $S \to T\}$). Then F'_a and F''_a are group functors of the good type extending $\hat{\mathbf{W}}$ by $\hat{\mathbf{G}}_m$ over ground ring S. Now since $\hat{\mathbf{G}}_m$ is smooth, one can choose a section s to π:

$$F \underset{\pi}{\overset{s}{\rightleftarrows}} \hat{\mathbf{W}} \times \hat{\mathbf{W}}$$

s restricts to morphisms $s_a : \hat{\mathbf{W}}/S \to F'_a$, for all $a \in \hat{\mathbf{W}}(S)$. Take $p_{F'_a}(s_a)$. As a varies, these fit together into a new section $p'(s)$ to π. But $p'(s)$ is now a homomorphism with respect to addition into the 2nd variable, i.e.

$$p'(s)(u, v) +_1 p'(s)(u, v') = p'(s)(u, v + v'). \qquad (*)'$$

Now switch the 2 factors: $p'(s)$ restricts to morphism

$p'(s)_a : \hat{\mathbf{W}}/S \to F''_a$, for all $a \in \hat{\mathbf{W}}(S)$. Take $p_{F''_a}(p'(s)_a)$. As a varies, these fit together into a new section $p''(p'(s))$ to π.

Then this satisfies :

$$p''(p'(s))(u, v) +_2 p''(p'(s))(u', v) = p''(p'(s))(u + u', v). \qquad (*)''$$

But now, using the functoriality of p, and the property of bi-extensions linking $+_1$ and $+_2$, it falls out that $p''(p'(s))$ still has property $(*)'$ enjoyed by $p'(s)$! So $p''(p'(s))$ preserves both group laws and splits the extension F. Q.E.D.

Definition. *A_R will denote the ring $W(R)[[F, V]]$ modulo the relations*

(a) $FV = p$

(b) $VaF = a^t$

(c) $Fa = a^\sigma F$

(d) $aV = Va^\sigma$, *all* $a \in W(R)$.

Every element in this ring can be expanded uniquely in the form:

$$P = a_0 + \sum_{i=1}^{\infty} V^i a_i + \sum_{i=1}^{\infty} a_{-i} F^i.$$

For every such P, let

$$P^* = a_0 + \sum_{i=1}^{\infty} a_i F^i + \sum_{i=1}^{\infty} V^i a_{-i}.$$

Then * is an anti-automorphism of $\bar{A}_k$ of order 2. We shall consider $\bar{A}_R$ as an $A_R \times A_R$-module via

$$(P, Q).x = P.x.Q^*. \tag{*}$$

PROPOSITION 3. Bi-hom$_R$ $(\hat{\mathbf{W}} \times \hat{\mathbf{W}}, \hat{\mathbf{G}}_m) \cong \bar{A}_R$.

Moreover, since $A_R = \mathrm{Hom}_R(\hat{\mathbf{W}}, \hat{\mathbf{W}})^0$, *the left-hand side is an* $A_R \times A_R$*-module; under the above isomorphism, this structure corresponds to the* $A_R \times A_R$*-module structure on* $\bar{A}_R$ *defined by* (*).

PROOF. Cartier [1] has shown that for all R, the Artin-Hasse exponential defines isomorphisms

$$\mathrm{Hom}_R(\hat{\mathbf{W}}, \hat{\mathbf{G}}_m) \cong \mathbf{W}(R)$$

where $\mathbf{W}$ is the full Witt functor

$$\begin{cases} \mathbf{W}(R) = \{(a_0, a_1, \dots) \mid a_i \in R\} \\ \text{group law} = \text{addition of Witt vectors.} \end{cases}$$

Therefore,

$$\text{Bi-Hom}_R(\hat{\mathbf{W}} \times \hat{\mathbf{W}}, \hat{\mathbf{G}}_m) \cong \mathrm{Hom}_R(\hat{\mathbf{W}}, \mathbf{W}).$$

Define a homomorphism

$$\bar{A}_R \xrightarrow{\phi} \mathrm{Hom}_R(\hat{\mathbf{W}}, \mathbf{W})$$

by

$$P \longrightarrow \text{the map } [b \longmapsto P(b)].$$

Here $P(b)$ means that V and F operate on Witt vectors in the usual way: note that the doubly infinite series P operates on b since b has only a finite number of components and all are nilpotent, whereas $P(b)$ is allowed to have all components non-zero.

Let

$$\hat{\mathbf{W}}_n(R) = \{(a_0, a_1, \dots) \mid a_i^{p^n} = 0, \text{ all } i\,; \text{ almost all } a_i = 0\}.$$

Notice that

$$\mathrm{Hom}_R(\hat{\mathbf{W}}, \mathbf{W}) \cong \varprojlim_n \mathrm{Hom}_R(\hat{\mathbf{W}}_n, \mathbf{W}),$$

and that ϕ factors through maps

$$\bar{A}_R/\bar{A}_R.F^n \xrightarrow{\phi_n} \mathrm{Hom}_R(\hat{\mathbf{W}}_n, \mathbf{W}).$$

It suffices to show that ϕ_n is an isomorphism for all n. But for $n=1$, $\bar{A}_R/\bar{A}_R.F \cong R[[V]]$, while

$$\mathrm{Hom}_R(\hat{\mathbf{W}}_1, \mathbf{W}) \cong \mathrm{Hom}_{p\text{-Lie algebras}}\ (\mathrm{Lie}\,(\hat{\mathbf{W}}),\ \mathrm{Lie}\,(\mathbf{W})).$$

Also Lie $(\hat{\mathbf{W}})$ is the free R-module on generators $\hat{e}_0, \hat{e}_1, \hat{e}_2, \ldots$ with $\hat{e}_i^{(p)} = \hat{e}_{i+1}$; and Lie $(\mathbf{W})$ is the R-module of all expressions $\sum_{i=0}^{\infty} a_i e_i$, $a_i \in R$, with same p^{th} power map. Moreover $\sum_{i=0}^{\infty} V^i a_i \in R[[V]]$ goes via ϕ_1 to the lie algebra map taking $\hat{e}_0$ to $\sum_{i=0}^{\infty} a_i e_i$. Thus ϕ_1 is an isomorphism. Now use induction on n, and the exact sequences

$$0 \longrightarrow \hat{\mathbf{W}}_{n-1} \longrightarrow \hat{\mathbf{W}}_n \xrightarrow{F^{n-1}} \hat{\mathbf{W}}_1 \longrightarrow 0.$$

This leads to the diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_R(\hat{\mathbf{W}}_1,\mathbf{W}) & \xrightarrow{\circ F^{n-1}} & \mathrm{Hom}_R(\hat{\mathbf{W}}_n,\mathbf{W}) & \longrightarrow & \mathrm{Hom}_R(\hat{\mathbf{W}}_{n-1},\mathbf{W}) & \\
& \uparrow \phi_1 & & \uparrow \phi_n & & \uparrow \phi_{n-1} & \\
0 \longrightarrow & \bar{A}_R/\bar{A}_R.F & \xrightarrow{\times F^{n-1}} & \bar{A}_R/\bar{A}_R.F^n & \longrightarrow & \bar{A}_R/\bar{A}_R.F^{n-1} & \longrightarrow 0.
\end{array}$$

The bottom line is easily seen to be exact, so if ϕ_1 and ϕ_{n-1} are isomorphisms, the diagram implies that ϕ_n is an epimorphism.

Q.E.D.

COROLLARY. *Let F_1 and F_2 be group functors isomorphic to $(\hat{\mathbf{W}})^{n_i}$ for some n_1, n_2. Let $M_i = \mathrm{Hom}_R(\hat{\mathbf{W}}, F_i)$ be the corresponding finitely generated, free A_R-module. Then there is a $1-1$ correspondence between bi-homomorphisms*

$$B : F_1 \times F_2 \to \hat{\mathbf{G}}_m$$

and maps

$$\beta: M_1 \times M_2 \to \bar{A}_R,$$

bi-linear in the following sense:

$$\beta(Pm, Qn) = P.(m,n).Q^*$$

(all $m \in M_1$, $n \in M_2$, $P, Q \in A_R$*).*

5. Applications. Putting Propositions 1, 2 and 3 together, we conclude the following

COROLLARY.

(a) *Let* Φ, Ψ *be formal groups over* R.

(b) *Let* M, N *be the corresponding Dieudonné modules.*

(c) *Let*

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow N \longrightarrow 0$$

be resolutions of M *and* N *by finitely generated, free* A_R*-modules. Then the group* $\text{Bi-ext}_R(\Phi \times \Psi, \hat{\mathbf{G}}_m)$ *of bi-extensions of formal groups can be computed as the set of pairs of bi-linear maps:*

$$\beta: F_0 \times G_1 \to \bar{A}_R,$$

$$\gamma: F_1 \times G_0 \to \bar{A}_R,$$

such that $\beta = \gamma$ *on* $F_1 \times G_1$, *taken modulo restrictions of bi-linear maps* $\alpha: F_0 \times G_0 \to \bar{A}_R$.

In another direction, bi-extensions can be linked to p-divisible groups, as defined by Tate [6].

PROPOSITION 4. *Let* F *and* F' *be formal groups over a* char p *ring* R. *Assume that the subgroups* G_n *(resp* G'_n*)* $= \text{Ker}(p^n$ *in* F *(resp* F'*)) form* p*-divisible groups over* R*(i.e.* F *and* F' *are "equi-dimensional", or of "finite height"). Then there is a* $1-1$ *correspondence between* (1) *bi-extensions of* $F \times F'$ *by* $\hat{\mathbf{G}}_m$ *and* (2) *sets of bi-homomorphisms* $\beta_n: G_n \times G'_n \to \mu_{p^n}$, *such that for all* $x \in G_{n+1}(S)$, $y \in G'_{n+1}(S)$,

$$\beta_n(px, py) = \beta_{n+1}(x,y)^p.$$

PROOF. We will use descent theory and existence of quotients by finite, flat equivalence relations: c.f. Raynaud's article in the same volume as Tate's talk [6]. Starting with the β_n's, let L_n be the quotient functor in the flat topology of $\hat{\mathbf{G}}_m \times G_n \times G'_{2n}$ by the equivalence relation:

$$(\lambda, x, y) \sim (\lambda . \beta_n(x, b), x, y + b)$$

where $\lambda \in \hat{\mathbf{G}}_m(S)$ $x \in G_n(S)$, $y \in G'_{2n}(S)$, $b \in G'_n(S)$. Then L_n is a bi-extension of $G_n \times G'_n$ by $\hat{\mathbf{G}}_m$. Moreover, L_n is a subfunctor of L_{n+1}, so if we let L be the direct limit of the functor L_n, then L is a bi-extension of $F \times F'$ by $\hat{\mathbf{G}}_m$.

Conversely, if we start with L, let L_n be the restriction of L over $G_n \times G'_n$. In the diagram

$$\begin{array}{ccc} & \overset{\phi}{\dashrightarrow} & L_n \\ & & \downarrow \pi \\ G_n \times G'_{2n} & \xrightarrow[1 \times p^n]{} & G_n \times G'_n, \end{array}$$

I want to define a canonical map ϕ which is a homomorphism in both variables, i.e. which splits the induced bi-extension over $G_n \times G'_{2n}$. Suppose $x \in G_n(S)$, $y \in G'_n(S)$ for some R-algebra S. Choose $z_1 \in L(S)$ such that $\pi(z_1) = (x,y)$. If we add z_1 to itself p^n times in the 1st variable, we obtain a point:

$$[p^n]_{+_1}(z_1) = z_2$$
$$\pi(z_2) = (0,y).$$

But $\pi^{-1}((0) \times F')$ is canonically isomorphic to $\hat{\mathbf{G}}_m \times (0) \times F'$, so $z_2 = (\lambda, 0, y)$, some $\lambda \in \hat{\mathbf{G}}_m(S)$. Now choose a finite flat S-algebra S' such that $\lambda = \mu^{p^n}$ for some $\mu \in \hat{\mathbf{G}}_m(S')$. Letting z_1 also denote the element of $L(S')$ induced by z_1, define $z'_1 = \mu^{-1}.z_1$. This is a new point of L over (x, y), which now satisfies $[p^n]_{+1}(z'_1) = (1,0,y)$. Now add z'_1 to itself p^n times in the 2nd variable. This gives a point

$$[p^n]_{+_2}(z_1') = z_3' \in L_n^*(S'),$$
$$\pi(z_3') = (x, p^n y).$$

Clearly, z_3' is independent of the choice of μ, so by descent theory, z_3' must be induced by a unique element $z_3 \in L_n(S)$. Define $\phi(x, y) = z_3$. It is easy to check that ϕ is a homomorphism in both variables.

We can use ϕ to set up a fibre product diagram:

$$\begin{array}{ccc} \hat{\mathbf{G}}_m \times G_n \times G'_{2n} & \xrightarrow{\alpha} & L_n \\ \big\downarrow \pi & & \big\downarrow \pi \\ G_n \times G'_{2n} & \xrightarrow[(1 \times p^n)]{} & G_n \times G'_n \end{array}$$

where α is a homomorphism of bi-extensions. Since p^n is faithfully flat, so is α, and L_n is therefore the quotient of $\hat{\mathbf{G}}_m \times G_n \times G'_{2n}$ by a suitable flat equivalence relation. For every $x \in G_n(S)$, $y \in G'_{2n}(S)$, $b \in G'_n(S)$ and $\lambda \in \hat{\mathbf{G}}_m(S)$, there is a unique element $\beta_n(x, y, b, \lambda) \in \hat{\mathbf{G}}_m(S)$ such that

$$\alpha((\lambda, x, y)) = \alpha((\lambda . \beta_n(x, y, b, \lambda), x, y + b)$$

and this function β_n describes the equivalence relation. Using the fact that α is a homomorphism of bi-extensions, we deduce

(1) that β_n does not depend on λ,

(2) $\beta_n(x, y, b).\beta_n(x, y+b, b') = \beta_n(x, y, b + b')$ (via associativity of equivalence relation),

(3) $\beta_n(x, y, b).\beta_n(x', y, b) = \beta_n(x+x', y, b)$ (α preserves $+_1$),

(4) $\beta_n(x, y, b).\beta_n(x, y', b') = \beta_n(x, y+y', b+b')$ (α preserves $+_2$).

By (4) and (2) with $b = y' = 0$,

$$\beta_n(x, y, 0).\beta_n(x, 0, b') = \beta_n(x, y, b') = \beta_n(x, y, 0).\beta_n(x, y, b'),$$

hence β_n is independent of y too. Then (3) and (4) show that β_n is a bi-homomorphism, so L_n is constructed from a β_n as required.

We leave it to the reader to check that if we start from a set of β_n's, and construct a bi-extension L, then the above procedure leads you back to these same β_n's. Q.E.D.

I think that with these results, bi-extensions can be applied to the problem of determining the local structure of the moduli space of polarized abelian varieties.

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