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Enriques' classification of surfaces in char p: I

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The principal assertion in Enriques' classification of surfaces is:

THEOREM. *Let F be a non-singular projective surface, without exceptional curves of the* 1^{st} *kind. Let K_F be the canonical divisor class on F. Then*

(i) *if* $|12K_F| = \emptyset$, *then F is ruled,*

(ii) *if* $|12K_F| \neq \emptyset$, *then either* $12K_F \equiv 0$, *or else* $|nK_F|$ *is a linear system without base points for some n.*

As a Corollary, if we introduce the notations:

(a) $\mathcal{K} = \operatorname{tr} d_k \sum_{n=0}^{\infty} H^0(F, o_F(nK)) - 1$

(b) F *elliptic* if $\exists$ a morphism $f: F \to C$, C a curve, with almost all fibres non-singular elliptic curves,

(c) F *quasi-elliptic* if $\exists$ a morphism $f: F \to C$, C a curve, with almost all fibres singular rational curves E with $p_a(E) = 1$,[1]

(d) F of *general type* if $\exists f: F \to F_0$, f birational F_0 normal with K_{F_0} ample. Then we find that there are 4 types of surfaces F,

$\mathcal{K} = -1$	F ruled
$\mathcal{K} = 0$	$12K_F \equiv 0$
$\mathcal{K} = 1$	F elliptic, or quasi-elliptic, with K_F of positive degree
$\mathcal{K} = 2$	F of general type .

All this, plus a detailed analysis of the case $\mathcal{K} = 0$, has been proven in char 0; it is due essentially to Enriques [1], and has been worked out in detail by Kodaira [4], [5], in Safarevic's seminar [7], and in Zariski's seminar at Harvard. The purpose of this note and its sequel is to supply some new ideas which make the proof work in char p. Some of the steps supply new proofs of parts of the theorem in char 0, which have, I believe, some interest.

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[1] In view of the results of Tate [11], such surfaces can only occur if $\operatorname{char}(k) = 2$ or 3; moreover, almost all fibres E have a single ordinary cusp.

LIST OF NOTATION

F = non-singular projective surface
K = canonical divisor class on F
c_2 = 2^{nd} Chern class of F (a number)
q = dimension of Albanese of F
B_2 = 2^{nd} Betti number of F, $\dim H^2(F, \mathbf{R})$
ρ = Base number of F
Ω_F^i = sheaf of i-forms on F,
$h^{p,q} = \dim H^q(\Omega_F^p)$
$p_g = h^{2,0} = h^{0,2}$
$p_a = h^{0,2} - h^{0,1} = \chi(o_F) - 1$
$p_a(D) = 1 + \frac{(D \cdot D + K)}{2}$, if D is a curve on F

1. $(K \cdot D) < 0$ for some effective divisor D

The situation $(K \cdot D) < 0$, some D, is known classically as the case "*adjunction terminates*." In this case, we shall prove that F is ruled.

Step (I). *There is an ample H such that $(K \cdot H) < 0$.*

PROOF. In fact, first replacing D by a suitable component of itself, we find an *irreducible* D such that $(K \cdot D) < 0$. If $(D^2) < 0$, then since

$$(K \cdot D) + (D^2) = 2p_a(D) - 2 \geqq -2 ,$$

it follows that $(K \cdot D) = (D^2) = -1$, i.e., D is exceptional of 1^{st} kind. This has been excluded, so $(D^2) \geqq 0$. Let H_1 be any ample divisor on F. Then for all $n \geqq 0$, $nD + H_1$ is ample by the Nakai-Moisezon criterion of ampleness [2]. But if $n \gg 0$, $(K \cdot nD + H_1) < 0$. q.e.d.

COROLLARY OF *Step* I. $|nK| = \emptyset$, *all* $n \geqq 1$.

Step (II). *If $(K^2) > 0$, then F is rational, hence ruled.*

PROOF. Use the general formulas

$$12(\chi(o_F)) = (K^2) + c_2$$
$$c_2 = 2 - 4q + B_2 .$$

In our case, $p_g = \dim |K| + 1 = 0$, hence it follows that the Picard scheme of F is reduced [6, lecture 26]. Therefore

$$\begin{aligned}\dim H^1(o_F) &= \dim (\text{tangent space to Picard scheme}) \\ &= \dim (\text{Picard scheme}) \\ &= q\,,\end{aligned}$$

and so

$$\begin{aligned}\chi(o_F) &= \dim H^0(o_F) - \dim H^1(o_F) + \dim H^2(o_F) \\ &= 1 - q + p_g \\ &= 1 - q\,.\end{aligned}$$

Therefore

$$12 - 12q = (K^2) + 2 - 4q + B_2\,,$$

or

$$(*) \qquad 10 = 8q + (K^2) + B_2\,.$$

But if ρ = base number of F, then $B_2 \geqq \rho > 0$ (Igusa [12]), so if $(K^2) > 0$, it follows that $q = 0$ or 1. If $q = 1$, then F admits a morphism onto an elliptic curve, hence $B_2 \geqq 2$ and $(*)$ still cannot be satisfied. But if $q = 0$, then since $|2K| = \emptyset$, the hypotheses of Castelnuovo's criterion are met, and by Zariski [9], it follows that F is rational. q.e.d.

Step III. (Kodaira). *If* $(K^2) \leqq 0$, *then for all n there are effective divisors D on F such that*

(a) $|D + K| = \emptyset$

(b) $\dim |D| \geqq n$.

PROOF. Let H be an ample divisor such that $(K \cdot H) < 0$. For all n, note that $(nH + mK.\, H) < 0$ and hence $nH + mK$ cannot be linearly equivalent to an effective divisor, if $m \gg 0$. Let m_n be a non-negative integer such that

$$\begin{aligned}|nH + m_nK| &\neq \emptyset \\ |nH + (m_n + 1)K| &= \emptyset\,.\end{aligned}$$

Let $D_n \in |nH + m_nK|$. Write $D_n = D'_n + D''_n$, where D'_n and D''_n are positive and the components E of D'_n satisfy $(E.K) < 0$, while those of D''_n satisfy $(E \cdot K) \geqq 0$. Note that $(E \cdot K) < 0 \Rightarrow (E^2) \geqq 0$ (cf. *Step* I) so $(D'^2_n) \geqq 0$. Next, note that $|K - D'_n| = \emptyset$. In fact if not, K itself would be effective, since it would be the sum of D'_n and an effective divisor in $K - D'_n$. So by Serre duality, $H^2(o_F(D'_n)) = (0)$. Now use Riemann-Roch.

$$\begin{aligned}\dim |D'_n| &= \dim H^0(o_F(D'_n)) - 1 \\ &\geqq \chi(o_F(D'_n)) - 1 \\ &= \frac{(D'_n \cdot D'_n - K)}{2} + \chi(o_F) - 1 \\ &\geqq -\frac{(D'_n \cdot K)}{2} + \chi(o_F) - 1 \\ &\geqq -\frac{(D_n \cdot K)}{2} + \chi(o_F) - 1 \\ &= -n\frac{(H \cdot K)}{2} - m_n\frac{(K^2)}{2} + \chi(o_F) - 1 \\ &\geqq \frac{n}{2} + \chi(o_F) - 1 .\end{aligned}$$

Since $|K + D'_n| = \emptyset$, D'_n has all the required properties. q.e.d.

Now comes the new idea to take care of char p.

Key Step IV. *If D is an effective divisor such that $|K+D| = \emptyset$, then the natural map*

$$\mathrm{Pic}^0_F \longrightarrow \mathrm{Pic}^0_D$$

is surjective.

PROOF. $H^0(o_F(K + D)) = (0)$ implies $H^2(o_F(-D)) = (0)$ by Serre duality. Using the exact sequence

$$0 \longrightarrow o_F(-D) \longrightarrow o_F \longrightarrow o_D \longrightarrow 0 ,$$

it follows that the natural map $H^1(o_F) \rightarrow H^1(o_D)$ is surjective. But these vector spaces are the tangent spaces at 0 to the connected and reduced schemes group Pic^0_F and Pic^0_D. (Pic^0_F is reduced since $p_g = 0$!). Hence Pic^0_F maps onto Pic^0_D. q.e.d.

Step (V). *If D is an effective divisor such that $|K + D| = \emptyset$, and if $D = \sum n_i E_i$, then*

(i) *all E_i are non-singular*

(ii) *if $n_i \geqq 2$, then*

(a) *E_i is rational or*

(b) *E_i is elliptic, $(E_i^2) = 0$ with non-trivial normal bundle or*

(c) $(E_2^i) < 0$

(iii) *the E_i are connected together without loops.*

PROOF. In fact, by Step (IV), Pic^0_D is an abelian variety. Since the natural map $\mathrm{Pic}^0_D \rightarrow \mathrm{Pic}^0_{E_i}$ is surjective, $\mathrm{Pic}^0_{E_i}$ is abelian too, so

E_i is non-singular. If (iii) were false, Pic_D^0 would have subgroups of type $\mathbf{G}_m$ coming from the loops. To check (ii), use the fact that if $k_i \geqq 2$, $\mathrm{Pic}_D^0 \to \mathrm{Pic}_{2E_i}^0$ is surjective, hence $\mathrm{Pic}_{2E_i}^0$ is abelian, hence $\mathrm{Pic}_{E_i}^0 = \mathrm{Pic}_{2E_i}^0$. Looking at tangent spaces at 0, this implies that $H^1(o_{2E_i}) \cong H^1(o_{E_i})$. But *via* the exact sequence

$$0 \longrightarrow o_{E_i}(-E_i^2) \longrightarrow o_{2E_i} \longrightarrow o_{E_i} \longrightarrow 0 ,$$

this implies

$$H^1(o_{E_i}(-E_i^2)) = (0) .$$

But if $(E_i^2) \geqq 0$, $o_{E_i}(-(E_i^2))$ has degree $\leqq 0$, hence $H^1 \neq (0)$ except in the cases E_i rational, $o_{E_i}(-(E_i^2))$ of degree -1, 0; or E_i elliptic, $(E_i^2) = 0$ and $o_{E_i}(-(E_i^2)) \not\cong o_{E_i}$. q.e.d.

Step (VI). *There is a non-singular rational curve on F passing through every point of F.*

PROOF. Suppose to the contrary. Then there are only a finite number of non-singular rational curves on F of each degree. Moreover, on any surface there are only a finite number of non-singular curves E of each degree such that either

(a) $(E^2) < 0$, or

(b) E elliptic, $(E^2) = 0$, with *non-trivial* normal bundle.

Taking all these curves together, we get a countable set of curves, which cannot exhaust F. Let $P_1, \cdots, P_q$ be distinct points of F on none of these curves. Let D be a divisor satisfying the requirements of *Step* (III) with $\dim |D| \geqq 3q$. It follows that there is a divisor $D' \in |D|$ with double points at each P_i, $1 \leqq i \leqq q$. Let $D' = \sum n_i E_i$. If $n_i \geqq 2$, then by *Step* (V), E_i would be in the countable set of curves just described, so E_i does not contain any P_j. In other words, each P_i lies only on *simple* components of D', and since each E_i is non-singular (*Step* (V)), each P_i lies on 2 components. Since the E_i's are connected together as a tree, this shows that there are in all at least $q + 1$ E_i's containing some P_j. Note that they are all curves of genus at least 1. Therefore

$$q = \dim(\mathrm{Pic}_F^0) \geqq \dim(\mathrm{Pic}_{D'}^0) \geqq \sum_i \dim(\mathrm{Pic}_{E_i}^0) \geqq q + 1 .$$

So our assumption was false. q.e.d.

Step (VII). *F is ruled.*

PROOF. This follows by any number of methods. If $q = 0$, there must be a *linear* system of positive dimension of non-singular rational curves. Apply Tsen's theorem[2] to a pencil of such curves. If $q > 0$, look at the Albanese map $\pi: F \to A$. Since π maps all rational curves on F to points, it follows from *Step* (VI) that π factors

$$F \xrightarrow{\pi'} C \hookrightarrow A$$

where C is a curve, and the fibres of π' are rational. Therefore F is ruled by Tsen's theorem. q.e.d.

2. $(K\cdot D) \geqq 0$, all effective divisors D; $(K^2) = 0$

In this section we will show that

$$\begin{Bmatrix} (K^2) = 0 \\ (K\cdot D) \geqq 0, \text{ eff. } D \end{Bmatrix} \Longrightarrow \begin{Bmatrix} \text{either } 2K \equiv 0\ , \\ \text{or } \exists \text{ a pencil of curves on } F \text{ with } p_a = 1 \end{Bmatrix}$$

(Note: in char 2 or 3, the pencil may consist entirely of rational curves with cusps; Bertini's theorem tells us only that almost all the curves are irreducible, cf. [8].)

First make the definition.

DEFINITION. A curve $D = \sum n_i E_i$ on F is *of canonical type* if $(K\cdot E_i) = (D\cdot E_i) = 0$, all i. D is *indecomposable of canonical type*, if D is connected, and g.c.d. $(n_i) = 1$.

Step (I). *Either* $2K \equiv 0$ *or* F *contains at least one indecomposable curve of canonical type.*

PROOF. (Enriques; cf. also Šafarevič [7, Lemmas 9, 10, pp. 71–73)]. If $D \in |2K|$, then D is either 0 or of canonical type. In fact, let $D = \sum n_i E_i$. Then

$$0 = (K\cdot D) = \sum n_i(K\cdot E_i)\ .$$

Since $(K\cdot E_i) \geqq 0$, all i, it follows that $(K\cdot E_i) = 0$, all i; hence $(D\cdot E_i) = 2(K\cdot E_i) = 0$, too. Decomposing D, we find an indecomposable curve of canonical type. On the other hand, suppose $|2K| = \emptyset$. In this case, $q = h^{0,1}$ and $q \neq 0$ or else F would be rational by Castelnuovo; and $q \geqq 2$ would contradict the formula

$$\chi(o_F) = 1 - \dim H^1(o_F) + \dim H^2(o_F) \leqq 1 - q$$
$$12\chi(o_F) = (K^2) + (c_2) = 2 - 4q + B_2 \geqq 3 - 4q\ .$$

There remains the one really subtle case of $q = 1$. Consider the

[2] If $\pi: F \to C$ is a morphism of a surface to a curve, with fibres of $p_a = 0$, then $k(F) \cong k(C)(X)$.

Albanese morphism $\pi: F \to \varepsilon$, ε an elliptic curve. Let $f = \pi^{-1}(P)$, some $P \in \varepsilon$, be an irreducible fibre of π. If $p_a(f) = 1$, f is of canonical type and we are through. If $p_a(f) > 1$, then $(K \cdot f) = 2p_a(f) - 2 \geqq 2$. We then use the following argument of Enriques: for all $Q \in \varepsilon$, $Q \neq P$ consider the exact sequence

$$0 \longrightarrow o_F(2K + \pi^{-1}(Q) - f) \longrightarrow o_F(2K + \pi^{-1}(Q)) \longrightarrow o_f(2K \cdot f) \longrightarrow 0 .$$

Note that $H^2(o_F(2K + \pi^{-1}(Q) - f)) = (0)$, since if $|f - \pi^{-1}(Q) - K|$ contained an effective divisor, then $(K \cdot f) \leqq 0$. Check by Riemann-Roch that $\chi(o_F(2K + \pi^{-1}(Q) - f)) = 0$. Therefore, for all $Q \neq P$, either

(i) $|2K + \pi^{-1}(Q) - f| \neq \emptyset$, or

(ii) $H^i(o_F(2K + \pi^{-1}(Q) - f)) = (0)$, all i, hence

$$H^0(o_F(2K + \pi^{-1}(Q))) \xrightarrow{\sim} H^0(o_f(2K \cdot f)) .$$

Suppose $|2K + \pi^{-1}(Q) - f| = \emptyset$ for all Q. Then fix a non-zero $s \in H^0(o_f(2K \cdot f))$ and let A be the Cartier divisor $s = 0$ on f. For all Q, s lifts to a unique $s'_Q \in H^0(o_F(2K + \pi^{-1}(Q)))$, and let D_Q be the divisor $s'_Q = 0$. This is a 1-dimensional algebraic family of divisors, such that $D_Q \cdot f = A$ for all $Q \neq P$. Moreover, all the D_Q's are distinct since they are not even linearly equivalent. Therefore,

$$F = \text{closure of } \bigcup_{Q \neq P} D_Q .$$

In particular, as $Q \to P$, D_Q must specialize to a divisor D_P containing the whole fibre f. Then

$$D_P = f + D_P^*$$
$$D_P \in |2K + f| .$$

So $|2K| \neq \emptyset$. So finally in all cases, $|2K + \pi^{-1}(Q) - f| \neq \emptyset$ for some $Q \in \varepsilon$ (possibly $Q = P$). But a divisor $D \in |2K + \pi^{-1}(Q) - f|$ is of canonical type just as before. q.e.d.

We would like to assert next that on any surface F and for any indecomposable D of canonical type, $\dim H^0(o_F(nD)) > 1$ and hence $|nD|$ is composite with a pencil, for some n. Unfortunately, this is false, as one sees by considering an elliptic ruled surface obtained by completing a line bundle of infinite order over an elliptic curve, the 0-section of the bundle being taken as the curve of canonical type. The rest of the proof is an exercise in avoiding this case. The following will be useful.

LEMMA. *Let $D=\sum n_i E_i$ be an indecomposable curve of canonical type, and let L be an invertible sheaf on D, (regarded as a 1-dimensional scheme). If* $\deg(L\otimes o_{E_i})=0$, *all i, then*

$$H^0(D, L)\neq(0)\Longleftrightarrow L\cong o_F$$

and moreover $H^0(D, o_D)=k$.

PROOF. Let $s\in H^0(D, L)$. It suffices to show that if $s\neq 0$, s generates L hence defines an isomorphism of o_D with L; in particular, if $L=o_D$, this shows that all non-zero elements of the algebra $H^0(D, o_D)$ are units, i.e., $H^0(D, o_D)$ is a field, hence is k. But consider the induced section of $L\otimes o_{E_i}$. Since this sheaf has degree 0, a section either generates $L\otimes o_{E_i}$ or is identically 0. If s vanishes on one E_i, it vanishes on one point of each E_j meeting E_i; since D is connected, s either generates L everywhere, or vanishes on all E_i's. Assume that s vanishes on all E_i's; let $k_i=$ order of vanishing of s on E_i. Then $1\leqq k_i\leqq n_i$. Whenever $k_i<n_i$, s defines a non-zero section of

$$L\otimes\big[o_F(-k_iE_i)/o_F\big(-(k_i+1)E_i\big)\big]\ .$$

This section vanishes to order at least $I\big(E_i, \sum_{j\neq i}k_jE_j; P\big)$ at every $P\in E_i$ ($I=$ intersection multiplicity). It follows that if $k_i<n_i$, then

$$(1)\qquad \begin{aligned}\big(E_i\cdot\textstyle\sum_{j\neq i}k_jE_j\big)&\leqq\deg\{L\otimes o_F(-k_iE_i)/o_F\big((-k_i+1)E_i\big)\}\\ &=\deg[o_F(-E_i)/o_F(-E_i^2)]^{k_i}\\ &=-k_i(E_i^2)\ .\end{aligned}$$

Note that if $k_i=n_i$, then since $(E_i\cdot D)=0$,

$$(2)\qquad \big(E_i\cdot\textstyle\sum_j k_jE_j\big)=-\big(E_i\cdot\sum_j(n_j-k_j)E_j\big)\leqq 0\ .$$

So if $D_1=\sum_j k_jE_j$, then *for all i*, using (1) or (2), according as $k_i<n_i$ or $k_i=n_i$, $(E_i\cdot D_1)\leqq 0$. On the other hand,

$$\sum n_i(E_i\cdot D_1)=(D\cdot D_1)=\sum k_i(D\cdot E_i)=0\ ,$$

so it follows that $(E_i\cdot D_1)=0$, all i. This shows that $D-D_1$ and D_1 are of canonical type too. I claim that D and D_1 are both multiples of a 3rd divisor of canonical type, which will show that D is decomposable, unless $D=D_1$, i.e., $s\equiv 0$. To prove this, we must show that k_i/n_i is independent of i. Let $a/b=\max(k_i/n_i)$ and let $Z=aD-bD_1$. Then Z is an effective cycle and E_i occurs in Z if and only if $k_i/n_i<a/b$. Then if $k_i/n_i=a/b$, E_i is not a component

of Z, and since $(Z\cdot E_i) = a(D\cdot E_i) - b(D_1\cdot E_i) = 0$, E_i does not even meet Z; so Z does not contain any E_j's meeting E_i. Since D is connected, this shows that $Z = 0$, hence k_i/n_i independent of i as needed. q.e.d.

COROLLARY 1. *If D is indecomposable of canonical type, then*

$$o_F(K + D) \otimes o_F \cong o_D .$$

PROOF. Let $\omega = o_F(K + D) \otimes o_D$: then ω is the dualizing sheaf on the Cohen-Macauley scheme D, so by Serre duality $\dim H^1(D, \omega) = \dim H^0(D, o_D) = 1$. *Via* the exact sequence

$$0 \longrightarrow o_F(K) \longrightarrow o_F(K + D) \longrightarrow \omega \longrightarrow 0 ,$$

we see that

$$\begin{aligned} \chi(\omega) &= \chi\big(o_F(K + D)\big) - \chi\big(o_F(K)\big) \\ &= \frac{(K + D\cdot D)}{2} \qquad \text{(Riemann-Roch on } F) \\ &= 0 , \end{aligned}$$

hence $\dim H^0(D, \omega) = 1$ too. Hence $\omega \cong o_D$ by the lemma. q.e.d.

COROLLARY 2. *If $D = \sum n_i E_i$ is an indecomposable divisor of canonical type, and D' is any effective divisor on F such that $(D'\cdot E_i) = 0$ all i, then*

$$D' = nD + D''$$

where $n \geqq 0$ and D'' is an effective divisor disjoint from D.

Incidentally, a complete list of all curves of canonical type can be found in Kodaira [3], [4], and the lemma could be checked case by case.

Step (II). *If $p_g = 0$ (and $(K^2) = 0$, $(K\cdot D) \geqq 0$* all eff. *D as always) and D is an indecomposable curve of canonical type, then $|nD|$ is composite with a pencil of curves of canonical type, for some n.*

PROOF. First, look at the sequences

$$0 \longrightarrow o_F\big(nK + (n - 1)D\big) \longrightarrow o_F(nK + nD) \longrightarrow o_D \longrightarrow 0$$

obtained by applying Corollary 1. If $n \geqq 2$, then

$$H^2\big(o_F\big(nK + (n - 1)D\big)\big) = H^2\big(o_F(nK + nD)\big) = (0) .$$

Since $H^1(o_D) \neq (0)$, we find $H^1\big(o_F(nK + nD)\big) \neq (0)$. But

$$\chi\big(o_F(nK + nD)\big) = \chi(o_F) = 0 \quad \text{or} \quad 1 \qquad \text{(cf. } \textit{Step } \text{I)} .$$

Therefore $H^0(o_F(nK + nD)) \neq (0)$. This shows that there is a divisor $D_n \in |nK + nD|$.

D_n is of canonical type. Note that if $D = \sum n_i E_i$, then

$$(D_n \cdot E_i) = n(K \cdot E_i) + n(D \cdot E_i) = 0 .$$

So by Corollary 2, $D_n = aD + \sum k_i F_i$, where the F_i are disjoint from D. Now $(K \cdot F_i) \geqq 0$ for all i, while

$$\begin{aligned} \sum k_i(K \cdot F_i) &= \left(K \cdot \sum k_i F_i + aD\right) \\ &= (K \cdot nK + nD) \\ &= 0 \end{aligned}$$

so $(K \cdot F_i) = 0$, all i. Finally

$$(D_n \cdot F_i) = n(K \cdot F_i) + n(D \cdot F_i) ,$$

and this is 0 since D does not meet F_i.

It would seem to follow that we have produced at least one indecomposable curve of canonical type disjoint from D. But in one case this would not be so; namely, if each D_n was a multiple of D. In that case, though, K itself would be a multiple of D, hence $p_g \neq 0$ contrary to hypothesis.

Now assume that D and D' are disjoint indecomposable curves of canonical type. Then look at the sequence

$$0 \longrightarrow o_F(2K + D + D') \longrightarrow o_F(2K + 2D + 2D') \longrightarrow o_D \oplus o_{D'} \longrightarrow 0$$

(again using Corollary 1). As before, all H^2's vanish, so now

$$\dim H^1(o_F(2K + 2D + 2D')) \geqq 2 ,$$

hence calculating Euler characteristics, we find

$$\dim H^0(o_F(2K + 2D + 2D')) \geqq 2$$

so $|2K + 2D + 2D'|$ is composite with pencil of curves of canonical type. Since the intersection multiplicity $(D \cdot 2K + 2D + 2D') = 0$, it follows that one of the fibres of this pencil must be a multiple of D. q.e.d.

The final step is a new approach, not in Enriques.

Step (III). If $p_g > 0$, and D is an indecomposable curve of canonical type, then $|nD|$ is composite with a pencil of curves of canonical type, for some n.

PROOF. Let $\mathcal{F}_n$ be the quotient sheaf $o_F(nD)/o_F$. In view of the exact sequences

$$H^0(o_F(nD)) \longrightarrow H^0(\mathcal{F}_n) \longrightarrow H^1(o_F) ,$$

it will suffice to show that $\dim H^0(\mathcal{F}_n) \to \infty$ as $n \to \infty$ in order to establish *Step* (III). Let L be invertible sheaf $\mathcal{F}_1$ on the scheme D. For all n, $\mathcal{F}_{n-1}$ is a subsheaf of $\mathcal{F}_n$ with quotient L^n:

$$(*) \qquad 0 \longrightarrow \mathcal{F}_{n-1} \longrightarrow \mathcal{F}_n \longrightarrow L^n \longrightarrow 0 .$$

This proves

(A) $\dim H^0(\mathcal{F}_n)$ *is a non-decreasing function of* n.

Next, by the Riemann-Roch theorem on F, you see that $\chi(o_F(nD)) = \chi(o_F)$. Therefore $\chi(\mathcal{F}_n) = 0$. Now use the exact sequence

$$H^1(\mathcal{F}_n) \longrightarrow H^2(o_F) \longrightarrow H^2(o_F(nD)) .$$

Since D is effective, $|K - nD|$ is empty for large n, hence by Serre duality, $H^2(o_F(nD)) = (0)$. But since $p_g > 0$, $H^2(o_F) \neq (0)$, hence $H^1(\mathcal{F}_n) \neq (0)$, for large n. This proves

(B) $\dim H^0(\mathcal{F}_n) > 0$ *for* $n \gg 0$, *and* $\chi(\mathcal{F}_n) = 0$, *all* n.

Now assume that $\dim H^0(\mathcal{F}_n)$ is bounded above, and let n be the largest integer for which $\dim H^0(\mathcal{F}_{n-1}) < \dim H^0(\mathcal{F}_n)$ (there is at least one such n by (B) and the fact that $\mathcal{F}_0$ is the 0 sheaf). Using exact sequence $(*)$, it follows that L^n has a non-0 section. But since D is of canonical type, L^n has degree 0 on each component of D. So by Corollary 1, $L^n \cong o_D$. Therefore $\mathcal{F}_n$ has a section s which generates L^n everywhere as an o_D-module, hence s generates $\mathcal{F}_n$ everywhere as o_F-module. In other words, s defines an isomorphism

$$o_F/o_F(-nD) \cong \mathcal{F}_n = o_F(nD)/o_F .$$

Taking powers of s, we obtain isomorphisms

$$o_F/o_F(-nD) \cong o_F(nmD)/o_F(n(m-1)D) = \mathcal{F}_{nm}/\mathcal{F}_{n(m-1)} .$$

Now consider the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & o_F & \longrightarrow & o_F(n(m-1)D) & \longrightarrow & \mathcal{F}_{n(m-1)} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & o_F & \longrightarrow & o_F(nmD) & \longrightarrow & \mathcal{F}_{nm} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{F}_n & = & \mathcal{F}_n & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

set up by means of the above isomorphisms. Taking cohomology, we get

$$\begin{array}{ccccccc} H^1(\mathcal{F}_{n(m-1)}) & \longrightarrow & H^1(\mathcal{F}_{nm}) & \longrightarrow & H^1(\mathcal{F}_n) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \\ H^2(o_F) & = & H^2(o_F) & & & & \\ \downarrow & & \downarrow & & & & \\ 0 & & 0 & & & & \end{array}$$

Since $H^2(o_F) \neq (0)$, it follows that the map from $H^1(\mathcal{F}_{n(m-1)})$ to $H^1(\mathcal{F}_{nm})$ is not zero. Therefore $\dim H^1(\mathcal{F}_{nm}) > \dim H^1(\mathcal{F}_n)$. Since $\chi(\mathcal{F}_{nm}) = \chi(\mathcal{F}_n) = 0$, this shows $\dim H^0(\mathcal{F}_{nm}) > \dim H^0(\mathcal{F}_n)$, contradicting our hypothesis on n. Hence $\dim H^0(\mathcal{F}_n)$ is unbounded. q.e.d.

This establishes the existence of an elliptic or quasi-elliptic pencil on F in all cases except when $2K \equiv 0$. A propos the general question raised by this last step—namely, given a curve C such that $(C^2) = 0$, when does nC lie in a pencil, for some n?—there is one curious result that shows the situation in char p is "better" than that in char 0.

PROPOSITION. *Let C be an irreducible curve on a non-singular projective surface F. Assume $(C^2) = 0$ and* char $(k) \neq 0$.

$$\left\{\begin{array}{l} nC \text{ lies in a pencil} \\ \text{for some } n \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} o_F(nC)/o_F \cong o_F/o_F(-nC) \\ \text{for some } n \end{array}\right\}.$$

The proof is left as a curiosity to the reader. To get a counter example in char 0, let F = completion of the non-trivial $\mathbf{G}_a$-bundle over an elliptic curve, let C = line at ∞.

3. All but one of the remaining steps

Note first that if $(K\cdot D) \geqq 0$, for all effective divisors D, then $(K^2) \geqq 0$ (cf. [6]). Therefore, Enriques' theorem will follow if we show

(a) $[(K\cdot D) \geqq 0$, all eff D, $(K^2) > 0] \Rightarrow |2K| \neq \emptyset$ and $|nK|$ has no base points for some n.

(b) $[F$ elliptic or quasi-elliptic, for the morphism $f\colon F \to C] \Rightarrow nK \equiv f^*(A)$ for some n and for some 0-cycle A on C.

(c) $[F$ elliptic and $(K\cdot D) \geqq 0$, all eff $D] \Rightarrow |12K| \neq \emptyset$

(d) $[F$ quasi-elliptic and $(K\cdot D) \geqq 0$, all eff $D] \Rightarrow$ either F elliptic too or $|2K| \neq \emptyset$.

PROOF OF (a). Suppose that $|2K| = \emptyset$. Then $p_g = 0$, hence $q = h^{0,1}$, and by the Riemann-Roch theorem and Serre duality,

$$
(*)\qquad \begin{aligned}(K^2) + \chi(o_F) &= \chi(o_F(2K)) \\ &= \dim H^0(o_F(2K)) + \dim H^0(o_F(-K)) \\ &\quad - \dim H^1(o_F(2K)) \\ &\leqq 0 .\end{aligned}
$$

But $\chi(o_F) = 1 - q$, and by the Riemann-Roch theorem,

$$
\begin{aligned}12(1-q) &= 12\chi(o_F) \\ &= (K^2) + c_2 \\ &= (K^2) + 2 - 4q + B_2 ,\end{aligned}
$$

hence,

$$
(**)\qquad 10 = (K^2) + 8q + B_2 .
$$

Equation (**) shows that $q \leqq 1$, hence $\chi(o_F) \geqq 0$, hence by (*), $(K^2) \leqq 0$ which contradicts our hypotheses. Finally, the fact that in case (a), $|nK|$ has no base points for some n was shown in my appendix to [10].

PROOF OF (b). Each of the fibres of f is a connected curve of canonical type, and almost all are irreducible with multiplicity 1. A finite number of them, say $f^{-1}(x_1), \cdots, f^{-1}(x_n)$, are of the form

$$
f^{-1}(x_i) = k_i D_i , \qquad k_i > 1
$$

where D_i is an indecomposable curve of canonical type. Let $k =$ l.c.m. $(k_1, \cdots, k_n)$.

Now K intersects the generic fibre of f in a divisor linearly equivalent to 0. Therefore K can be represented by a divisor E_i whose support is disjoint from the generic fibre, i.e.,

$$
\operatorname{supp}(E_i) \subset f^{-1}(y_1, \cdots, y_m) .
$$

Therefore for a suitable 0-cycle A on C,

$$
K + f^{-1}(A) \equiv E_2 ,
$$

where E_2 is effective and is a sum of components of various fibres of f. Applying Corollary 2 § 2 above, it follows that

$$
E_2 = a_1 D_1 + \cdots + a_n D_n + f^{-1}(A')
$$

for some integers a_i, and a 0-cycle A' on C. Therefore

$$kK \equiv f^{-1}\left(-KA + KA' + \frac{ka_1}{k_1}x_1 + \cdots + \frac{ka_n}{k_n}x_n\right).$$

PROOF OF (c). This is the step which we defer to the sequel of this paper!

PROOF OF (d). Assume $|2K| = \emptyset$. Then if $q = 0$, F is rational by Castelnuovo's criterion and this contradicts the hypotheses. And $q \leqq 1$ just as in the proof of (a). Therefore $q = 1$. Let

$$f: F \longrightarrow \varepsilon$$

be the Albanese map. Since the fibres of the quasi-elliptic fibration are rational, they are mapped to points under f, hence f must be the quasi-elliptic fibration. Since almost all fibres of f have a unique singular point, the set of points $x \in F$ where $df = 0$ contains a curve $\varepsilon' \subset F$ mapped generically one-to-one to ε. Now $\operatorname{res}(f): \varepsilon' \to \varepsilon$ is flat and inseparable. Let its degree be p^n ($p = 2$ or 3 of course). Therefore

$$p^n = (\varepsilon' \cdot f^{-1}(x)), \qquad \text{all } x \in \varepsilon.$$

Now a non-singular branch can meet a cusp only with multiplicity 2 or 3, so choosing a non-singular point of ε', it follows that $(\varepsilon' \cdot f^{-1}(x)) = 2$ or 3. Therefore $n = 1$. Moreover, if ε' had a singular point y, then y would also be singular on $f^{-1}(f(y))$, so ε' would meet $f^{-1}(f(y))$ with multiplicity at least 4. Therefore ε' is non-singular and elliptic: especially it is of canonical type! So now apply the results of *Steps* II and III, § 2. We conclude that $|n\varepsilon'|$ varies in a pencil for some n, hence F is an elliptic surface.

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