

PERIODS OF A MODULI SPACE OF BUNDLES ON CURVES.

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We will work over the complex numbers in this paper. For all curves C , and for all integers (n, d) , the problem arises of determining the structure of the "space" of all vector bundles E , with rank n and degree ($= \deg c_1(E)$) d . The problem has been considerably clarified recently by the introduction of the concept of stable and semi-stable bundles: [4], [6], [10]. It has been proven, in particular, that for each n and each line bundle L on C such that n and $\deg L$ are relatively prime, then the set:

$$S_{n,L}(C) = \left\{ \begin{array}{l} \text{set of all stable vector bundles } E \text{ on } C \text{ of} \\ \text{rank } n \text{ such that } \Lambda^n E \cong L \end{array} \right.$$

has a natural structure of a non-singular projective variety of dimension $(n^2 - 1) \cdot (g - 1)$, where $g = \text{genus } (C)$. It is important to note that the map

$$E \mapsto E \otimes M$$

for a line bundle M induces an isomorphism

$$S_{n,L}(C) \xrightarrow{\cong} S_{n,L \otimes M^n}(C)$$

hence the variety $S_{n,L}(C)$ depends essentially only on the residue class of $\deg L \pmod n$.

We wish to look at the case $g \geq 2$, $n = 2$, $\deg L$ odd. In this case, we may assume for simplicity that a base point $x_0 \in C$ has been chosen that L is taken to be the line bundle whose sections form the sheaf $O_C(x_0)$. We abbreviate $S_{2,L}(C)$ now to $S_2^-(C)$. The topology of these varieties has been described in [7] and when the genus of C is 2, their complete structure is described in [8]. $S_2^-(C)$ has dimension $3g - 3$ and is known to be birationally equivalent to \mathbf{P}_{3g-3} . In particular, it is simply connected and the invariants $h^{0,p} = h^{p,0}$ are all 0, ([9]). In [7], it is also proven that $B_2 = 1$, $B_3 = 2g$. Now for non-singular projective varieties X with $h^{0,3} = h^{3,0} = 0$, a very interesting invariant is Weil's "intermediate jacobian" attached to $H^3(X)$. This is an abelian variety, which we shall denote $J^2(X)$, which is by definition:

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which is easily seen to be complex-analytic using the fact that w is of type $(2, 2)$ in the Hodge decomposition of H . Note the obvious fact:

LEMMA 1. ϕ_w is an isomorphism if and only if $w_{3,1}$ is "unimodular," (i. e., written out as a matrix in terms of bases of $H^3(X, \mathbf{Z})/\text{torsion}$, $H^1(Y, \mathbf{Z})$, it is a square matrix with $\det = \pm 1$).

1. In the sequel, we abbreviate $S_2(C)$ by S . The first step in our proof is to construct a universal vector bundle E on $S \times C$, i. e., one whose restriction to $\{t\} \times C$, for any $t \in S$, is exactly the vector bundle E_t on C corresponding to the point $t \in S$. This is a problem in descent theory. In fact, S can be described as a quotient $R/PGL(v)$, where R is a non-singular quasi-projective variety, and $PGL(v)$ acts freely on R ; and where there is a vector bundle F on $R \times C$ whose restriction to $\{t\} \times C$, any $t \in R$, is the vector bundle on C corresponding to the image of t in S : cf. [10], p. 321. However, the difficulty is that the action of $PGL(v)$ on R does not, a priori, lift to an action on F . Instead, $GL(v)$ acts on F satisfying

- 1) $G_m = \text{center}(GL(v))$ acts on F by homotheties
- 2) if $\pi: GL(v) \rightarrow PGL(v)$ is the canonical map, and T_g represents the action of an element g , then the diagram

$$\begin{array}{ccc}
 F & \xrightarrow{\quad T_g \quad} & F \\
 \downarrow & & \downarrow \\
 R \times C & \xrightarrow{\quad T_{\pi(g)} \times 1_C \quad} & R \times C
 \end{array}
 \quad \text{commutes.}$$

The way out of this type of impasse is to find a "functorial" way of associating to every vector bundle E on C (of the type being considered) a 1-dimensional vector space $\lambda(E)$ such that multiplication by α in E induces multiplication by α in $\lambda(E)$. By functorial we mean that the procedure extends to families of such vector bundles: if E is a vector bundle on $T \times C$ (for any algebraic scheme T) whose restriction to $\{t\} \times C$ is of the type under consideration, then we should get a line bundle $\lambda(E)$ on T . Moreover, for any diagram of vector bundles

$$\begin{array}{ccc}
 E_1 & \xrightarrow{\quad g \quad} & E_2 \\
 \downarrow & & \downarrow \\
 T_1 \times C & \xrightarrow{\quad f \times 1_C \quad} & T_2 \times C
 \end{array}$$

making E_1 into a fibre product of E_2 and $T_1 \times C$ over $T_2 \times C$, we should be given a definite isomorphism of $\lambda(E_1)$ with $f^*(\lambda(E_2))$. For example, if $T_1 = T_2 = \text{Spec}(\mathbf{C})$, $E_1 = E_2 = E$, and g is multiplication by a scalar $\alpha \neq 0$, we are then given an induced automorphism of $\lambda(E)$: we want this automorphism to be multiplication by α too (it might turn out to be multiplication by α^n instead). All this data is subject to an obvious co-cycle condition: compare [5], p. 64. If we can find such data, we get as a consequence a line bundle $\lambda(F)$ on R , plus an action of $GL(v)$ on $\lambda(F)$ in which the center acts by homotheties. If we then define

$$F' = F \otimes p_1^*(\lambda(F)^{-1}),$$

we get a new vector bundle on $R \times C$ with the same restrictions to the fibres $\{t\} \times C$ as before; but where in the natural action of $GL(v)$ on F' , the action of the center \mathbf{G}_m on F and $p_1^*(\lambda(F)^{-1})$ cancel each other out, i.e., $PGL(v)$ acts on F . Then $F/PGL(v)$ is the sought-for universal vector bundle on $S \times C$.

Here's how to construct λ . We limit ourselves to the case $T = \text{Spec}(\mathbf{C})$, E a vector bundle on C , since the generalization of λ to an arbitrary base will be clear. Recall E has rank 2, degree 1, and is stable:

a) $H^1(E \otimes (\Omega_C^1)^k) = (0)$, if $k \geq 1$.

Proof. This group is dual to $H^0(\hat{E} \otimes (\Omega_C^1)^{1-k})$ and if this were non-zero, we would get a non-zero homomorphism

$$(\Omega_C^1)^{k-1} \rightarrow \hat{E}$$

hence a sub-line-bundle $G \subset \hat{E}$ of degree $\geq 2(k-1)(g-1) \geq 0$. This contradicts the stability of E .

b) If $V_k(E) = H^0(E \otimes (\Omega_C^1)^k)$, then

$$\text{Dim } V_k(E) = (2g-2)(2k-1) + 1.$$

Proof. Riemann-Roch.

c) Set $\lambda(E) = [\Lambda^{2g-1}V_1(E)]^{\otimes(3g-1)} \otimes [\Lambda^{g-5}V_2(E)]^{\otimes(-g)}$.

Then multiplication by α in E induces the endomorphism, multiplication by α , in each $V_k(E)$, hence it induces multiplication by α to the power

$$(3g - 1)(2g - 1) + (6g - 5)(-g)$$

in $\lambda(E)$. This number happens to be 1!

We now know that E exists. Next consider the chern classes of E . We have

$$\begin{aligned} c_2(E) \in H^4(S \times C, \mathbf{Z}) &\cong (H^2(S, \mathbf{Z}) \otimes H^2(C, \mathbf{Z})) \\ c_1(E) \in H^2(S \times C, \mathbf{Z}) &\cong H^2(C, \mathbf{Z}) \otimes H^2(S, \mathbf{Z}) \\ &\quad \otimes (H^3(S, \mathbf{Z}) \otimes H^1(C, \mathbf{Z})) \\ &\quad \otimes H^4(S, \mathbf{Z}). \end{aligned}$$

Note that any bundle $E \otimes p_1^*M$, M a line bundle on S , would have the same universal property that E does, so $c_1(E)$ is not very interesting. However, let

$$\alpha = (c_2(E))_{s,1} = [\text{component of } c_2(E) \text{ in } H^3(S, \mathbf{Z}) \otimes H^1(C, \mathbf{Z})].$$

A simple computation of chern classes shows that α is independent of this modification of E . According to [7], $H^3(S, \mathbf{Z})$ and $H^1(C, \mathbf{Z})$ have the same rank. In fact:

PROPOSITION 1. α is unimodular.

This will be proven in § 2. Assuming this, it follows from Lemma 1 that if $W =$ the algebraic 2nd chern class of E , then Weil's map $\phi_W: \text{Alb}(C) \rightarrow J^2(S)$ is an isomorphism, as required. Although it is not essential, it will be convenient in § 2 to know that $H^3(S, \mathbf{Z})$ is torsion-free. In fact, the torsion subgroup of $H^3(X, \mathbf{Z})$ —for any non-singular complete variety X over \mathbf{C} —is a birational invariant of X known as the “topological Brauer group” (cf. [12], Cor. (7.3) and equation (8.9), p. 59). And S is birationally equivalent to \mathbf{P}_{3g-3} which has no H^3 at all!

2. We start by recalling the results of [6]. In fact, let S_0 be the subset of $SU(2)^{2g}$ consisting of points (A_1, \dots, A_{2g}) such that

$$\prod_{i=1}^g A_{2i-1} A_{2i} A_{2i-1}^{-1} A_{2i}^{-1} = -I.$$

Then S_0 is an orientable submanifold of $SU(2)^{2g}$ and there is a natural map

$$p: S_0 \rightarrow S,$$

which is a principal fibration with group $PU(2)$. The map p may be determined as follows. Let \tilde{C} be the simply-connected covering of C which is

ramified over x_0 with ramification index 2. The group π of this covering is generated by elements a_1, \dots, a_{2g} subject to the single relation

$$\left[\prod_{i=1}^g a_{2i-1} a_{2i} a_{2i-1}^{-1} a_{2i}^{-1} \right]^2 = e.$$

Thus a point of S_0 may be regarded as a representation of π , and this representation defines a stable bundle E over C of rank 2, with $\Lambda^2 E \cong L$, and hence a point of S . So we get a map $p: S_0 \rightarrow S$. Notice that the a_i determine elements of $\pi_1(C)$ and hence of $H_1(C; \mathbf{Z})$, and that these elements form a basis for $H_1(C; \mathbf{Z})$; let $\{\alpha_i\}$ be the dual basis of $H^1(C; \mathbf{Z})$.

LEMMA 2. $p^*: H^3(S; \mathbf{Z}) \rightarrow H^3(S_0; \mathbf{Z})$ is an isomorphism.

Proof. Since $H^1(PU(2); \mathbf{Z}) = 0$ and $H^1(S; \mathbf{Z}) = 0$ (S is simply-connected), the spectral sequence of the fibration p gives rise to an exact sequence

$$H^0(S; H^2(PU(2); \mathbf{Z})) \rightarrow H^3(S; \mathbf{Z}) \xrightarrow{p^*} H^3(S_0; \mathbf{Z}) \rightarrow H^0(S; H^3(PU(2); \mathbf{Z})).$$

Now the first group in this sequence is \mathbf{Z}_2 and the last is \mathbf{Z} . Moreover $H^3(S; \mathbf{Z})$ is torsion-free (see § 1) and has the same rank as $H^3(S_0; \mathbf{Z})$ by the results of [7]. The lemma now follows.

LEMMA 3. The homomorphism $H^3(SU(2)^{2g}; \mathbf{Z}) \rightarrow H^3(S_0; \mathbf{Z})$ induced by the inclusion of S_0 in $SU(2)^{2g}$ is an isomorphism.

Proof. Lemma 3 of [7] shows that the homomorphism

$$H_3(S_0; \mathbf{Z}) \rightarrow H_3(SU(2)^{2g}; \mathbf{Z})$$

is surjective, except possibly for some 2-primary torsion. However, in this simple case, the same argument can be used to prove that the homomorphism is really surjective. It follows at once that $H^3(SU(2)^{2g}; \mathbf{Z})$ is contained in $H^3(S_0; \mathbf{Z})$ as a direct summand. The lemma now follows from the fact that the ranks of these two groups are equal (see [7]) and that $H^3(S_0; \mathbf{Z})$ is torsion-free by Lemma 2.

Now let $p_i: S_0 \rightarrow SU(2)$ denote the projection on the i -th factor and let

$$\beta_i = p_i^*[\text{generator of } H^3(SU(2); \mathbf{Z})].$$

Then by Lemma 3 the β_i form a basis for $H^3(S_0; \mathbf{Z})$. In view of Lemma 2, it is now sufficient to prove:

PROPOSITION 2. $c_2[(p \times 1_C)^* E]_{3,1} = \sum_{i=1}^{2g} \beta_i \otimes \alpha_i.$

Now choose embedding $s_i: S^1 \rightarrow C - x_0$ which represent the generators a_i of π . Then

$$s_i^*(\alpha_j) = 0 \quad i \neq j$$

$$= \text{generator of } H^1(S^1; \mathbf{Z}) \quad i = j.$$

Hence Proposition 2 will follow at once from

PROPOSITION 3.

$$c_2[(1_{S_0} \times s_i)^*(p \times 1_C)^*E]_{3,1} = \beta_i \otimes [\text{generator of } H^1(S^1; \mathbf{Z})].$$

We now need to recall a few more details from [6]. Let E_ρ be the bundle over C corresponding to the representation $\rho \in S_0$. Then ([6] Remark 6.2) we can write down coordinate transformations for E_ρ as follows. Choose a finite open covering $\{U_i\}$ ($i = 0, 1, \dots, m$) of C such that every non-empty intersection of the sets U_i is contractible. Assume $x_0 \in U_0, x_0 \notin U_i$ for $i \neq 0$. Assume moreover that there exist discs D_i in \tilde{C} such that U_0 is the quotient of D_0 by \mathbf{Z}_2 and that for $i \neq 0, D_i$ maps homeomorphically onto U_i . For every i, j, k , where $k = i$ or j , let $W_{ij,k}$ be a connected component of $v^{-1}(U_i \cap U_j) \cap D_k$ (where $v: \tilde{C} \rightarrow C$ is the covering map). If $U_i \cap U_j = \emptyset, i \neq j, W_{ij,k}$ maps homeomorphically onto $U_i \cap U_j$; let γ_{ij} be the element of π such that $\gamma_{ij}W_{ij,j} = W_{j,i}$. Then a set of coordinate transformations g_{ij} for E_ρ is given by

$$g_{ij} = \rho(\gamma_{ij}) \text{ on } U_i \cap U_j, i \neq 0, j \neq 0$$

$$= f_i \cdot \rho(\gamma_{0i}) \text{ on } U_0 \cap U_i, i \neq 0,$$

where f_i is an analytic scalar function on $U_0 \cap U_i$ which is independent of ρ . Note that the coordinate transformations depend differentially on ρ , so that the same g_{ij} (now regarded as functions on $S_0 \times U_i \cap S_0 \times U_j$) define a differentiable bundle E' over $S_0 \times C$ which is a differentiable family of analytic bundles over C .

Now $E' | \{\rho\} \times C \cong E_\rho \cong (p \times 1_C)^*E | \{\rho\} \times C$ for all $\rho \in S_0$. Since E is stable, it follows that

$$\dim H^0(C; \text{Hom}_{\text{Anal.}}(E', (p \times 1_C)^*E) | \{\rho\} \times C) = 1$$

for all ρ . So by Proposition 2.7 of [2],

$$\bigcup_{\rho \in S_0} H^0(C; \text{Hom}_{\text{Anal.}}(E', (p \times 1_C)^*E) | \{\rho\} \times C)$$

has a natural structure of differentiable line bundle over S_0 . Let L be the induced line bundle over $S_0 \times C$. There is then an obvious isomorphism

$$E' \otimes L \cong (p \times 1_C)^*E.$$

So

$$c_2[(p \times 1_C)^*E]_{3,1} = c_2(E')_{3,1}.$$

Using the above explicit description of the bundle E' , we see that for any continuous map $s: S^1 \rightarrow C - x_0$, $(1_{S_0} \times s)^*E'$ can be described as follows: take a trivial bundle of rank 2 over $S_0 \times [0, 1]$ and glue its ends together by means of the map

$$S_0 \rightarrow SU(2),$$

defined by

$$\rho \mapsto \rho(a), \text{ where } a \in \pi \text{ corresponds to } s.$$

Apply this when $s = s_i$, $a = a_i$, and $\rho(a_i) = p_i(\rho)$; so Proposition 3 will follow at once from

LEMMA 4. *Let W be a space and let F be the bundle over $W \times S^1$ obtained by glueing together the two ends of the trivial bundle of rank 2 over $W \times [0, 1]$ by means of the map $f: W \rightarrow SU(2)$. Then*

$$c_2(F) = f^*[\text{generator of } H^2(SU(2); \mathbf{Z})] \otimes [\text{generator of } H^1(S^1; \mathbf{Z})].$$

Proof. F is the bundle induced by f from the bundle obtained by taking $W = SU(2)$, $f = 1_{SU(2)}$ in the construction. Hence it is sufficient to prove the lemma for this special case. But then it follows from the fact that $H^*(BSU(2))$ is generated by c_2 .

This completes the proof of Proposition 3 and hence of our theorem.

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