[Note: Need to add suitable references to the undated letter from Mumford to Grothendieck and to the Bowdoin lectures, in the first footnote.]

Degeneration of algebraic theta functions [1]

§1. 2-adic theta functions, values in a complete valued field [2]

Problem:

Given K: complete algebraically closed valued field; integers $\mathcal{O}$, residue field $k = \mathcal{O}/\mathfrak{m}$, absolute value $|\ |: k^{\times} \to \mathbb{R}_{>0}$, char.$(k) \neq 2$.

Given V: $2g$-dimensional vector space over $\mathbb{Q}_2$, plus e, e_* , Λ.[3]

[1] This is a lightly edited version of a set of hand written notes by Mumford. It contains an essentially complete proof of the results in the undated letter [???] to Grothendieck, despite the following disclaimer in that letter: '*I say "think" because I haven't written down the details systematically. In fact one should get a rather complete "structure theorem" for these abelian varieties (I hope).*' Mumford lectured on these results in the 1967 Summer School at Bowdoin, see paper [???] in this volume. Appendix II in the 1984 Ph. D. dissertation of C.-L. Chai, London Math. Soc. Lecture Notes Series 107, 1985, pp. 237–286 is an adulterated version of the same set of notes.

[2] The notes come in two groups, reproduced as two sections. The first section contains the key results on the structure of 2-adic theta functions associated to abelian varieties over a local field. This structure theory is applied in § 2 to 2-adic monodromy of abelian varieties over local fields. Two pages of the original notes are essentially the same as the last section of the Bowdoin lecture notes [???]; they are not reproduced here.

[3] The notations and results in *Equations defining abelian varieties I, II, III*, abbreviated as [Eq I, II, III] in the footnotes, are used extensively in this set of notes. In particular

$$e\colon V \times V \longrightarrow \mu_{2^\infty}(K)$$

is a skew-symmetric bi-multiplicative non-degenerate pairing from $V \times V$ to the group of all roots of unity whose order is a power of 2, Λ is a maximal isotropic $\mathcal{O}$-lattice in V, and

$$e_*\colon \frac{1}{2}\Lambda/\Lambda \longrightarrow \{\pm\}$$

is a quadratic character such that

$$e_*(\alpha+\beta)\, e_*(\alpha)\, e_*(\beta) = e(\alpha,\beta)^2 \qquad \forall \alpha,\beta \in \frac{1}{2}\Lambda\,.$$

Given $\Theta : V \to K$, a *theta function* w.r.t. e, e_*,[4] coarse support$(\Theta) = V$.[5]

Analyze structure of θ

(I) All values $\Theta(\alpha)$ are integrally dependent on $\{\Theta(\beta) \,|\, \beta \in \frac{1}{2}\Lambda\}$. Hence $\max|\Theta(\alpha)|$ exists and is taken on for some $\alpha \in \frac{1}{2}\Lambda$.[6] So multiply Θ by a constant s.t.

(a) $\Theta(\alpha) \in \mathcal{O}$ for all $\alpha \in \frac{1}{2}\Lambda$

(b) $\exists \alpha \in \frac{1}{2}\Lambda$ with $\Theta(\alpha) \notin \mathfrak{m}$, or equivalently $|\Theta(\alpha)| = 1$.

$\therefore$ Get a non-zero theta function $\overline{\Theta}(\alpha) := [\Theta(\alpha) \mod \mathfrak{m}] \in k$.

(II) Say coarse support$(\overline{\Theta}) = W + \frac{1}{2}\Lambda$, $W \subseteq V$ a *cusp*.[7]

(*) Choose a symplectic translation T of V s.t. $T(\Lambda) = \Lambda$, $e_* \equiv 1$ on $T(W^\perp) \cap \frac{1}{2}\Lambda$. Change Λ by this: Then 0 is an *origin*[8] for W. Later, will have to apply T in reverse to the structure Th. we get for Θ.

$\rightsquigarrow$ OK:

$$\overline{\Theta}(\alpha) = e_*(\eta/2)\, e(\eta/2, \alpha)\, \overline{\Theta}^*(\alpha_0^*)$$

$$\text{if} \quad \begin{array}{rcl} \alpha & = & \eta + \alpha_0, \ \eta \in \Lambda, \ \alpha_0 \in W \\ \alpha_0^* & = & \text{image of } \alpha_0 \text{ in } W/W^\perp \\ \overline{\Theta}^* & = & k\text{-valued non-degen. theta fcn. on } W/W^\perp \end{array}$$

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[4]That $\Theta : V \longrightarrow K$ is a theta function for (V, Λ, e, e_*) means that it satisfies

- (theta transformation law) $\Theta(\alpha + \beta) = e_*(\beta/2)\, e(\beta/2, \alpha)\, \Theta(\alpha) \quad \forall \alpha \in V, \forall \beta \in \Lambda$.
- (symmetry) $\Theta(-\alpha) = \Theta(\alpha) \quad \forall \alpha \in V$.
- (Riemann theta relation) For all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in V$ we have

$$\prod_{i=1}^{4} \Theta(\alpha_i) = 2^{-g} \cdot \sum_{\eta \in \frac{1}{2}\Lambda/\Lambda} e(\gamma, \eta) \prod_{i=1}^{4} \Theta(\alpha_i + \gamma + \eta)$$

where $\gamma = -\frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$.

See [Eq 3], p. 216.

[5]The *coarse support* of an algebraic theta function $\Theta \colon V \to K$ is the set of all for every $\alpha \in V$, there exists $\eta \in \frac{1}{2}\Lambda$ such that $\Theta(\alpha + \eta) \neq 0$.

[6]See Prop. 1 of *Abstract theta functions*, paper [???] in this volume.

[7]See Theorem on p. 230 of [Eq III] for this assertion. A *cusp* is a vector subspace $W \subseteq V$ such that $W^\perp \subseteq W$; see page 229 of *loc. cit.*

[8]An *origin* of a cusp W is an element $\eta_0 \in \frac{1}{2}\Lambda$ such that $e_*(\eta_0) = 1$ and $e_*(\alpha) = e(\alpha, \eta_0)^2$ for all $\alpha \in W^\perp \cap \frac{1}{2}\Lambda$; see p. 229 of *loc. cit.*

Choose:

$$\left.\begin{array}{rcl} V &=& W_1 \oplus W_2 \\ \Lambda &=& \Lambda_1 \oplus \Lambda, \quad \Lambda_i = \Lambda \cap W_i \\ e_* &=& 1 \text{ on } \frac{1}{2}\Lambda_i \end{array}\right\} \text{standard decomp. of } V$$

s.t. $(0) \subset W^\perp \subset W_1 \subset W \subset V$, so $W = W_1 \oplus \widetilde{W}_2$, $\widetilde{W}_2 \subset W_2$.

Given $V = W_1 \oplus W_2$ and $W = W_1 \oplus \widetilde{W}_2$ as above:
$\exists$ 1-1 correspondence between

(a) $\mathcal{O}$-valued theta fcns. Θ on V s.t. coarse $\operatorname{supp}(\overline{\Theta}) = W + \frac{1}{2}\Lambda$.

(b) $\mathcal{O}$-valued *Gaussian measures*[9] μ on W_2 s.t. $\operatorname{supp}(\overline{\mu}) = \widetilde{W}_2$

In fact[10]

$$\boxed{\begin{array}{rcll} \mu(a_2 + 2^n\Lambda_2) &=& 2^{-ng} \displaystyle\sum_{a_1 \in 2^{-n}\Lambda_2/\Lambda_2} e(a_2, a_1/2) \cdot \overline{\Theta}(a_1 + a_2) & \forall a_2 \in W_2,\ \forall n \in \mathbb{N} \\ \overline{\Theta}(a_1 + a_2) &=& e(a_1, a_1/2) \displaystyle\int_{a_2+\Lambda_2} e(a_1, \beta) \cdot d\mu(\beta) & \forall a_1 \in W_1,\ \forall a_2 \in W_2 \end{array}}$$

Esp.

$$\begin{aligned} &\sup\left\{ |\mu(U)| \colon U \subset a_2 + \widetilde{W}_2 + \frac{1}{2}\Lambda_2,\ U \text{ compact open}\right\} \\ &= \sup\left\{ |\mu(a_2' + 2^n\Lambda_2)| : a_2' \in a_2 + \widetilde{W}_2 + \frac{1}{2}\Lambda_2,\ n \geq 0\right\} \\ &= \sup\left\{ |\Theta(a_1 + a_2')| \colon a_1 \in W_1,\ a_2 \in \widetilde{W}_2 + \frac{1}{2}\Lambda_2\right\} \\ &= \sup\left\{ \Theta(b)| \colon b \in a_2 + W + \frac{1}{2}\Lambda \right\} \end{aligned}$$

(III) Show that $\forall\Theta$ or μ, and $\forall a_2$, this *sup* is a *max*.
Proof. Associate Φ to Θ s.t.

$$\Phi(\alpha)\,\Phi(\beta) = \sum_{\zeta \in \frac{1}{2}\Lambda_1/\Lambda_1} e(\alpha, \zeta)\,\Theta(\alpha + \beta + \zeta)\,\Theta(\alpha - \beta + \zeta)$$

$$2^g\,\Theta(2\alpha)\,\Theta(2\beta) = \sum_{\zeta \in \frac{1}{2}\Lambda_2/\Lambda_2} e(\alpha, \zeta)^2\,\Phi(\alpha + \beta + \zeta)\,\Phi(\alpha - \beta + \zeta)$$

[9] An k-valued *even* measure on W_2 is a *Gaussian measure* if there exists a k-valued measure ν on W_2 such that $(\mu\times\mu)(U) = (\nu\times\nu)(\xi(U))$ for all compact open subsets U in $W_2 \times W_2$, where $\xi\colon W_2\times W_2 \longrightarrow W_2\times W_2$ is defined by $\xi(x, y) = (x + y, x - y)$; see page 118 of [Eq II].
[10] See pp. 116–117 of [Eq II].

$$\therefore |\Phi(\alpha)| \cdot |\Phi(\beta)| \leq \max_{\zeta \in \frac{1}{2}\Lambda_1} |\Theta(\alpha+\beta+\zeta)| \cdot |\Theta(\alpha-\beta+\zeta)|$$

$$|\Theta(\alpha+\beta) \cdot |\Theta(\alpha-\beta)| \leq \max_{\zeta \in \frac{1}{2}\Lambda_2} |\Phi(\alpha+\zeta)| \cdot |\Phi(\beta+\zeta)|$$

$$\therefore \max_{\zeta \in \frac{1}{2}\Lambda_1} |\Theta(\alpha+\beta+\zeta)| \cdot |\Theta(\alpha-\beta+\zeta)| = \max_{\zeta \in \frac{1}{2}\Lambda_2} |\Phi(\alpha+\zeta)| \cdot |\Phi(\beta+\zeta)|$$

So

$$\max_{\zeta \frac{1}{2}\Lambda_1} |\Theta(\alpha+\beta+\zeta)| \cdot \max_{\zeta \in \frac{1}{2}\Lambda_1} |\Theta(\alpha+\beta+\zeta)| = \max_{\zeta \in \frac{1}{2}\Lambda_2 + \frac{1}{4}\Lambda_1} |\Phi(\alpha+\zeta)| \cdot |\Phi(\beta+\zeta)|$$

Now assume $\beta \in W$. Use $\begin{cases} \forall x \in W + \Lambda_2 \\ \exists \eta \in \frac{1}{4}\Lambda_1 \quad s.t. \quad |\Phi(x+\eta)| = 1 \end{cases}$.

$$\text{Let} \quad \tau(\gamma) = \max_{\zeta \in \frac{1}{4}\Lambda_1} |\Theta(\gamma+\zeta)|$$

$$\therefore \tau(\alpha+\beta)\,\tau(\alpha-\beta) = \max_{\zeta_1,\zeta_2 \in \frac{1}{2}\Lambda_2 + \frac{1}{8}\Lambda_1,\ \zeta_1+\zeta_2 \in \frac{1}{4}\Lambda_1} |\Phi(\alpha+\zeta_1)| \cdot |\Phi(\beta+\zeta_2)|$$

Def. $\alpha \in V$ is *normal* if $\max_{\zeta \in \frac{1}{8}\Lambda_1} |\Phi(\alpha+\zeta)| = \max_{\zeta \in \frac{1}{2}\Lambda_2 + \frac{1}{8}\Lambda_1} |\Phi(\alpha+\zeta)|$.

$$[\ \forall \alpha\ \exists \eta \in \frac{1}{2}\Lambda_2 \ \text{ s.t. } \alpha+\eta \text{ is normal}\ ]$$

So if α normal, $\beta \in W$, then

$$\tau(\alpha+\beta)\,\tau(\alpha=\beta) = x \max_{\zeta \in \frac{1}{8}\Lambda_1 + \frac{1}{2}\Lambda_2} |\Phi(\alpha+\zeta)| =: \rho(\alpha)$$

Esp.

$$\tau(\alpha+\beta)\,\tau(\alpha-\beta) = \tau(\alpha)^2\,.$$

Note: If $\eta \in \frac{1}{2}\Lambda_2$, $\alpha+\eta$ normal, then $\tau(\alpha+\eta) \geq \tau(\alpha)$.

PROOF OF NOTE.

$$\begin{aligned} \tau(\alpha)^2 &= \max_{\zeta_1,\zeta_2 \in \frac{1}{2}\Lambda_2 + \frac{1}{8}\Lambda_1,\ \zeta_1+\zeta_2 \in \frac{1}{4}\Lambda_1} |\Phi(\alpha+\zeta_1)| \cdot |\Phi(\zeta_2)| \\ &\leq \max_{\zeta \in \frac{1}{2}\Lambda_2 + \frac{1}{8}\Lambda_1} |\Phi(\alpha+\zeta)| \ = \ \rho(\alpha) \end{aligned}$$

$\alpha+\eta$ normal $\implies \tau(\alpha+\eta)^2 = \rho(\alpha+\eta) \geq \tau(\alpha)^2$. Q.E.D.

Now suppose $\alpha_n \in a + W + \frac{1}{2}\Lambda$ s.t.

$$|\Theta(\alpha_n)| \longrightarrow \sup\left\{ |\Theta(\beta)| \colon \beta \in a + W + \frac{1}{2}\Lambda \right\} =: s\,.$$

W.l.o.g. can assume $|\Theta(\alpha_n)| = \tau(\alpha_n)$ & α normal (in view of Note above).
OK : Pass to subsequence s.t.

$$\alpha_n - \alpha_m \in W + \Lambda \quad (\text{all } \; n, m)\,.$$

W.l.o.g. may assume $\alpha_n - \alpha_m \in W$ for all n, m. Now if

$$\tau(\alpha_n) = |\Theta(\alpha_n)| > \sqrt{s \cdot |\Theta(\alpha_1)|} = \text{geom. mean of } s, |\Theta(\alpha_1)|\,,$$

then

$$s \cdot \tau(\alpha_1) < \tau(\alpha_n)^2 = \tau(\alpha_n + (\alpha_1 - \alpha_n)) \cdot \tau(\alpha_n - (\alpha_1 - \alpha_n)) < \tau(\alpha_1) \cdot s\,,$$

contradiction. $\therefore\ s = \tau(\alpha_n)$ for all n. Step (III.) is proved.

We conclude
Proposition 1. $\forall$ $\mathcal{O}$-valued Gaussian measure μ on W_2, let $\widetilde{W}_2 = \operatorname{supp}(\overline{\mu})$. Then $\forall$ compact open subgroup $\Lambda_2' \subset V_2$ and $\forall a \in W_2$,

$$\sup\left\{ |\mu(U)| \colon U \subseteq \widetilde{W}_2 + \Lambda_2' + a \right\}$$

is attained.[11]

(IV) Theorem 2. Let $\mathcal{V}$ be a finite-dimensional vector space over $\mathbb{Q}_2$, W be a vector subspace of $\mathcal{V}$, and let $\Lambda \subset \mathcal{V}$ be a compact open subgroup.[12] Let μ be a Gaussian measure on $\mathcal{V}$ with values in $\mathcal{O}$. Let ν be the dual Gaussian measure of μ, i.e. $\xi_*(\mu \times \mu) = \nu$.[13] Assume that

(1) $\overline{\mu}, \overline{\nu}$ have support $W \subset \mathcal{V}$.

(2) $\forall\, w \in W$

$$\begin{aligned} \max\{\,|\mu(V)| \colon V \subset w + \Lambda + W\,\} &=: \sigma(w) \\ \max\{\,|\nu(V)| \colon V \subset w + \Lambda + W\,\} &=: \tau(w) \end{aligned}$$

exist.

If $w \in \mathcal{V}$, $c \in \mathcal{O}$, and $|c| = \sigma(w) = \max\limits_{\eta \in \frac{1}{2}\Lambda} (\sigma(w + \eta))$, then

$$\operatorname{supp}\left\{ \overline{\frac{\mu}{c}\Big|_{w+\Lambda+W}} \right\} = w + \eta_0 + W$$

for some $\eta_0 \in \Lambda$.

[11] Prop. 1 has been proved for $\Lambda' = \frac{1}{2}\Lambda_2$. Apply an automorphism A of V_2 such that $A(\frac{1}{2}\Lambda_2) \subset \Lambda_2'$

[12] The general notation for §1 is suspended in Step (IV). In application the triple $(\mathcal{V}, \Lambda, W)$ in this theorem will be $(W_2, \Lambda_2, \widetilde{W})$. Also the meaning of τ is *different* from that in the proof of Step (III).

[13] As before, $\xi \colon \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V} \times \mathcal{V}$, $\;\xi \colon (x, y) \mapsto (x + y, x - y)$.

PROOF. **Claim 1:** $\tau(w)^2 = \sigma(w)$.

(1) $\exists U \subset w + \Lambda + W$ s.t. $|\mu(U)| = \sigma(W)$.[14]

$$\therefore\ |\mu \times \mu(U \times \Lambda)| = \sigma(w)\,, \qquad \therefore\ |\nu \times \nu(\xi(U \times \Lambda))| = \sigma(w)\,.$$

But $\xi(U \times \Lambda) \subset (w + \Lambda + W) \times (w + \Lambda + W)$,

$$\therefore\ \exists U_1, U_2 \subset w + \Lambda + W \quad \text{s.t. } |\nu(U_1)| \cdot |\nu(U_2)| = |\nu \times \nu(U_1 \times U_2)| = \sigma(w)$$

$$\therefore\ \tau(w) \geq \max\{|\nu(U_1)|,\, |\nu(U_2)|\} \geq \sqrt{\sigma(w)}$$

(2) $\exists U \subset w + \Lambda + W$ s.t. $|\nu(U)| = \tau(W)$.

$$\therefore\ |\nu \times \nu(U \times U)| = \tau(w)^2$$

But

$$(w+\Lambda+W) \times (w+\Lambda+W) = \bigcup_{\text{disjoint}} \xi\left((w + \Lambda + W + \eta) \times (\Lambda + W + \eta)\right) \qquad \eta \in \frac{1}{2}\Lambda$$

$$\therefore\ \exists\, \eta, U_1 \subset w + \Lambda + W + \eta,\ \exists\, U_2 \in \Lambda + W + \eta \ \text{ s.t.}$$

$$\begin{aligned} \tau(w)^2 &= |\nu \times \nu(\xi(U_1 \times U_2))| = |\mu(U_1)| \cdot |\mu(U_2)| \\ &\leq\ |\mu(U_1)| \ \leq\ \sigma(w + \eta) \ \leq\ \sigma(w)\,. \end{aligned}$$

We have proved Claim 1.

Look at measures

$$\left.\overline{\frac{\mu}{c}}\right|_{w+\Lambda+W} =: \mu_w\,, \qquad \left.\overline{\frac{\nu}{\sqrt{c}}}\right|_{w+\Lambda+W} =: \nu_w\,.$$

Claim 2. (a) $\xi_*(\mu_w \times \overline{\mu}) = (\nu_w \times \nu_w)|_{\xi((w+\Lambda+W)\times(\Lambda+W))}$.
(b) The restriction of the measure $\nu \times \nu$ to

$$(w + \Lambda + W) \times (w + \Lambda + W) - \xi((w + \Lambda + W) \times (\Lambda + W))$$

has absolute values strictly less than $\sigma(w)$.
(c) $\xi_*(\mu_w \times \overline{\mu}) = \nu_w \times \nu_w$ as measures on $(w + \Lambda + W) \times (w + \Lambda + W)$.

[14] Here U is compact open; the same for the U_1, U_2 and U below.

Clearly (a) holds, and (b) implies (c). To see (b), suppose that $U_1 \subset w + \Lambda + W + \eta$, $U_2 \subset \Lambda + W + \eta$, U_1, U_2 compact open, $\eta \in \frac{1}{2}\Lambda$, $\eta \notin \Lambda + W$. Then

$$\begin{array}{rcl} |\nu \times \nu(\xi(U_1 \times U_2)| & = & |\mu(U_1)| \cdot |\mu(U_2)| \\ & \leq & \sigma(w+\eta) \cdot \sigma(\eta) \\ & < & \sigma(w+\eta) \qquad (\because\ \eta \notin \Lambda + W) \\ & \leq & \sigma(w) \qquad (\because\ \text{assumption on } w) \end{array}$$

Claim (b) is proved.[15]

Theorem 2 is a formal consequence of (c): Let $S := \mathrm{supp}(\mu_w)$, $T := \mathrm{supp}(\nu_w)$. So

$$T \times T = \xi(S \times W) = T \times T = \{\, (a+u, a-u) \mid a \in S,\ u \in W \,\}$$

because $\xi_*(\mu_w \times \mu_w) = \nu_w \times \nu_w$. Start with $a \in S$, $u \in W$. Then $a \pm u \in T$, so $(a+u, a+u) \in T \times T$. and $a + u \in S$ too because $\xi^{-1}(a+u, a+u) = (a+u, 0)$. We have shown that $a + W \subset S$ for all $a \in S$. If $b \in S$ also, then $b \pm u \in T$ as before, and $(a+u, b+u) \in T \times T$.

$$\therefore\ \frac{1}{2}(a-b) \in W \quad \because \xi^{-1}(a+u, b+u) \in S \times W\,.$$

So $a - b \in W$ for all $a, b \in S$. Step (IV) is proved.

(V) We reformulate what have been proved so far, and what is expected. Let $\mathcal{V}$ be a vector space over $\mathbb{Q}_2$, $W \subset \mathcal{V}$ a vector subspace, $\pi\colon \mathcal{V} \twoheadrightarrow \mathcal{V}/W$, $\dim \mathcal{V} = g$, $\dim W = g - r$. Let μ be an $\mathcal{O}$-valued Gaussian measure on $\mathcal{V}$ s.t. $\mathrm{supp}(\overline{\mu}) = W$. We have proved:

(1) For all compact open subset $U \subset \mathcal{V}/W$,

$$\sup \left\{\, |\mu(U')|\colon U' \subset \pi^{-1}(U),\ U' \text{ compact open} \,\right\}$$

is reached by some compact open subset U'.

(2) For all compact open $U \subset \mathcal{V}/W$, let

$$\sigma_U = \max\{\, |\mu(U')|\colon U' \subset \pi^{-1}(U),\ U' \text{ compact open} \,\}\,,$$

let $c_U \in K$ be s.t. $|c_U| = \sigma_U$, and let

$$\mu_U = \overline{\left[\left. \frac{\mu}{c_U} \right|_{\pi^{-1}(U)} \right]}\,.$$

Then $\mathrm{supp}(\mu_U)$ is a finite union of cosets of W.

[15] Because each compact open subset of $(w + \Lambda + W) \times (w + \Lambda + W) - \xi((x + \Lambda + W) \times (\Lambda + W))$ is a finite disjoint union of subsets of the form $\xi(U_1 \times U_2)$ satisfying the above conditions.

Expectation 3: $\exists\, S \subset \mathcal{V}/W$,
$$\begin{array}{ccc} S & \hookrightarrow & V/W \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{Z}[1/2]^h & \hookrightarrow & \mathbb{Q}_2^h \end{array}$$
and $\exists$ a function[16] $\sigma\colon S \longrightarrow \mathbb{R}$ of the form $\sigma(x) = e^{-Q(x,x)}$, Q a pos. def. quad. form on S, s.t. $\forall U \subset \mathcal{V}/W$ compact open, let $\sigma_U = \max_{x\in U\cap S} \sigma(x)$, and let $c_U \in K$ be s.t. $|c_U| = \sigma_U$. Then[17]

(a) $|\mu(U')| \leq \sigma_U$ for all compact open $U' \subset \pi^{-1}(U)$.

(b) $\mu_U = \overline{\left[\frac{\mu}{c_U}\Big|_{\pi^{-1}(U)} \right]}$ is a k-valued measure whose support is exactly
$$\bigcup_{y\in S\cap U,\, \sigma(y)=\sigma_U} \pi^{-1}(y)\,.$$

Def. The *singular set* $S = S(\mu)$ of μ is defined by
$$S = S(\mu) := \left\{\, x \in \mathcal{V}/W \colon \exists\,\text{open neigh. } U \ni x \in \mathcal{V}/W \text{ s.t. } \operatorname{supp}(\mu_U) = \pi^{-1}(x) \,\right\}$$

Def. Define $\sigma\colon S \longrightarrow \mathbb{R}$ by
$$\sigma(x) = \max\left\{\, |\mu(U')| \colon U' \subset \pi^{-1}(U) \,\right\}$$
for any $x \in S$, where U is an open neighborhood of x in $\mathcal{V}/W$ s.t. $\operatorname{supp}(\mu_U) = \pi^{-1}(x)$. This definition is independent of the choice of U.

It remains to show that
$$\begin{array}{ccc} S & \hookrightarrow & V/W \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{Z}[1/2]^r & \hookrightarrow & \mathbb{Q}_2^r \end{array}$$
and
$$\sigma(x) = e^{-Q(x)}\,, \quad Q \text{ pos. def.}$$

Proof of Expectation 3. Let ν be the $\mathcal{O}$-valued Gaussian measure dual to μ, i.e. $\xi_*(\mu\times\mu) = \nu\times\nu$. Let $T = S(\nu) \subset \mathcal{V}/W$ be the singular set of ν, and let $\tau = \sigma(\nu) : T \longrightarrow \mathbb{R}$ be the sup. map for ν. Then
$$\xi(S\times S) = T\times T$$
and
$$\sigma(x)\cdot\sigma(y) = \gamma(x+y)\cdot\gamma(x-y) \qquad \text{for all } x,y\in S\,.$$
From these we deduce

[16]The function σ on $S \subset \mathcal{V}/W$ here is *different* from the function σ on W in Theorem 2.

[17]In terms of the function $\sigma : S \longrightarrow \mathbb{R}$ here, for $w \in \mathcal{V}$, the positive number $\sigma(w)$ in Theorem 1 is equal to $\max_{x\in S\cap\pi(w+\Lambda+W)} \sigma(x)$.

(a) S is a subgroup of $\mathcal{V}/W$ and $2S = S$.

(b) $Q := -\log\sigma$ is a quadratic form from S to non-negative real numbers.[18]

Let $\Lambda \subset \mathcal{V}/W$ be a neighborhood of 0, and let $S_0 := \Lambda \cap S$, a subgroup of S s.t.

$$\bigcup_{n\in\mathbb{N}} 2^{-n} S_0 = S\,.$$

(c) Let $x_1, \ldots, x_n \in S_0$ be a $\mathbb{Z}$-linearly independent subset in S_0. Look at the maximal H s.t. $\exists$

$$\begin{array}{ccccc} \mathbb{Z}^n & \hookrightarrow & H & \hookrightarrow & \mathbb{Q}^n \\ \downarrow & \swarrow_{\phi} & & & \\ S_0 & & & & \end{array} \qquad \phi((a_1, \ldots, a_n) = \sum a_i\, x_i$$

Let Q' be the quadratic form on $\mathbb{Q}^n$ s.t.

$$Q'(a,a) := -\log\sigma(\phi(a_1, \ldots, a_n)) \qquad \text{for } a = (a_1, \ldots, a_n) \in H$$

Note that

$$\left.\begin{array}{l} Q \text{ is a pos. semi-definite quad. form on } \mathbb{Q}^n \\ \\ Q(a) = 0, \quad a \in \mathbb{Q}^n \quad\Longrightarrow\quad a = 0 \end{array}\right\}$$

(c$_1$) $[H : \mathbb{Z}^n] < \infty$.
If not, $\exists$ $\mathbb{Q}$-vector subspace $L \subset \mathbb{Q}^n$ s.t. $H \cap L$ is dense in L in classical topology. But $\forall a \in \mathbb{Z}^n$, $\phi(a) \in S_0$,

$$\therefore \text{ in } \phi(a) + 2^m\Lambda,\ \sigma(\phi(a)) \geq \sigma(b) \ \text{ for all } b \in \phi(b) + 2^m\Lambda$$

if m is large enough. Thus

$$Q'(a,a) \leq Q'(b,b) \qquad \text{for all } b \in a + 2^m H\,.$$

Take $a \in L \cap \mathbb{Z}^n$ and $b \in (a + 2^m H) \cap L$ in particular: then the possible b's are dense in L. So there are some b's for which $Q'(b,b) <$ any given ϵ, and get a contradiction.

Corollary. H is a finitely generated abelian group: w.l.o.g. $H = \mathbb{Z}^n$.

(c$_2$) Q' is positive definite.
If not, get

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow[\pi]{\text{proj.}} & \mathbb{R}^m \\ \uparrow & & \\ H & & \end{array} \qquad (m < n)$$

[18] For $s \in S$, we have $\sigma(s) = 1 \Leftrightarrow s \in W = \mathrm{supp}(\overline{\mu})$.

and a quadratic form Q'' on $\mathbb{R}^m$ s.t. $Q'(a) = Q''(\pi(a))$ for all $a \in \mathbb{R}^n$, and $\pi(H) \subset \mathbb{R}^m$ is *not* discrete. i.e. $\exists$ $\mathbb{R}$-vector subspace $L \subset \mathbb{R}^m$ s.t. $\pi(H) \cap L$ is dense in L. Get the same contradiction as above.

(d) S_0 is a free abelian group of rank $r = \dim(V/W)$.

PROOF. Define $r := \dim(V/W)$, $d := \dim_{\mathbb{Q}}(S_0 \otimes \mathbb{Q})$. Then

$$d \text{ finite } \Leftrightarrow S_0 \text{ fin. gen. by } (c_1) \text{ and } (c_2).$$

If $d < r$, then S_0 is too small to be dense[19] in Λ, OUT. If $d > r$, well

$$S_0/2S_0 \subset \Lambda/2\Lambda \cong (\mathbb{Z}/2\mathbb{Z})^r .$$

Q.E.D.

$$\therefore \quad \begin{array}{ccc} S & \xrightarrow{\cong} & \mathbb{Z}[1/2]^r \\ \uparrow & & \uparrow \\ S_0 & \xrightarrow{\cong} & \mathbb{Z}^r \end{array}$$

and $\sigma = e^{-Q(a,a)}$, Q pos. def. quad. form on $\mathbb{R}^r$. Expectation 3 is proved.

(VI) Theorem. 4 (1) Every Gaussian measure μ on $\mathcal{V}$ (as above) can be written as

$$\mu = \sum_{x \in S} \mu_x$$

where each μ_x[20] is an $\mathcal{O}$-valued measure on $\mathcal{V}$ with

$$\left.\begin{array}{ll} \operatorname{supp}(\mu_x) = \pi^{-1}(x) & \forall x \in S \\ \sup\{ |\mu_x(U) \colon U \subset \mathcal{V} \text{ compact open} \} & \forall x \in S \end{array}\right\}$$

Similarly, the dual measure ν can be written as

$$\nu = \sum_{x \in S} \nu_x$$

with similar properties as above. Moreover

$$\xi_*(\mu_x \times \mu_y) = \nu_{x+y} \times \nu_{x-y} \qquad \forall x, y \in S .$$

[19] S is dense in $\mathcal{V}/W$ because $\operatorname{supp}(\mu) = \mathcal{V}$.

[20] For $x \in S$, the measure μ_x is the push-forward to $\mathcal{V}$ of a measure μ'_x on $\pi^{-1}(x)$, defined as follows. For any compact open subset U' of $\pi^{-1}(x)$, let $\{U_i\}_{i \in \mathbb{N}}$ be a decreasing family of compact open subsets of $\mathcal{V}$ such that $\cap_{i \in \mathbb{N}} U_i = U'$. Then $\mu'_x(U') = \lim_{i \to \infty} \mu(U_i)$.

(2) Correspondingly: if $(0) \subset W^\perp \subset W \subset V$ is the set-up for $\pi : V \twoheadrightarrow V/W \subset S$,[21] and Θ is an $\mathcal{O}$-valued theta function w.r.t. e, e_*, Λ as in the beginning of §1. Then

$$\Theta(\alpha) = \sum_{x \in S} \Theta_x(\alpha)\,,$$

where each Θ_x[22] is a function on V such that

(a) $\Theta_x(\alpha+\beta) = e_*(\beta/2)\, e(\beta/2,\alpha)\, \Theta_x(\alpha) \qquad \forall \beta \in \Lambda$,

(b) $\mathrm{supp}(\Theta_x) \subset \pi^{-1}(x) + \Lambda$,

(c) $\Theta_x(\alpha+\beta) = e(\beta, \gamma_x - \alpha/2)\, \Theta_x(\alpha) \qquad \forall \beta \in W^\perp$ if $\gamma_x \in V$ satisfies $\pi(\gamma_x) = x$.

Defining an associated tower of toroidal groups

Θ on V gives

$$\begin{cases} (0) \subset W^\perp \subset W \subset V,\ \ \pi\colon V \twoheadrightarrow (V/W) & (\text{we assume } e_*(\alpha/2) = 1, \forall \alpha \in W^\perp \cap (1/2)\Lambda\,) \\ S \subset V/W & \\ \Theta_x \ \text{ on } \ \pi^{-1}(x) + \Lambda & \end{cases}$$

(1) Θ_0 on $W/W^\perp$ defines a tower of abelian varieties

B_α indexed by compact open subsets $U_\alpha \subset W/W^\perp$

(2) If $U \subset V$ is a compact open subgroup, get

(a) $U_\alpha = (U \cap W)/(U \cap W^\perp)$, hence B_α.

(b) $\pi(U) \cap S = S_0$, a lattice in S.

$\forall x \in S_0$, choose $\gamma_x \in U \cap \pi^{-1}(x)$. Set

$$\Phi_x(\alpha) = e(\gamma_x/2, \alpha) \cdot \Theta_x(\alpha + \gamma_x)\,, \qquad \alpha \in W\,,$$

a function on $W/W^\perp$ "related to" Θ_0.

$\therefore$ Φ_x defines a point $P_\alpha(x) \in B_\alpha$.

[If $\gamma'_x = \gamma_x + \eta$, $\eta \in U + W$, then $\phi'_x(\alpha) =$ const. $e(\eta/2,\alpha)\,\phi_x(\alpha+\eta)$, so $P_\alpha(x)$ doesn't change.]

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[21] The quotient V/W here corresponds to the quotient $\mathcal{V}/W$ in (1).

[22] The functions Θ_x is related to the measures μ_x as in (II):

$$\mu_x(a_2 + 2^n\Lambda_2) = 2^{ng} \sum_{a_1 \in 2^n\Lambda_1/\Lambda_1} e(a_2, a_1/2)\, \Theta_x(a_1 + a_2)\,.$$

$$\Theta_x(a_1 + a_2) = e(a_2, a_1/2) \int_{a_2 + \Lambda_2} e(a_1, \beta) \cdot d\mu_x(\beta) \qquad a_1 \in V_1, a_2 \in V_2\,.$$

Get a homomorphism $\begin{array}{ccc} S_0 & \xrightarrow{P_\alpha} & B_\alpha \\ x & \longmapsto & P_\alpha(x) \end{array}$

(c) $G_\alpha = \mathbf{Spec}_{B_\alpha}\left(\bigoplus_{x \in S_0} \{T^*_{P_\alpha(x)} L_\alpha \otimes L_\alpha^{-1}\}\right)$

A class of rigid analytic maps

Given: $K =$ complete valued field, $C = \widehat{\overline{K}}$,

Given: G, a comm. alg. grp. over K of type

$$\begin{array}{c} G \\ \mathbb{G}_m^r \Big\downarrow \pi \\ A \end{array} \qquad \text{abel. var.,}$$

L, ample inv. sheaf on A, all rational over K.

Now

$$G \cong \mathbf{Spec}_A \left\{ \bigoplus_{n \in \mathbb{Z}^r} (K_1^{n_1} \otimes \cdots \otimes K_r^{n_r}) \right\}$$

where $K_1, \ldots, K_r$ are invertible sheaves on A, alg. equiv. to $\mathcal{O}_A$.

To define a rigid analytic map $\phi\colon G_C \longrightarrow \mathbb{P}^m_C$, need $m+1$ analytic sections of $\pi^*(L)$ over G_C.

$$\rightsquigarrow \quad \text{m+1 Laurent-type expressions } L_i = \sum_{n \in \mathbb{Z}^r} s(n,i)\,, \quad 0 \le i \le m$$
$$s(n,i) \in \Gamma\,(A, L \otimes K_1^{n_1} \otimes \cdots \otimes K_r^{n_r})$$

Convergence: $\forall x \in G_C$, get $\pi(x) = y \in A_C$, plus $K_i(y) \xrightarrow{\sim} C$ for $i = 0, 1, \ldots m$. Then *evaluate*:

$$s(n,i) \longmapsto \mathrm{Val}_x[s(n,i)] \in L(y)$$

Ask that

$$\sum_n \mathrm{Val}_x[s(n,i)] \qquad \begin{cases} \text{exists in } L(y) \text{ for all } i \\ \text{\& not be } 0 \text{ for all } i \end{cases} .$$

Hence ϕ comes out.

§2. Application to monodromy: method of theta functions

Given

(a) an abelian variety X over K $\longrightarrow$ get $T_2(X)$, a module over $\mathbb{Z}_2\big[\mathrm{Gal}(\overline{K}/K)\big]$,

(b) a principal polarization on X plus an even symmetric theta-divisor D_θ representing it $\longrightarrow$ get a theta function $\Theta\colon V_2(X) \to \overline{K}$ s.t. $\Theta(\sigma x) = \Theta(x)^\sigma$ $\forall \sigma \in \mathrm{Gal}(\overline{K}/K)$.

[State converse: all such $(V, \Lambda, e, e_*, \Theta)$ come from (X, D_θ).]

Problem is:

if $K =$ local field, alg. cl. res. field k, $\mathrm{char}(k) \neq 2$,

& if $\Gamma := \mathrm{Gal}(\overline{K}/K)$ acts on $T_2(X)$ via its tamely ramified quotient Γ_{tame}.[23],

show that

$\exists$ an open subgroup $U \subset \Gamma_{\text{tame}}$ s.t. γ operates unipotently on $T_2(X)$ $\forall \gamma \in U$

Method: a complete description to the solutions of the theta functional equations over a local field. viz. $\exists$

(i) $(0) \subset W^\perp \subset W \subset V$ subspaces, $\pi\colon V \to V/W$

(ii)
$$\begin{array}{ccc} S & \hookrightarrow & V/W \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{Z}[1/2]^r & \hookrightarrow & \mathbb{Q}_2^r \end{array}$$

(iii) $Q\colon S \longrightarrow \mathbb{R}$ pos. def. quad. form

s.t.
$$\Theta = \sum_{s \in S} \Theta_x$$

(a) $\mathrm{supp}(\Theta_x) \subset \pi^{-1}(x) + \Lambda$.

(b) $\max_y |\Theta_x(y)| = e^{-Q(x,x)}$.

(c) $\Theta_x(\alpha + \beta) = e_*(\beta/2) \cdot e(\beta/2, \alpha) \cdot \Theta_x(\alpha)$ for all $\beta \in \Lambda$.

(d) $\Theta_x(\alpha + \beta) = e(\beta, \gamma_x - \frac{\alpha}{2}) \cdot \Theta_x(\alpha)$ for all $\beta \in W^\perp$ if $\gamma_x \in \pi^{-1}(x)$.

[23] $\Gamma_{\text{tame}} \cong \prod_\ell \mathbb{Z}_\ell(1)$, where ℓ runs through all prime numbers which are invertible in k.

Claim: It follows that

$$\gamma = \text{id. on } W \text{ and on } V/W^{\perp} \qquad \forall\, \gamma \in U\,,$$

i.e. the matrix representation of γ has the form

$$\begin{pmatrix} I & 0 & * \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

This Claim will be proved in two steps below.

Step 1. Assume that $\Theta\colon V \to k$ is an algebraic theta function for (V, Λ, e, e_*), $\sigma \in \mathrm{Sp}(V, \Lambda)$ such that

- $\forall x \in V$, $\exists \eta \in \frac{1}{2}\Lambda$ s.t. $\Theta(x+\eta) \neq 0$,
- $\Theta(\sigma x) = \Theta(x)$ for all $x \in V$

Then σ is of finite order.

Proof of Step 1. Replace σ by a suitable tower so that $(\sigma - 1)\Lambda \subseteq 4\Lambda$. We will show that[24] for any $n \geq 2$,

$$(\sigma - 1)\Lambda \subseteq 2^n\Lambda \implies (\sigma - 1)\Lambda \subseteq 2^{2n-1}\Lambda$$

For any $x \in 2^{-n}\Lambda$, we have

$$\Theta(x) = \Theta(\sigma x) = \Theta(x + (\sigma x - x)) = e_*(\frac{\sigma x - x}{2}) \cdot e(\frac{\sigma x - x}{2}, x) \cdot \Theta(x)$$

$\therefore\ e(\sigma x - x, x) = 1$ if $\Theta(x) \neq 0$. Pick an $\eta \in \frac{1}{2}\Lambda$ such that $\Theta(x+\eta) \neq 0$. Then

$$1 = e((\sigma-1)(x+\eta), (x+\eta)) = e((\sigma-1)x, x) \cdot e((\sigma-1)\eta, x) \cdot e((\sigma-1)x, \eta) \cdot e((\sigma-1)\eta, \eta))\,.$$

The last term is 1 because $(\sigma - 1)\eta \in 2^{n-1}\Lambda \subset 2\Lambda$. The product of the two middle terms is

$$\begin{aligned} e((\sigma-1)\eta, x) \cdot e((\sigma-1)x, \eta) &= e(\sigma\eta, x) \cdot e(x, \sigma^{-1}\eta) = e((\sigma^2 - 1)\eta, \sigma x) \\ &= e((\sigma-1)^2\eta + 2(\sigma-1)\eta, \sigma x) = 1 \end{aligned}$$

because $(\sigma-1)^2\eta \in 2^{2n-1}\Lambda \subseteq 2^n\Lambda$ and $2(\sigma-1)\eta \in 2^n\Lambda$. So

$$Q(x) := e((\sigma-1)x, x) = 1 \qquad \forall x \in 2^{-n}\Lambda\,.$$

Now we have

$$1 = \frac{Q(x+y)}{Q(x)\cdot Q(y)} = e((\sigma-1)x, y) \cdot e((\sigma-1)y, x) = e(x, \sigma^{-1}y) \cdot e(\sigma y, x) = e(x, \sigma^{-1}y - \sigma y)$$

for all $x, y \in 2^{-n}\Lambda$, therefore $\sigma^{-1}y - \sigma y \in 2^n\Lambda$ for all $y \in 2^{-n}\Lambda$. Write $\sigma = 1 + \tau$, we have

$$2^n\Lambda \ni \sigma^2 y - y = 2\tau y + \tau^2 y \qquad \forall y \in 2^{-n}\Lambda.$$

But $\tau^2 y \in 2^n\Lambda$, therefore $\tau y \in 2^{n-1}\Lambda$ for all $y \in 2^{-n}\Lambda$, i.e. $(\sigma-1)\Lambda \subseteq 2^{2n-1}\Lambda$. Q.E.D.

[24] This statement was formulated for $n = 2$ in the original notes.

We go back to the algebraic theta function Θ for $(V = V_2(X), \Lambda = T_2(X), e, e_*)$ attached to (X, D_θ). Let $W \subset V$ be the associated cusp, $W^\perp \subset W$. An element $\gamma \in \Gamma_{\text{tame}}$ operates on V via an element of $\mathrm{Sp}(V,\Lambda)$ s.t. $\gamma(W) \subseteq W$, $\gamma(W^\perp) \subseteq W^\perp$. By Step 1, there exists an open subgroup $U \subset \Gamma_{\text{tame}}$ such that every $\gamma \in U$ has a matrix representation of the form

$$\begin{pmatrix} {}^tA^{-1} & B & * \\ 0 & \mathrm{I} & C \\ 0 & 0 & A \end{pmatrix}$$

i.e. γ operates on $W/W^\perp$ as the identity.

Step 2. $A = I$, $B = 0$, i.e. $\gamma|_W = \mathrm{id}_W$.

We need the following facts for the proof of Step 2; they are consequences of the results in §1, summarized at the beginning of this section.

Fact (a). $\sigma(x) := \sup_{\eta \in \frac{1}{2}\Lambda} |\Theta(x+\eta)$ depends only on the image of x in $V/(W + \frac{1}{2}\Lambda)$.

Fact (b). $\forall x \in V$, $\exists \xi_x \in V$, depending only on the image of $x \in V/W$, s.t.

$$|\Theta(x+u) - e(\xi_x, u)\,\Theta(x)| < \sigma(x) \qquad \forall u \in W^\perp\,.$$

We know that $\Theta(\gamma x) = \Theta(x)^\gamma \quad \forall x \in V$, $\gamma(\Lambda) \subseteq \Lambda$ and $(\gamma - 1)W \subseteq W^\perp$. Replacing U by an open subgroup, we may assume[25] that $U \cong \mathbb{Z}_2$ and

$$(\sigma - 1)(\Lambda) \subseteq 8\Lambda + W^\perp, \quad (\sigma - 1)(\Lambda \cap W) \subseteq 8\Lambda$$

i.e. $P(3)$ holds, where $P(n)$ stands for the statement

$$(\sigma - 1)(\Lambda) \subseteq 2^n\Lambda + W^\perp \ \text{ and } \ (\sigma - 1)(\Lambda \cap W) \subseteq 2^n\Lambda \quad \forall \gamma \in U\,.$$

It is clear that Step 2 follows from Claim 2 below.

Claim 2. Suppose $P(n)$ holds, $n \geq 3$, then $P(2n-1)$ holds.[26]

The first part of $P(2n-1)$ implies that if $x \in \Lambda \cap W$, $\xi = 2^{-2n+1}\lambda + w$, $\lambda \in \Lambda$, $w \in W$, then $\gamma^{-1}\xi \in \Lambda + W^\perp$, so

$$e(\gamma x, \xi) = e(x, \gamma^{-1}\xi) = 1\,.$$

i.e. the second part of $P(2n-1)$ follows from the first part.

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[25]Because $\mathrm{Sp}(V,\Lambda)$ is an extension of a finite group by a pro-2 group.

[26]Claim 2 was formulated for $n = 3$ in the original notes.

Let $x \in 2^{-n}\Lambda$, $n \geq 3$. Changing x by an element of $\frac{1}{2}\Lambda$, we may assume that $|\Theta(x)| = \sigma(x)$. Write $(\gamma - 1)x = \eta + u$, $\eta \in \Lambda$, $u \in W^\perp$. Then

$$\Theta(x)^\gamma = \Theta(\gamma x) = \Theta(x + \eta + u) = e_*(\eta/2)\, e(\eta/2, x+u)\, e(\xi_x, u)\, \Theta(x)\,.$$

Change x to x' with $w := x' - x \in 2^{-n}\Lambda \cap W$. Then $|\Theta(x')| = \sigma(x') = \sigma(x)$ too by Facts (a), (b) above. We know[27] that

$$\frac{\Theta(x')^\gamma}{\Theta(x')} = \frac{\Theta(x)^\gamma}{\Theta(x)}\,.$$

$$\therefore\ e_*(\eta/2)\, e(\eta/2, x+u)\, e(\xi_x, u) = e_*(\eta/2)\, e(\eta'/2, x'+u)\, e(\xi_x, u)$$

$$\therefore\ e(\eta, x+u) = e(\eta', x'+u)\,.$$

We have $(\gamma-1)x = \eta + u$, $(\gamma-1)x' = \eta' + u$, $\eta' - \eta = (\gamma-1)w \in W^\perp \cap \Lambda$ by $P(n)$.

$$\therefore\ e(\eta, x) = e(\eta', x') = e(\eta + (\gamma-1)w, x+w)$$

$$\therefore 1 = e(\eta, w)\, e((\gamma-1)w, x) = e(w, -\eta)\, e(w, (\gamma^{-1}-1)x) = e(w, -\eta + (\gamma^{-1}-1)x)$$

for all $w \in W \cap 2^{-n}\Lambda$.

$$\therefore\ -\eta + (\gamma^{-1}-1)x \in 2^n\Lambda + W^\perp$$

$$\therefore\ \gamma\eta + (\gamma-1)x \in 2^n\Lambda + W^\perp$$

$$\therefore\ \gamma\eta + \eta \in 2^n\Lambda + W^\perp \quad \because u \in W^\perp$$

Hence

$$2\eta \in 2^n\Lambda + W^\perp$$

by $P(n)$, i.e.

$$\eta \in 2^{n-1}\Lambda + W^\perp\,.$$

We have shown that $\forall y \in 2^{-n}\Lambda$, $\exists x \in y + \frac{1}{2}\Lambda$ s.t. $(\gamma-1)x \in 2^{n-1}\Lambda + W^\perp$ for all $\gamma \in U$. So

$$(\gamma - 1)\Lambda \subset 2^{2n-1}\Lambda + W^\perp\,.$$

Q.E.D.

[27] from the structure of tamely ramified extensions of local fields.