

PATHOLOGIES III.*

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This note continues, more or less, our two previous papers [2] and [3] in which we have been presenting unpleasant facts of (algebro-geometric) life. The specific topic in this note is Kodaira's vanishing theorem, or as it is called classically, the regularity of the adjoint linear system. Except for one slight fudge (our example is a normal surface, not a non-singular one) our result is that K. V. Theorem is *false* in characteristic p . In fact, this example is presented in § 2. In § 1, we give an outline of the true result in characteristic 0, and in particular extend the theorem to singular varieties in an attempt to clarify the role played by non-singularity.

The classical form of Kodaira's Vanishing Theorem (in the surface case) may be found in Zariski's book [7], p. 144, where, in essence, it is stated:

(*) If F is a non-singular projective surface, in characteristic 0, K the canonical divisor class of F , and H a hyperplane section of F , then the linear system $|K + H|$ is regular (Picard, 1906). More generally, if H is an irreducible curve which is part of an algebraic family of curves other than an irrational pencil, then $|K + H|$ is regular (Severi, 1908).

In terms of cohomology, $|K + H|$ regular means that

$$H^1(O_F(K + H)) = H^2(O_F(K + H)) = (0).$$

Or, by Serre duality, that

$$H^0(O_F(-H)) = H^1(O_F(-H)) = (0).$$

This is the form generalized by Kodaira [1]. He showed:

(**) If V is a non-singular projective variety of dimension n , characteristic 0, and H is an ample¹ divisor class on V , then

$$H^i(O_V(-H)) = (0), \quad i = 0, 1, \dots, n-1.$$

1. The fact that $H^0(O_V(-H)) = (0)$ is absolutely trivial in all characteristics. Thus the reader can check:

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¹ Ample in Grothendieck's sense, of course.

PROPOSITION 1. *If V is any complete variety, and L is any invertible sheaf on V such that $\Gamma(L^n) \neq 0$ for some $n \geq 1$, then $\Gamma(L^{-1}) = (0)$ unless $L \cong \mathcal{O}_V$.*

On the other hand, the vanishing of H^1 already is very subtle. For H^1 's I claim the following which generalizes most of Severi's assertion too:

THEOREM 2. *Let V be a complete normal variety of dimension at least 2, characteristic 0. Let L be an invertible sheaf on V such that, for large n , L^n is spanned by its sections. Let these sections define the morphism*

$$V \xrightarrow{\phi} W$$

(Recall that if $n \gg 0$, then $\phi_*(\mathcal{O}_V) = \mathcal{O}_W$, W is normal and the fibres of ϕ are connected). Then

$$H^1(L^{-m}) = (0) \iff \dim W > 1.$$

for all $m \geq 1$.

Proof. The implication \Rightarrow is nearly obvious. By definition of ϕ , there is a positive integer m and a very ample invertible sheaf M on W such that $L^m \cong \phi^*(M)$. Use Leray's spectral sequence:

$$H^p(W, R^q\phi_*(L^{-m})) \Rightarrow H^*(V, L^{-m}).$$

By definition of W , $\phi_*(\mathcal{O}_V) = \mathcal{O}_W$; hence $\phi_*(L^{-2m}) = M^{-2}$. The spectral sequence gives:

$$0 \rightarrow H^1(W, M^{-2}) \rightarrow H^1(V, L^{-2m}) \rightarrow \dots$$

But if W is a curve, and M has positive degree, then $H^1(W, M^{-2}) \neq (0)$, hence $H^1(V, L^{-2m}) \neq (0)$.

To prove the implication \Leftarrow we first reduce to the case where V is non-singular and projective. To do this, introduce a projective de-singularization

$$\pi: \tilde{V} \rightarrow V.$$

Let $\tilde{L} = \pi^*L$. Since V is normal, $\pi_*(\mathcal{O}_{\tilde{V}}) = \mathcal{O}_V$, hence $\pi_*(\tilde{L}^{-m}) = L^{-m}$. Leray's spectral sequence again gives:

$$0 \rightarrow H^1(V, L^{-m}) \rightarrow H^1(\tilde{V}, \tilde{L}^{-m}) \rightarrow \dots$$

Therefore, $H^1(\tilde{V}, \tilde{L}^{-m}) = (0)$ implies $H^1(V, L^{-m}) = (0)$.

Now assume V is non-singular and projective. We can, of course, assume that the ground field is the field \mathbb{C} of complex numbers. As above, let M be

a very ample invertible sheaf on W such that $\phi^*(M) \cong L^m$, some positive m . Let $i: W \rightarrow \mathbf{P}_n$ be an immersion such that M is the restriction of $O(1)$ to W . Let \mathbf{L} , \mathbf{M} , and \mathbf{O}_1 be the line bundles on V , W and \mathbf{P}_n corresponding to L , M and $O(1)$. Next, equip \mathbf{O}_1 with its standard Hermitian structure: by pull-back, this puts a Hermitian structure on \mathbf{L}^m and \mathbf{M} , and by taking m -th roots, this puts a Hermitian structure on \mathbf{L} . Let Ω_0 be the curvature form of \mathbf{O}_1 : this is well known to be *positive definite*. The curvature form of \mathbf{L}^m is then $(i \circ \phi)^*\Omega_0$ and the curvature form Ω of \mathbf{L} is just

$$\Omega = \frac{1}{m} (i \circ \phi)^*\Omega_0,$$

which is, therefore, *positive semi-definite*.

Now recall the fundamental inequality of Kodaira's paper. Choose a Kähler metric on V , given in local coordinates by

$$ds^2 = \sum_{\alpha, \beta=1}^g g_{\alpha\beta} dz^\alpha d\bar{z}^\beta.$$

Choose local trivializations of \mathbf{L} and assume that the hermitian structure on \mathbf{L} is given by

$$\|s\|^2 = a_j^{-1} \cdot |f_j|^2$$

if a section s of \mathbf{L} is defined by the function f_j in terms of our local trivialization. By definition

$$\Omega = \frac{i}{2\pi} \bar{\partial} \partial (\log a_j) \quad \text{locally} = \frac{i}{2\pi} \sum_{\alpha, \beta} X_{\alpha\beta} dz^\alpha d\bar{z}^\beta$$

Now assume that an element $\phi \in H^{n-1}(V, \Omega^n \otimes L)$ is given by a harmonic \mathbf{L} -valued $(n, n-1)$ -form Φ , where

$$\Phi = \sum_{\text{locally } \alpha_i \text{'s}} u_{12 \dots n \alpha_1 \dots \alpha_{n-1}} dz^1 \wedge \dots \wedge dz^n \wedge d\bar{z}^{\alpha_1} \wedge \dots \wedge d\bar{z}^{\alpha_{n-1}}.$$

Then Kodaira proves that:

$$0 \geq \int_V \frac{1}{a_j} \left\{ \sum_{\substack{\alpha_1, \dots, \alpha_{n-1} \\ \gamma_1, \dots, \gamma_{n-1} \\ \beta, \epsilon}} X_{\epsilon \alpha_1} u_{12 \dots n \beta \alpha_2 \dots \alpha_{n-1}} \cdot \bar{u}_{12 \dots n \gamma_1 \dots \gamma_{n-1}} g^{\epsilon \beta} \cdot g^{\gamma_1 \alpha_1} \dots g^{\gamma_{n-1} \alpha_{n-1}} \right\} dV.$$

(The reader may check that this integral is intrinsic.) This is easier to see in terms of the dual harmonic form

$$\Phi^\dagger = \frac{1}{\text{locally } a_j} * \Phi$$

which is an L^{-1} -valued $(0, 1)$ -form. Express Φ^\dagger :

$$\Phi^\dagger = \sum_{\alpha} v_{\alpha} d\bar{z}^{\alpha}.$$

locally

Then one checks that

$$v_{\alpha} = \frac{1}{a_j \cdot g} \sum_{\epsilon} \bar{u}_{12 \dots n \ 12 \dots \hat{\epsilon} \dots n} g_{\epsilon \alpha}$$

and, after some sweat, one also can rearrange Kodaira's integral into:

$$0 \geq \int_V g \cdot a_j \{ \sum_{\gamma, \beta} X_{\gamma \beta} g^{\gamma \beta} \cdot \sum_{\alpha, \epsilon} v_{\alpha} \bar{v}_{\epsilon} g^{\epsilon \alpha} \\ - \sum_{\gamma, \epsilon} X_{\gamma \epsilon} \cdot [\sum_{\beta} g^{\gamma \beta} v_{\beta}] \cdot [\sum_{\alpha} g^{\alpha \epsilon} \bar{v}_{\alpha}] \} dV.$$

To see exactly what we have here, examine the integrand at one point P of V . Choose local coordinates so that the Kähler metric is given by

$$g_{\alpha \beta}(P) = \delta_{\alpha \beta},$$

and the curvature form of L is diagonalized:

$$X_{\alpha \beta}(P) = \lambda_{\alpha} \cdot \delta_{\alpha \beta}.$$

Since X is positive semi-definite, $\lambda_{\alpha} \geq 0$ for all α . The integrand is then:

$$(i) \quad \sum_{\alpha \neq \beta} \lambda_{\alpha} \cdot |v_{\beta}|^2.$$

Since this is non-negative, we conclude that the integral is non-positive only if the integrand is identically 0. If $\Phi \not\equiv 0$, moreover, then $\Phi(P) \neq 0$ at a dense set of points P . From the form of (i), we conclude that at most one λ_{α} is not zero at all such points P ; hence, in fact, at most one λ_{α} is not zero at all points. In terms of ϕ , this means that $\dim W = 1$ (i.e. use of Sard's theorem and the positive definiteness of Ω_0). Q.E.D.

Actually a completely algebro-geometric proof of this result can be given. Because this proof is so simple and because it indicates the reason why the theorem will fail in characteristic p , it seems worth giving:

Second Proof. As above, we need only prove " \Leftarrow ." Therefore assume $\dim W > 1$. In this proof, we need to make a preliminary reduction not to the case where V is non-singular, but to the case where $L = O_V(D)$, D a reduced effective Cartier divisor on V . But by our hypothesis on L , it is clear that for large n , $L^n = O_V(D_n)$ for some reduced Cartier divisor D_n on V . Pick an affine open covering $\{U_i\}$ of V such that L is defined by the co-cycle $\{a_{ij}\}$,

$$a_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_V)$$

and D_n is defined by local equations f_i in U_i :

$$f_i \in \Gamma(U_i, \mathcal{O}_V)$$

$$f_i = a_{ij}^n \cdot f_j.$$

Define a n -fold cyclic covering:

$$\pi: \tilde{V} \rightarrow V$$

by local equations:

$$z_i^n = f_i$$

$$z_i = a_{ij} \cdot z_j.$$

Then if $L = \pi^*(L)$, the equations $z_i = 0$ define a Cartier divisor \tilde{D} on V such that

$$\tilde{L} = \mathcal{O}_{\tilde{V}}(\tilde{D}).$$

Moreover, suppose we prove that $H^1(\tilde{V}, \tilde{L}^{-1}) = (0)$. Then

$$\begin{aligned} H^1(V, \pi_*(\mathcal{O}_{\tilde{V}}) \otimes L^{-1}) &\cong H^1(V, \pi_*(\tilde{L}^{-1})) \\ &\cong H^1(\tilde{V}, \tilde{L}^{-1}) \\ &\cong (0). \end{aligned}$$

But \mathcal{O}_V is a direct summand of the coherent sheaf $\pi_*(\mathcal{O}_{\tilde{V}})$ of \mathcal{O}_V -modules:

$$\begin{array}{ccc} & \alpha & \\ \mathcal{O}_V & \xrightleftharpoons{\hspace{1.5cm}} & \pi_*(\mathcal{O}_{\tilde{V}}) \\ & \beta & \end{array}$$

where α is the canonical map π^* taking functions on V to functions on \tilde{V} ,

and $\beta = \frac{1}{n}(\text{Trace})$. Therefore $H^1(V, L^{-1}) = (0)$ also.

Now assume $L = \mathcal{O}_V(D)$. The next step is to show that D is connected: Let H be any hyperplane section of W in the embedding defined by M . Then, since $\dim W > 1$, H is connected, hence $\phi^{-1}(H)$ is connected. But since the morphism $i \circ \phi: V \rightarrow \mathbf{P}_n$ is defined by the complete linear $\Gamma(L^m)$, the divisor mD equals the divisor $\phi^*(H)$ for some hyperplane section H of W . Therefore D is connected.

Now consider the exact sequence:

$$0 \rightarrow L^{-1} \rightarrow \mathcal{O}_V \rightarrow \mathcal{O}_D \rightarrow 0.$$

This gives:

$$H^0(\mathcal{O}_V) \rightarrow H^0(\mathcal{O}_D) \rightarrow H^1(L^{-1}) \rightarrow H^1(\mathcal{O}_V) \rightarrow H^1(\mathcal{O}_D).$$

Since D is reduced and connected, $H^0(\mathcal{O}_D)$ consists only in constant functions, all of which lift to $H^0(\mathcal{O}_V)$. Therefore,

$$H^1(L^{-1}) \cong \text{Ker}\{H^1(\mathcal{O}_V) \rightarrow H^1(\mathcal{O}_D)\}.$$

Let $\text{Pic}^0(V)$, $\text{Pic}^0(D)$ be the connected components of the origin of the Picard schemes of V and D . Recall that $H^1(\mathcal{O}_V)$, $H^1(\mathcal{O}_D)$ are canonically isomorphic to the Zariski-tangent spaces to $\text{Pic}^0(V)$ and $\text{Pic}^0(D)$ at their origins, so that the map from $H^1(\mathcal{O}_V)$ to $H^1(\mathcal{O}_D)$ is just the differential of the canonical homomorphism:

$$\text{Pic}^0(V) \xrightarrow{\alpha} \text{Pic}^0(D)$$

(cf. [4], Lecture 24). Since the characteristic is 0, the kernel of α , as a subgroupscheme of $\text{Pic}^0(V)$, must be reduced (cf. [4], Lecture 25). Therefore, if the differential of α has a non-trivial kernel, α itself must have a positive-dimensional kernel. Therefore, every non-trivial subgroupscheme has non-trivial points of finite order on it. Therefore, we conclude:

$$H^1(L^{-1}) \neq (0) \Rightarrow \left(\begin{array}{l} \exists \delta \in \text{Pic}^0(V) \text{ of finite order } n, n > 1, \\ \text{such that } \alpha(\delta) = 0. \end{array} \right)$$

Now δ defines, in the usual way, an unramified Galois covering:

$$\begin{array}{c} V' \\ \pi \downarrow \\ V \end{array}$$

with covering group $\mathbf{Z}/n\mathbf{Z}$. $\alpha(\delta) = 0$ implies that this covering splits over D , i. e., $\pi^{-1}(D) = D_1 \cup \dots \cup D_n$ (the D_i 's disjoint and isomorphic to D). But let $L' = \pi^*L$. Then it is clear that the pair (V', L') satisfy all the requirements imposed on (V, L) . Therefore the identical argument used to prove that D is connected also proves that $D' = \pi^{-1}(D)$ is connected. This is a contradiction. Q. E. D.

Note that the only place where $\text{char} = 0$ has been used is in the step where we used the fact that $\ker(\alpha)$ must be reduced. Incidentally, this proof also can be generalized to further classes of invertible sheaves $L = \mathcal{O}_V(D)$, which do not necessarily have the property that L^m is spanned by its sections

for large m . But I don't know of any really definitive statement in this direction. One further point: it is not clear whether or not the normality of V is essential in Theorem 2. By Grothendieck's duality theorem, it follows that V must at least have the "property S2": $\forall x \in V$, $\text{depth } (O_x) \geq 2$ (x a closed point). But I don't know whether or not singularities in codimension 1 can be allowed.

2. Now consider the question of whether or not

$$H^1(V, O_V(-D)) = (0)$$

when V is a normal variety in characteristic p , and D is an ample effective divisor. As in characteristic 0, we can analyze this via the exact sequence:

$$H^0(O_V) \rightarrow H^0(O_D) \rightarrow H^1(O_V(-D)) \rightarrow H^1(O_V) \rightarrow H^1(O_D).$$

The contribution from the left is easy to dispose of:

PROPOSITION 3. *If V is a complete normal variety, $\dim V \geq 2$, and D is an effective ample Cartier divisor on V , then $H^0(O_D)$ consists only in constants.*

Proof. In characteristic 0, this follows from Theorem 2: Now assume that the characteristic is $p > 0$. Let D be defined by local equations $f_i = 0$ with respect to a covering $\{U_i\}$ of V . Since D is ample, D is connected. Therefore, if $s \in H^0(O_D)$, s is constant on the scheme D_{red} . Assume that O_D has a non-zero section s which is zero on D_{red} . Let s be represented in U_i by a function

$$s_i \in \Gamma(U_i, O_V).$$

Then $s_i - s_j = a_{ij}f_i$, $a_{ij} \in \Gamma(U_i \cap U_j, O_V)$. It follows that for all positive ν ,

$$s_i^{p\nu} - s_j^{p\nu} = a_{ij}^{p\nu} \cdot f_i^{p\nu}.$$

Therefore the collection of functions $\{s_i^{p\nu}\}$ defines a section s_ν of O_{D_ν} where D_ν is the Cartier divisor defined by $\{f_i^{p\nu}\}$. Notice that $s_\nu \neq 0$: for if $s_\nu = 0$, then we would have:

$$s_i^{p\nu} = b_i \cdot f_i^{p\nu}, \quad b_i \in \Gamma(U_i, O_V).$$

Then (s_i/f_i) would be a rational function on V that was integrally dependent on $\Gamma(U_i, O_V)$. Since V is normal, this would imply that

$$s_i \in f_i \cdot \Gamma(U_i, O_V)$$

i. e., $s = 0$. Therefore $s_\nu \neq 0$, and by the exact sequence

$$H^0(O_V) \rightarrow H^0(O_{D_\nu}) \rightarrow H^1(O_V(-D_\nu))$$

this implies that $H^1(O_V(-D_\nu)) \neq (0)$. But

$$O_V(-D_\nu) \cong [O_V(-D)]^{\otimes p^\nu}.$$

Since we can take ν arbitrarily large, and since $O_V(D)$ is ample, this contradicts the lemma of Enriques-Severi-Zariski (cf. [5], Th. 4, p. 270). Q. E. D.

Our question is, therefore, equivalent to the injectivity of $H^1(O_V) \rightarrow H^1(O_D)$. We can further restrict this kernel by examining the Frobenius cohomology operation:²

$$H^1(O_V) \xrightarrow{F} H^1(O_V)$$

defined by $F(\{a_{ij}\}) = \{a_{ij}^p\}$ in terms of Čech co-cycles. The point is that if we consider F on the subspace $H^1(O_V(-D))$, it factors as follows:

$$(i) \quad \begin{array}{ccc} H^1(O_V) & \xrightarrow{F} & H^1(O_V) \\ \cup & & \cup \\ H^1(O_V(-D)) & & H^1(O_V(-D)) \\ & \searrow F & \\ & & \cup \\ & & H^1(O_V(-pD)). \end{array}$$

But, for large ν , $H^1(O_V(-p^\nu D)) = (0)$ by the lemma of Enriques-Severi-Zariski. Diagram (i) implies, by induction, that

$$F^\nu\{H^1(O_V(-D))\} \subset H^1(O_V(-p^\nu D)).$$

Therefore:

PROPOSITION 4. F is nilpotent on the image of $H^1(O_V(-D))$ in $H^1(O_V)$.

This Proposition can also be proven by using the fact that elements of $H^1(O_V)$ idempotent for F die in p -cyclic unramified coverings of V (cf. [6], § 16) and by imitating the second proof of Theorem 2. On the other hand, the elements of $H^1(O_V)$ killed by F die in principal coverings of V with infinitesimal structure group α_p . These facts may be expressed in the language of Grothendieck cohomologies by:

$$\begin{aligned} \{\alpha \in H^1(O_V) \mid F\alpha = \alpha\} &\cong H_{\text{étale topology}}^1(V, \mathbf{Z}/p\mathbf{Z}), \\ \{\alpha \in H^1(O_V) \mid F\alpha = 0\} &\cong H_{\text{flat topology}}^1(V, \alpha_p). \end{aligned}$$

² This fact was pointed out to me by Serre, and has also been noticed by Grauert.

But principal coverings of V with structure group α_p , even if they are varieties at all, may be non-normal and must be purely inseparable over V . This is very awkward and is what really leads to the debacle. To present the counter-example, I want first to make positive use of the preceding remarks via:

LEMMA 5. *Let $f: V' \rightarrow V$ be a finite surjective morphism of normal varieties corresponding to a separable function field extension. Let $\alpha \in H^1(V, O_V)$ be non-zero and such that $F\alpha = 0$. Then $f^*\alpha \in H^1(V', O_{V'})$ is not zero.*

Proof. Let α be represented by the Čech co-cycle $\{a_{ij}\}$ with respect to some open affine covering $\{U_i\}$ of V . Then $F\alpha = 0$ implies that there are functions $g_i \in \Gamma(U_i, O_V)$ such that

$$a_{ij}^p = g_i - g_j.$$

If $f^*\alpha = 0$, then there are also functions $h_i \in \Gamma(f^{-1}(U_i), O_{V'})$ such that

$$f^*(a_{ij}) = h_i - h_j.$$

It follows that

$$h_i^p - f^*(g_i) = h_j^p - f^*(g_j),$$

i.e., there is a constant β such that

$$f^*(g_i) = h_i^p + \beta, \text{ all } i.$$

Therefore $f^*(g_i) \in k(V')^p$. Since $k(V')$ is separable over $k(V)$, this implies that $g_i \in k(V)^p$, for all i . If $g_i = k_i^p$, $k_i \in k(V)$, then since V is normal, it follows that

$$k_i \in \Gamma(U_i, O_V).$$

Then

$$a_{ij} = k_i - k_j,$$

so $\alpha = 0$: a contradiction. Q.E.D.

Example 6. A normal complete algebraic surface V with an ample invertible sheaf L such that

$$H^1(V, L^{-1}) \neq (0).$$

Start with any normal projective algebraic surface V_0 and a non-zero element $\alpha \in H^1(V_0, O_{V_0})$ such that $F\alpha = 0$: e.g. the product of a super-singular elliptic curve with any other curve. Let H be a hyperplane section of V_0 and let $L_0 = O_{V_0}(H)$. We shall let V be the normalization of V_0 in

a suitable finite separable extension field E of $k(V_0)$. If $\pi: V \rightarrow V_0$ is the projection, let $L = \pi^*(L_0) = \mathcal{O}_V(\pi^{-1}(H))$: an ample sheaf on V .

Let α be represented, as in the lemma, by $\{a_{ij}\}$ and let $g_i \in \Gamma(U_i, \mathcal{O}_{V_0})$ satisfy

$$a_{ij}^p = g_i - g_j.$$

Let $h_i = 0$ be a local equation of H in U_i (replacing $\{U_i\}$ by a finer covering if necessary). Define an extension E_i of $k(V_0)$ by the separable equation:

$$z_i^p - h_i^p z_i = g_i.$$

We shall let E be a join of the extensions E_i . Then I claim that for this E and the corresponding V and L , the element $\pi^*\alpha$ is actually in the sub-space:

$$H^1(V, \mathcal{O}_V(-\pi^{-1}(H))) \subset H^1(V, \mathcal{O}_V).$$

Cover V by the open affines $U_i^* = \pi^{-1}(U_i)$. Then $z_i \in \Gamma(U_i^*, \mathcal{O}_V)$ and

$$\begin{aligned} \pi^*\alpha &= \text{coho. class } [a_{ij}] \\ &= \text{coho. class } [a_{ij} - z_i + z_j] \end{aligned}$$

and

$$\begin{aligned} \left\{ \frac{a_{ij} - z_i + z_j}{h_i} \right\}^p &= \frac{a_{ij}^p - z_i^p + z_j^p}{h_i^p} \\ &= \frac{(g_i - z_i^p) - (g_j - z_j^p)}{h_i^p} \\ &= -z_i + (h_j/h_i)^p z_j \\ &\in \Gamma(U_i^* \cap U_j^*, \mathcal{O}_V). \end{aligned}$$

Therefore, $a_{ij} - z_i + z_j$ is actually in $\Gamma(U_i^* \cap U_j^*, \mathcal{O}_V(-\pi^{-1}(H)))$. Q. E. D.

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