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ABELIAN QUOTIENTS OF THE TEICHMÜLLER MODULAR GROUP

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1967

ABELIAN QUOTIENTS
OF THE TEICHMÜLLER MODULAR GROUP*

By

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in Cambridge, Mass., U.S.A.

Let D be the Teichmüller space of genus g : it is a $3g-3$ -dimensional complex analytic manifold isomorphic to a bounded and contractible domain of holomorphy in \mathbb{C}^{3g-3} . Let Γ be the Teichmüller modular group: it is a discrete group acting discontinuously on D . This is the analytic side of the moduli problem.

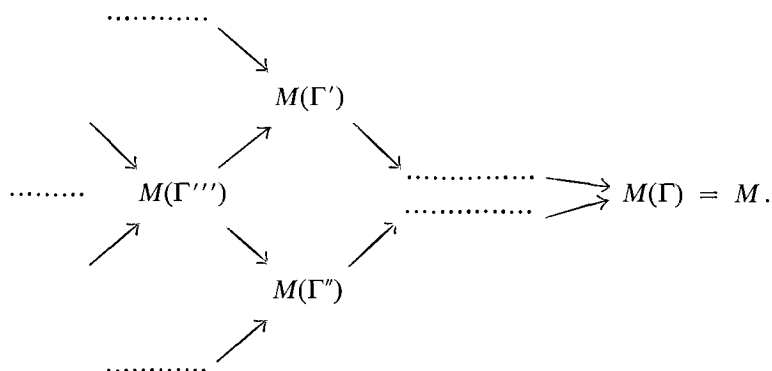
On the other hand, let $M = D/\Gamma$: this analytic space has a canonical structure of algebraic variety. In fact, it turns out to be a Zariski-open subset of a projective variety: a so-called *quasi-projective* variety. Therefore, for all subgroups $\Gamma' \subset \Gamma$ of finite index, the analytic spaces $M(\Gamma') = D/\Gamma'$, being coverings of M , are also quasi-projective algebraic varieties. This follows from the

Generalized Riemann existence theorem.** *If X is any normal algebraic variety, Y any normal analytic space, and $f: Y \rightarrow X$ is a proper holomorphic map with finite fibres, and if there is a Zariski-open set $U \subset X$ such that $f^{-1}(U)$ is dense in Y and $\text{res } f: f^{-1}(U) \rightarrow U$ is unramified, then Y has one and only one structure of algebraic variety making f into a morphism.*

Thus from the algebraic standpoint, one has an inverse system of quasi-projective varieties:

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** In this form, the theorem is due to J. P. Serre and M. Artin. It can be reduced to the comparison theorem of GAGA [9], using either the paper of Grauert-Remmert [2], or resolution of singularities and elementary arguments; or, by methods of Artin and Grothendieck it can be reduced to the 1-dimensional case where it is classical.



Moreover all sufficiently small $\Gamma' \subset \Gamma$ of finite index act freely on D , hence all $M(\Gamma')$ sufficiently “far up” in this inverse system are non-singular varieties, with D as their common universal covering space. The purpose of this note is to prove two closely related results:

Theorem 1. *If $[\Gamma, \Gamma]$ is the commutator subgroup of Γ , then $\Gamma/[\Gamma, \Gamma]$ is a finite cyclic group, whose order divides 10. (?!)*

Theorem 2. *The Albanese variety of M is trivial, i.e. there are no non-trivial rational maps from M to an abelian variety.**

Moreover, in the terminology of [6], these results are also equivalent to:

Theorem. *The Picard group of the moduli problem is a finitely generated abelian group isomorphic to $H^2(\Gamma, \mathbf{Z})$.*

(Cf. §7 of [6]). Analogy with the many calculations that have been made for arithmetic groups acting on symmetric domains, as well as the general feeling that M should be quantitatively similar to projective space, leads me to conjecture that the rank of $H^2(\Gamma, \mathbf{Z})$ is one.

I want to thank Professor Magnus for a very informative letter acquainting me with the literature on the Teichmüller modular group and explaining what seems to be really proven, and what is not too clear. I also want to

* It is equivalent to consider morphisms or rational maps from M to abelian varieties 0, since M is locally a quotient of a non-singular variety by a finite group.

thank Professors Maskit and Grothendieck for helpful comments and for checking my computations.

§1. The connection between Theorem 1, 2.

Recall that if V is any algebraic variety, and $x \in V$ is a base point, then among all morphisms:

$$\begin{aligned} \phi: V &\rightarrow A \\ \phi(x) &= 0 \end{aligned}$$

where A is an abelian variety, O its origin, there is a “universal” one, i.e. one such morphism:

$$\phi_0: V \rightarrow A_0,$$

such that if $\psi: V \rightarrow B$ is any other, there is a unique morphism $\chi: A_0 \rightarrow B$ (which is necessarily a homomorphism) such that $\psi = \chi \circ \phi_0$. A is called the Albanese variety of V . We want to describe the Albanese variety of M .

Theorem 3. *Let V be a non-singular projective variety and let*

$$f: V \rightarrow V_0$$

be a birational morphism of V onto a normal projective variety V_0 . Let $Z \subset V$ be a Zariski closed subset such that

$$\text{codim} f(Z) \geq 2.$$

Then

$$H_1(V - Z, \mathbb{Q}) \xrightarrow{\sim} H_1(V, \mathbb{Q}).$$

Proof. Since $H_1(V - Z, \mathbb{Z})$ and $H_1(V, \mathbb{Z})$ are finitely generated abelian groups, it suffices to show that for almost all primes p ,

$$H_1(V - Z, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} H_1(V, \mathbb{Z}/p\mathbb{Z}).$$

But, for any reasonable topological space X ,

$$H_1(X, \mathbf{Z}/p\mathbf{Z}) \cong \pi_1(X)/\pi_1(X)^p \cdot [\pi_1(X), \pi_1(X)].$$

Therefore the isomorphism we need amounts to showing:

(a) all p -cyclic unramified coverings of $V-Z$ extend to p -cyclic unramified coverings of V

(b) a p -cyclic unramified covering of V that splits over $V-Z$ (i.e., restricted to $V-Z$, it is isomorphic to the disjoint union of p copies of $V-Z$), also splits over V .

However, p -cyclic unramified topological coverings of V and $V-Z$ are canonically algebraic varieties in view of the generalized Riemann Existence Theorem already quoted. Therefore if V is normal, connected coverings of V are obtained by taking the normalization of V in suitable finite extension fields L of the function field $\mathbb{C}(V)$. But such coverings do not split over *any* Zariski-open subset of V . Therefore (b) holds for all p .

To prove (a), we need a preliminary step:

Lemma (Matsumura). *If D_1, \dots, D_n are the components of Z of codimension 1, then the fundamental classes of the D 's are independent in $H^2(V, \mathbf{Z})$.*

Proof. Suppose there was a relation

$$\sum_{i=1}^n n_i \cdot \text{class}(D_i) = 0.$$

Assume $n_i \neq 0$, if $1 \leq i \leq k$, and $n_i = 0$, $i > k$. Let $Z^* = f(D_1 \cup \dots \cup D_k)$. Then so long as $\dim(Z^*) > 0$, cutting V_0 by a sufficiently general hyperplane H , we can replace V_0 by $V_0 \cap H$, V by $f^{-1}(V_0 \cap H)$, and Z by $f^{-1}(f(Z) \cap H)$, and obtain the same pathology with a V of lower dimension. If $\dim Z^* = 0$, then so long as $\dim(V) \geq 3$ we can cut V by a sufficiently general hyperplane H , replace V by $V \cap H$, V_0 by the normalization of $f(V \cap H)$ and Z by the exceptional locus of

$$\text{res}f: V \cap H \rightarrow f(V \cap H)$$

(which still includes $D_1 \cap H, \dots, D_k \cap H$ since they are collapsed to point in V_0). Finally, we obtain a situation contradicting the lemma, with $\dim V = \dim V_0 = 2$. Then by [7], p. 6, the intersection matrix (D_i, D_j) is negative definite, hence the D 's must be homologically independent. Q.E.D.

To complete the proof of (a), we use Kummer Theory. If

$$\pi: W' \rightarrow V - Z$$

is an unramified algebraic p -cyclic extension, it is obtained by normalizing $V - Z$ in the extension of $C(V)$ given by $\sqrt[p]{f}$, some $f \in C(V)$. Define W to be the normalization of V in this extension field. Let D_1, \dots, D_n be the divisor components of Z , and let

$$(f) = \sum_{i=1}^n n_i D_i + \sum_{i=1}^m m_i E_i.$$

Since $\sqrt[p]{f}$ defines an unramified covering outside of Z , it follows that $p \mid m_i$ all i . Therefore in $H^2(V, \mathbf{Z})$,

$$0 = \text{class}((f)) = \sum n_i \cdot \text{class}(D_i) + p \cdot \text{class} \left(\sum \frac{m_i}{p} E_i \right).$$

But since by the lemma, $\text{class}(D_i)$, $1 \leq i \leq n$, in $H^2(V, \mathbf{Z})$ are independent, for almost all primes p this relation implies that p divides all the n_i as well. In that case

$$(f) = p \cdot D$$

for some divisor D on V , and $\sqrt[p]{f}$ defines an everywhere unramified covering of V , i.e. W/V is unramified. Q.E.D.

Corollary 1. *Let V be a non-singular algebraic variety. Assume that V is isomorphic to $V^* - Z^*$ where V^* is a projective variety and $\text{codim } Z^* \geq 2$. Let*

$$\phi: V \rightarrow A$$

be the Albanese morphism of V . Then

$$\phi_*: H_1(V, \mathcal{Q}) \xrightarrow{\sim} H_1(A, \mathcal{Q}).$$

Proof. Let V' be a desingularization of V^* obtained by blowing up only points of Z^* , and let

$$f: V' \rightarrow V$$

be the birational morphism. Let $Z' = f^{-1}(Z^*)$. Then $V' - Z' \cong V$. The morphism ϕ extends to a morphism ϕ' :

$$\begin{array}{ccc} V' & & \\ \cup & \searrow \phi' & \\ V' - Z' & & A \\ \cong & & \\ V & \xrightarrow{\phi} & \end{array}$$

since V' is non-singular ([3], p. 20), and ϕ' is the Albanese morphism for V' . Then it is well-known[#] that

$$\phi'_*: H_1(V', \mathcal{Q}) \xrightarrow{\sim} H_1(A, \mathcal{Q}).$$

This, plus the Theorem applied to $f: V' \rightarrow V$, and Z' , imply the Corollary. Q.E.D.

This Corollary does not apply directly to the moduli variety M since M is not non-singular. However, if $g \geq 3$ M does meet the other condition of the Corollary. In fact, let M_A denote the moduli variety for g -dimensional principally polarized abelian varieties, and let M_A^* denote the Satake compactification of M_A . Let

$$\theta: M \rightarrow M_A$$

[#] This is proven, for example by constructing A as

$$H_1(V', \mathbb{R}) / \text{Image } [H_1(V', \mathbb{Z})]$$

and constructing ϕ' by integrating the differentials $H^0(V', \Omega_V^1)$ cf. [10], p. 82.

be the morphism induced by associating the Jacobian variety and its Θ -polarization to each curve of genus g . In [8], A. Mayer and I determined the set $\overline{\theta(M)} - \theta(M)$ (closure in M_A^*). It turns out to be what you expect — g -dimensional products of lower dimensional jacobians in M_A itself, as Matsusaka and Hoyt had shown; h -dimensional products of lower dimensional jacobians ($h < g$), in $M_A^* - M_A$. In particular, for each type of product decomposition we get a locally closed subset of $\overline{\theta(M)} - \theta(M)$. There is no piece of dimension $3g - 4$ (codimension 1). The only piece of dimension $3g - 5$ is the locus corresponding to products

$$J_1 \times J_{g-1}$$

J_1, J_{g-1} being 1 and $(g-1)$ -dimensional jacobians respectively. The other pieces, such as:

- a) $J_i \times J_j, i + j = g, 1 < i, j < g$
- b) J_{g-1} (giving a point in $(M_A^* - M_A) \cap \overline{\theta(M)}$) have codimension at least 3.

To make the Corollary apply, choose a normal subgroup $\Gamma' \subset \Gamma$ of finite index that acts freely on the Teichmüller space D . Let $M' = D/\Gamma'$ and let $\pi: M' \rightarrow M$ be the canonical morphism. Then M' is non-singular and compactifiable in codimension 2 (take the normalization of $\overline{\theta(M)}$ in the function field of M'). Therefore, we get

$$(\Gamma'/[\Gamma', \Gamma']) \otimes \mathcal{Q} \cong H_1(M', \mathcal{Q}) \cong H_1(A', \mathcal{Q})$$

if $\phi': M' \rightarrow A'$ is the Albanese morphism. Furthermore, the finite group (Γ/Γ') acts on $\Gamma'/[\Gamma', \Gamma'] \otimes \mathcal{Q}$, on M' , and hence on A' .

(I). The canonical homomorphism

$$(\Gamma'/\Gamma', \Gamma') \otimes \mathcal{Q} \rightarrow (\Gamma/[\Gamma, \Gamma]) \otimes \mathcal{Q},$$

is surjective and its kernel is generated by elements

$$x - x^y, y \in \Gamma/\Gamma', x \in (\Gamma'/[\Gamma', \Gamma']) \otimes \mathcal{Q}.$$

(This is well-known; for example, use the Lyndon Spectral Sequence, [4], p. 354, formula (10.6)).

(II). If $\phi: M \rightarrow A$ is the Albanese of M , there is a canonical diagram:

$$\begin{array}{ccc} M' & \xrightarrow{\phi'} & A' \\ \pi \downarrow & & \downarrow \chi \\ M & \xrightarrow{\phi} & A \end{array}$$

for some homomorphism $\chi: A' \rightarrow A$. $\chi: A' \rightarrow A$ can be characterized as the universal homomorphism $\psi: A' \rightarrow B$, B abelian, such that $\psi(x^\gamma) = \psi(x)$, all $x \in A'$, $\gamma \in \Gamma/\Gamma'$. Therefore

$$A \cong A'/[\text{Subgroup generated by points } x^\gamma - x, x \in A', \gamma \in \Gamma/\Gamma'].$$

Using this, it is easy to see that

$$H_1(A, \mathcal{Q}) \cong H_1(A', \mathcal{Q})/[\text{Subgroup generated by points } x^\gamma - x, x \in H_1, \gamma \in \Gamma/\Gamma'].$$

Putting all this together, we finally conclude:

Corollary 2. *If $\phi: M \rightarrow A$ is the Albanese morphism of the moduli variety M and $g \geq 3$, then there is a canonical isomorphism:*

$$(\Gamma/[\Gamma, \Gamma]) \otimes \mathcal{Q} \xrightarrow{\sim} H_1(A, \mathcal{Q}).$$

§2. Dehn's presentation of Γ .

Everything that follows depends on the fundamental paper [1] of Dehn. Let F be a fixed differentiable surface of genus g . We shall picture F as follows:

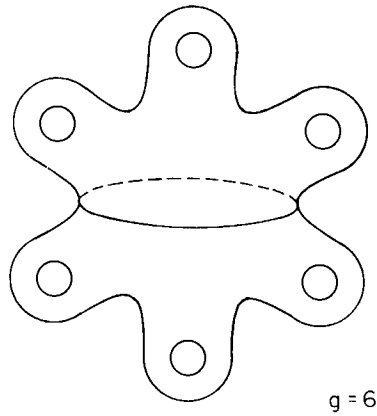


Figure 1

The first basic fact is that:

$$\Gamma \cong \frac{\{\text{Group of orientation preserving homeomorphisms of } F\}}{\{\text{Those homotopic to identity}\}}$$

$$\cong \frac{\{\text{Group of automorphisms of } \pi_1(F, p), \det = +1\}}{\{\text{Inner Automorphisms}\}}.$$

Here the determinant of an automorphism $\alpha: \pi_1(F) \rightarrow \pi_1(F)$ refers to the determinant of the induced map on $\pi_1/[\pi_1, \pi_1]$, which is a lattice of rank $2g$. To obtain generators of Γ , Dehn introduced particular homeomorphism classes of F into itself that he called “*Schraubungen*” — we shall call them “*screw maps*”. Let γ be a simple closed curve on F : Draw a small collar around γ , i.e. choose a continuous injective orientation-preserving map $f: [0, 1] \times S^1 \rightarrow F$ such that $\gamma = f[(\frac{1}{2}) \times S^1]$. The screw map S_γ associated to γ is defined by

$$\begin{cases} S_\gamma(x) = x & \text{if } x \notin \text{Im}(f). \\ S_\gamma(f(\alpha, \theta)) = f(\alpha, \theta + 2\pi\alpha). \end{cases}$$

In other words, if γ_α is the curve $f[(\alpha) \times S^1]$, we rotate γ_α through an angle $2\pi\alpha$, ranging from 0 on γ_0 , to π on γ itself, to 0 again on γ_1^* . The main theorem

* Note that this screw map depends, modulo homotopy, only on γ and the orientation of F , and *not* on an orientation for γ .

of his paper is that Γ is generated by a finite number of these screw maps (p. 203). To get more precise information about the group Γ — we should have called it Γ_g , where $g = \text{genus}$ — it is not enough to refer to the groups Γ_h , where $h < g$. More generally, we have to look at an oriented surface $F_{g,n}$ of genus g with n holes. For all g, n the appropriate Teichmüller modular group $\Gamma_{g,n}$ is:

$$\Gamma_{g,n} = \left\{ \begin{array}{l} \text{Group of homeomorphisms of } F_{g,n} \text{ leaving the} \\ \text{\quad } n \text{ boundary curves pointwise fixed} \\ \text{Those homotopic to identity, leaving the bo-} \\ \text{\quad } \text{undary pointwise fixed along the way} \end{array} \right\}.$$

Note that whenever we choose a continuous, injective orientation preserving map $f: F_{h,n} \rightarrow F$, we obtain a map $f_*: \Gamma_{h,n} \rightarrow \Gamma$, since homeomorphisms of $F_{h,n}$ extend uniquely to homeomorphisms of F which are the identity on $F - f(F_{h,n})$.

Dehn's main result is that $\Gamma_{g,n}$ is generated by the screw maps along the set of curves β_k and γ_{ij} :

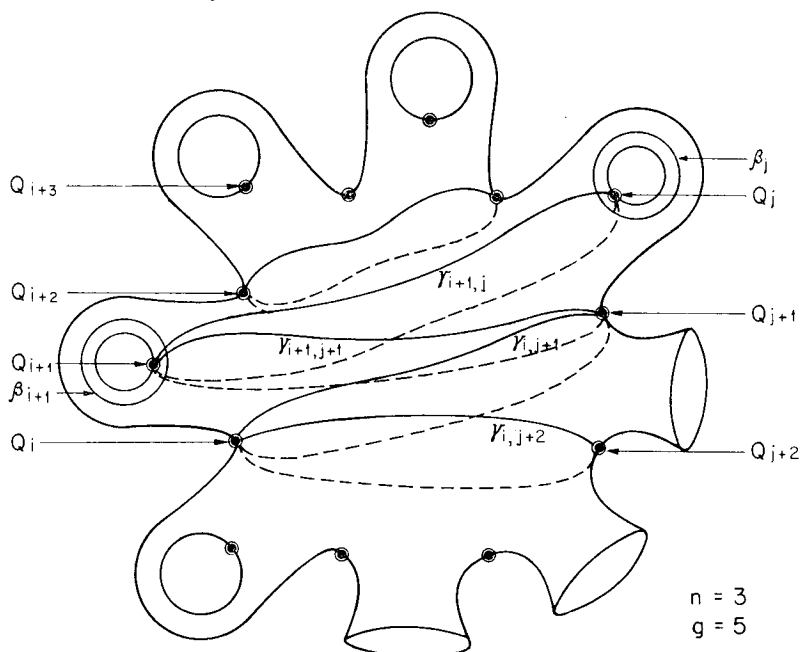


Figure 2

This result can easily be improved. On our surface F , consider the following piece homeomorphic to $F_{2,1}$:

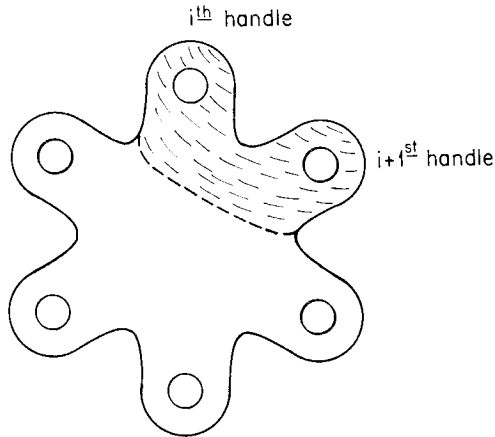


Figure 3

For all i , $1 \leq i \leq g$, this defines $f_{i,*}: \Gamma_{2,1} \rightarrow \Gamma$.

Theorem 4. Γ is generated by the images of $\Gamma_{2,1}$ under $f_{1,*}, \dots, f_{g,*}$.

Before beginning to prove this, let me list what is known about the groups $\Gamma_{g,n}$ for low g and n :

- (a) $\Gamma_{0,0} = \Gamma_{0,1} = (e)$
- (b) $\Gamma_{0,2} = \mathbf{Z}$: $F_{0,2}$ is just a collar, and $\Gamma_{0,2}$ is a free group on the screw map along its midline.
- (c) $\Gamma_{0,3} = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$, with generators given by screw maps along curves which are the 3 boundary curves, pulled slightly in:

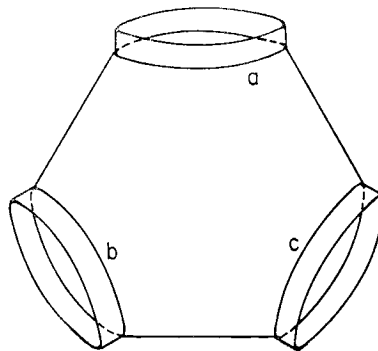


Figure 4

(d) $\Gamma_{1,1}$ = free group on screw maps Δ_a, Δ_b along a, b , modulo the relation

$$\Delta_b \Delta_a \Delta_b \Delta_a^{-1} \Delta_b^{-1} \Delta_a^{-1} = e$$

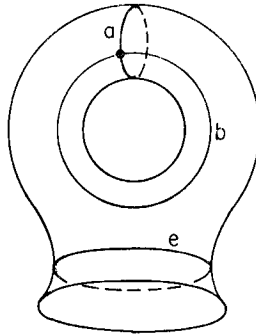


Figure 5

Moreover, $\Delta_e = (\Delta_a \Delta_b \Delta_a)^4$. (Dehn, pp. 156 and 172).

(e) $\Gamma_{0,4}$ is generated by screw maps $\Delta_a, \Delta_b, \Delta_{r_1}, \Delta_{r_2}, \Delta_{r_3}, \Delta_{r_4}$ where the Δ_{r_i} 's are in the center. It appears that there are no other relations, i.e. $\Gamma_{0,4} \cong \mathbf{Z}^4 \oplus$ (free group on 2 elements).

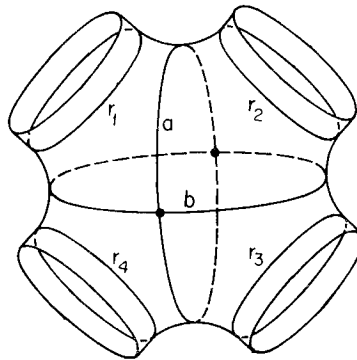


Figure 6

(f) $\Gamma_{1,2}$ is generated by screw maps $\Delta_a, \Delta_b, \Delta_c, \Delta_e$ and Δ_f :

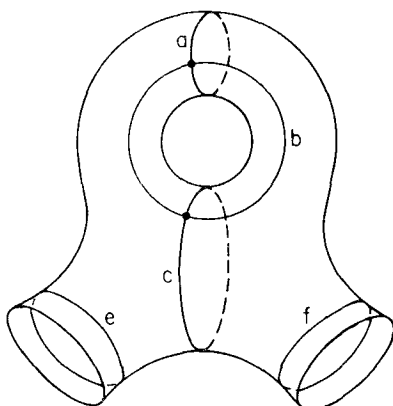


Figure 7

(Dehn, p. 188). The relations among these were worked out by Magnus [5], in a somewhat different form. To begin with, one has the obvious relations:

$$\Delta_e, \Delta_f \in \text{Center}$$

$$\Delta_a \Delta_c = \Delta_c \Delta_a.$$

Much subtler is the following*

$$\Delta_e \Delta_f = (\Delta_a \Delta_b)^2 \Delta_c \Delta_b (\Delta_a \Delta_b)^2 \Delta_b \Delta_c \Delta_b \Delta_c^{-1}.$$

There are still further relations that we will not list. $\Gamma_{1,2}$ is the first really complicated group and it is related to Artin's Braid group.

Proof of Theorem 4. Because of (f), the mapping class group $\Gamma_{1,2}$ is generated by $\Delta_a, \Delta_b, \Delta_c$ and either Δ_e or Δ_f , whichever is convenient. We use this to prove:

Lemma. $\Gamma_{2,2}$ is generated by

- (a) $\Gamma_{2,1}$
- (b) Δ_{c_1} or Δ_{c_2} , whichever you want
- (c) Δ_{e_1} or Δ_{e_2} whichever you want (cf. Figure 8).

* This can be readily derived from the formula

$$[\Theta^{-1}u\Theta, \Theta^{-1}a\Theta, \Theta^{-1}b\Theta] = s^{-4}\rho\tau^{-1}\rho^{-1}\tau$$

on p. 638 of Magnus' article.

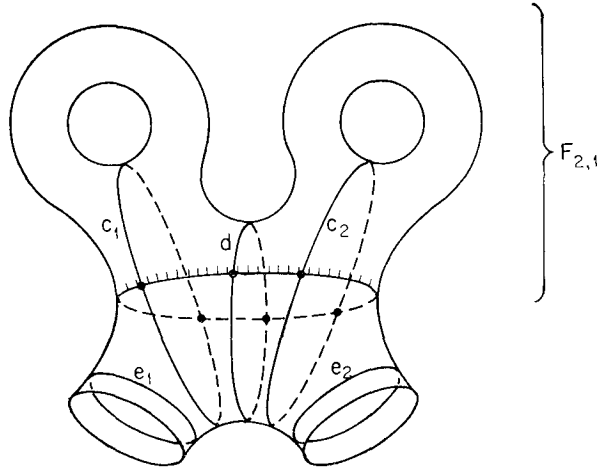


Figure 8

Proof. According to Dehn's result, $\Gamma_{2,2}$ is generated by $\Gamma_{2,1}$, Δ_{c_1} , Δ_{c_2} , Δ_d , Δ_{e_1} and Δ_{e_2} . But first of all Δ_{c_1} and Δ_{c_2} are conjugate with respect to a suitable element of $\Gamma_{2,1}$ (Dehn, §10, (1), p. 200; it is not hard to work this case out explicitly). Secondly, suppose you cut the surface open along d : then $F_{2,2} = F'_{1,2} \cup F''_{1,2}$. Applying the results of (f) to each piece, it follows that

- 1) $\Delta_{e_1} \cdot \Delta_d \in$ group generated by $\Delta_{c_1}, \Gamma_{2,1}$
- 2) $\Delta_d \cdot \Delta_{e_2} \in$ group generated by $\Delta_{c_2}, \Gamma_{2,1}$.

Q.E.D.

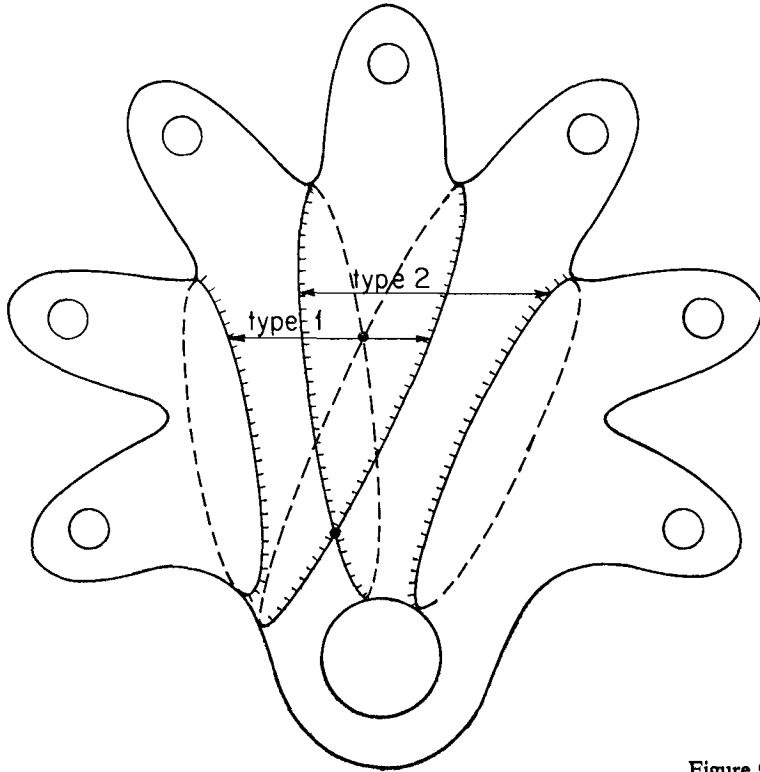


Figure 9

Apply this lemma to the $\Gamma_{2,2}$'s of types 1, 2 in Figure 9. It follows that any of the screw maps $\Delta_{\gamma_{ij}}$ of Fig. 2 can be written successively as products involving various elements of the $f_{k,*}(\Gamma_{2,1})$'s and screw maps $\Delta_{\gamma_{i',j'}}$ where $|i' - j'| < |i - j|$. Loosely speaking, modulo terms in the $f_{k,*}(\Gamma_{2,1})$'s, one replace $\Delta_{\gamma_{ij}}$ by $\Delta_{\gamma_{i',j'}}$'s with $\gamma_{i',j'}$ wrapping round a smaller part of F . This eventually eliminates the Δ_{γ} 's altogether. Q.E.D.

Corollary 1. Γ is generated by the four homeomorphisms $\Delta_a, \Delta_b, \Delta_d$ and R (cf. Fig. 10).

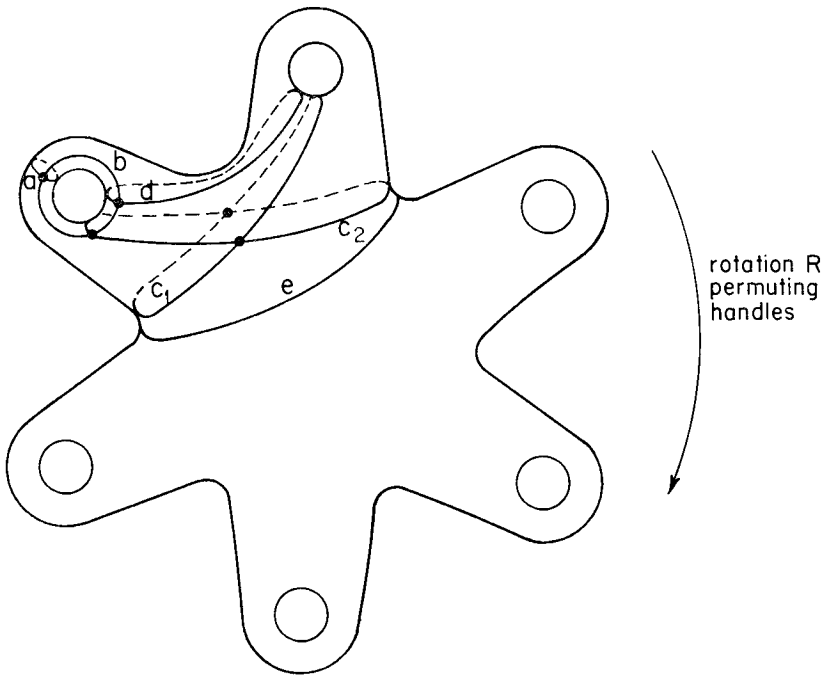


Figure 10

Proof. According to the Theorem, R plus $\Delta_a, \Delta_b, \Delta_{c_1}, \Delta_{c_2}, \Delta_d$ and Δ_e certainly suffice. On the other hand, cutting out an $F_{1,2}$ as follows:

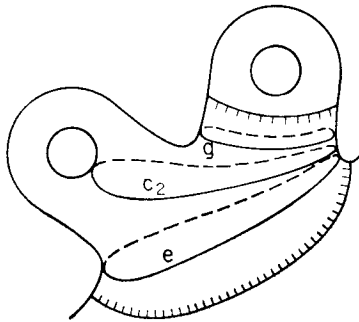


Figure 11

it follows from (f) that Δ_e can be expressed as a product of the elements Δ_a , Δ_b , Δ_{c_2} and Δ_g ; and Δ_g is a product of the elements $R\Delta_aR^{-1}$, $R\Delta_bR^{-1}$. This gets rid of Δ_e . Secondly, cutting out an $F_{1,2}$ like this:

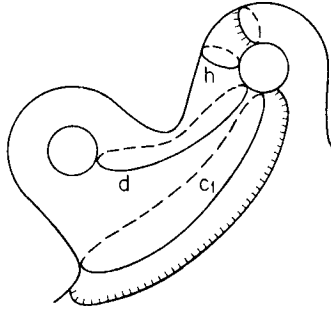


Figure 12

it follows that Δ_{c_1} can be expressed as a product of the elements Δ_a , Δ_b , Δ_d and Δ_h . But $\Delta_h = R\Delta_aR^{-1}$. Q.E.D.

Finally, if a_1, a_2 are any 2 simple closed curves on F neither of which divides F into 2 components, it is clear that $a_1 = T(a_2)$ for some orientation preserving homeomorphism T of F (i.e. since there must be an orientation preserving homeomorphism of $F - a_1$ and $F - a_2$). Therefore Δ_{a_1} and Δ_{a_2} are always conjugate in Γ in this case.

Corollary 2. Γ is generated by Δ_a and its conjugates (cf. Figure 10).

Proof. In fact, $\Delta_a, \Delta_b, \Delta_d$ all lie in the same conjugacy class by the remark just made, since none of the curves a, b, d decomposes F . And looking at the proof of Corollary 1, we see that we generated Γ by conjugates of $\Delta_a, \Delta_b, \Delta_d$ with respect to powers of R . Q.E.D.

Therefore, $\Gamma/[\Gamma, \Gamma]$ has one generator, the image of the conjugacy class δ of screw maps with respect to non-decomposing simple closed curves. Now refer back to the relation in (f) on $\Gamma_{1,2}$. We can embed $F_{1,2}$ in any surface F of genus $g (g \geq 2)$ so that a, b, c, e, f go into non-decomposing curves. Then $\Delta_a, \Delta_b, \Delta_c, \Delta_e$ and Δ_f all have image δ in $\Gamma/[\Gamma, \Gamma]$, and the relation says that $10\delta = 0$. This completes the proof of Theorem 1.

Theorem 2 follows from Theorem 1 by the results of §1 if $g \geq 3$. If $g = 2$, Theorem 1 is known from the results of Igusa [11].

Actually, when $g = 2$, it is not hard to check that $\Gamma/[\Gamma, \Gamma]$ is exactly $\mathbf{Z}/10\mathbf{Z}$. Closely related to this is the fact that the Picard group $\text{Pic}(\mathcal{M})$ of the moduli problem of genus 2 is also $\mathbf{Z}/10\mathbf{Z}$.

Proof. The methods in [6], §7 carry over without modification to prove that

$$\{\text{Torsion subgroup of } \text{Pic}(\mathcal{M})\} \cong \text{Hom}(\Gamma/[\Gamma, \Gamma], \mathbf{Q}/\mathbf{Z}).$$

(cf. 1st Corollary, p. 77; Corollary, p. 81, [6]). On the other hand, if M is the moduli space of genus 2, then, *modulo torsion*, $\text{Pic}(M)$ and $\text{Pic}(\mathcal{M})$ are isomorphic. Igusa showed ([11], p. 638) that there is a finite morphism $\pi: A^3 \rightarrow M$. So if $d = \text{degree}(\pi)$, and D is any Cartier divisor on M , then

$$\begin{aligned} d.D &= \pi(\pi^{-1}(D)) \\ &= \pi((f)), \text{ some } f \in \Gamma(A^3, \mathcal{O}_A) \\ &= (Nm f). \end{aligned}$$

Therefore $\text{Pic}(M)$ is torsion and $\text{Pic}(\mathcal{M})$ and $\Gamma/[\Gamma, \Gamma]$ are dual finite abelian groups. But we can easily check that 2 and 5 divide the order of $\text{Pic}(\mathcal{M})$. To do this, use the element $\delta \in \text{Pic}(\mathcal{M})$ gotten by attaching to each curve C of genus 2 the one-dimensional vector space:

$$\delta(C) = \Lambda^2[H^0(C, \Omega_C^1)]$$

($\Omega^1 =$ sheaf of 1-forms on C ; cf. [6], §5). Taking C_1 to be the curve $y^2 = x^5 - 1$, the automorphisms

$$\begin{cases} x \rightarrow \eta x \\ y \rightarrow y \\ \eta^5 = 1, \eta \neq 1 \end{cases}$$

of C_1 all act non-trivially on $\delta(C_1)$. Therefore $5 \mid \text{order}(\delta)$. Taking C_2 to be the curve $y^2 = x \cdot (x^4 - 1)$, the automorphisms

$$x \rightarrow \eta^2 \cdot x$$

$$y \rightarrow \eta \cdot x$$

$$\eta^8 = 1, \eta^4 \neq 1$$

all act non-trivially on $\delta(C_2)$, although their squares act trivially. Therefore $2 \mid \text{order}(\delta)$. Q.E.D.

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