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Reprinted from

Proceedings of Symposia in Pure Mathematics, Volume 9

ALGEBRAIC GROUPS AND DISCONTINUOUS SUBGROUPS

Families of Abelian Varieties

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In previous lectures, Kuga, Shimura, and Satake have considered various families of abelian varieties parametrized by the quotients of bounded symmetric domains by arithmetic subgroups. In particular, Shimura characterized certain of these families by means of the structure of the ring of endomorphisms—the "PEL-types." My purpose here is to show that an even larger class of Kuga's families can be characterized by intrinsic properties of the abelian varieties occurring in them. The properties in question involve the Kählerian geometry of the abelian varieties, but, assuming a famous conjecture of Hodge, they are equivalent to purely "algebro-geometric" properties of the abelian varieties. The results of this lecture are partly joint work with J. Tate.

1. **The Hodge group of a complex torus.** To give a complex torus A of dimension g is the same thing as giving

(i) a $2g$-dimensional rational vector space V;
(ii) a complex structure on $V_{\boldsymbol{R}} = V \otimes_{\boldsymbol{Q}} \boldsymbol{R}$;
(iii) a lattice $L \subset V$.

Here $V = H_1(A, \boldsymbol{Q})$, $L = H_1(A, \boldsymbol{Z})$, and the complex structure on $V_{\boldsymbol{R}}$ is induced by the natural isomorphism between $V_{\boldsymbol{R}}$ and the universal covering space of A. If we are only interested in the type of A up to isogenies, we can omit L. The datum (ii) is equivalent to either of the following objects:

(ii′) an endomorphism $J: V_{\boldsymbol{R}} \to V_{\boldsymbol{R}}$ such that $J^2 = -I$,
(ii″) a homomorphism of algebraic groups,

$$\phi: T \to \mathrm{GL}(V)$$

defined over $\boldsymbol{R}$ where T is the compact 1-dimensional torus over $\boldsymbol{R}$, i.e.,

$$T_{\boldsymbol{R}} = \{z \in \boldsymbol{C} \mid |z| = 1\};$$

and such that ϕ, as a representation of G_m, has weights $+1$ and -1, each with multiplicity g.

Starting with a complex structure on $V_{\boldsymbol{R}}$, we get data (ii′) and (ii″) as follows:

$J =$ multiplication by i.
$\phi(e^{i\theta}) =$ the element of $\mathrm{GL}(V)_{\boldsymbol{R}}$ given by multiplying in the complex structure on $V_{\boldsymbol{R}}$ by $e^{i\theta}$.
esp: $J = \phi(i)$.

DEFINITION. The *Hodge group* of A, written $Hg(A)$, is the smallest algebraic subgroup of $\mathrm{GL}(V)$ defined over $\boldsymbol{Q}$ and containing $\phi(T)$.

Since T is connected, it follows immediately that $Hg(A)$ is a connected algebraic group. A few more definitions:

DEFINITION. Let A be a complex torus, and let

$$H^k(A, \boldsymbol{C}) \cong \sum_{p+q=k} H^{p,q}(A)$$

be the Kähler decomposition of the cohomology of A. Then the *Hodge ring* of A is

$$H_0^*(A) = H^*(A, \boldsymbol{Q}) \cap \sum_{p=0}^{\dim A} H^{p,p}(A).$$

Hodge's conjecture asserts that $H_0^*(A)$ is the subring of $H^*(A, \boldsymbol{Q})$ given by the $\boldsymbol{Q}$-linear combinations of the fundamental classes of algebraic subvarieties of A.

Note that: if the complex torus A equals $V_{\boldsymbol{R}}/L$, then there is a canonical isomorphism:

$$H^i(A \times \cdots \times A, Q) \cong \wedge^i(V^* \oplus \cdots \oplus V^*).$$

Therefore, there is a natural representation of $Hg(A)$ on $H^*(A^k, \boldsymbol{Q})$, defined over $\boldsymbol{Q}$.

PROPOSITION 1. *For all k, the Hodge ring of A^k is the ring invariants of $Hg(A)$ in $H^*(A, \boldsymbol{Q})$.*

Using this Proposition, it is easy to give examples of abelian varieties A such that their Hodge ring is not generated by elements of degree 2 [cf. §3 for the existence of abelian varieties with various Hodge groups].

2. **The structure of the Hodge group of an abelian variety.** The result is the following:

THEOREM. *If A is an abelian variety, then*

(i) *$Hg(A)$ is a connected reductive group,*

(ii) *$\phi(-1)$ is the center of G, and centralizer $[\phi(i)]$ = centralizer $[\phi(T)]$: call this group Z,*

(iii) *$Z_{\boldsymbol{R}}^0$ is a maximal compact subgroup of $Hg_{\boldsymbol{R}}^0$ and $Hg_{\boldsymbol{R}}^0/Z_{\boldsymbol{R}}^0$ is a bounded symmetric domain.*

COROLLARY (OF (i)). *$Hg(A)$ is the largest subgroup of $\mathrm{GL}(V)$ which leaves invariant the Hodge rings of A^k for all k. Hence the Hodge group $Hg(A)$ as a subgroup of $\mathrm{GL}(V)$ and the collection of Hodge rings $H_0^*(A^k)$ as subrings of*

$$\wedge^*[V^* \oplus \cdots \oplus V^*]$$

are "equivalent" invariants of the abelian variety A: i.e., each can be computed from the other by linear algebra.

DEFINITION. The *Hodge type* of an abelian variety A of dimension g consists in the set of "equivalent" diagrams

$$T \xrightarrow{\phi} Hg(A) \subset \mathrm{GL}(2g)$$

obtained by identifying $\mathrm{GL}[H_1(A, \boldsymbol{Q})]$ with $\mathrm{GL}(2g)$ rationally over $\boldsymbol{Q}$; where two diagrams

$$T \xrightarrow[\phi_1]{} H_1 \subset \mathrm{GL}(2g),$$

$$T \xrightarrow[\phi_2]{} H_2 \subset \mathrm{GL}(2g),$$

are considered equivalent if there are elements $\alpha \in \mathrm{GL}(2g)_{\boldsymbol{Q}}$, $\beta \in (H_1)_{\boldsymbol{R}}$ such that

$$H_2 = \alpha H_1 \alpha^{-1},$$
$$\phi_2(\lambda) = \alpha\beta\phi_1(\lambda)\beta^{-1}\alpha^{-1}.$$

One should notice that once the $\boldsymbol{Q}$-rational subgroup $H \subset \mathrm{GL}(2g)$ is given, there are only a finite number of Hodge types (H, ϕ) extending H. This follows easily from the conjugacy of maximal compact tori in H via points of $H_{\boldsymbol{R}}$, and from the restriction on the weights of $\phi(T)$ in this representation.

DEFINITION. Let (H, ϕ) and (H', ϕ') be two Hodge types. Then (H, ϕ) is a refinement of (H', ϕ') if these types are represented by diagrams

$$T \xrightarrow{\phi} H \subset \mathrm{GL}(2g),$$

$$T \xrightarrow{\phi'} H' \subset \mathrm{GL}(2g),$$

where $H \subset H'$ and $\phi' = \phi$.

3. **The families.** Now suppose that a Hodge type (H, ϕ) is given. We will see that the set of all abelian varieties of this Hodge type, plus the limits which have finer Hodge types will be a family over a bounded symmetric domain, such that the action of certain arithmetic groups on the domain lifts to an action on the family.

DEFINITION. *A Hermitian symmetric pair* $(\mathscr{G}, J)$ is a real connected Lie group $\mathscr{G}$ with compact center, and an element $J \in \mathfrak{G}$, its Lie algebra, such that

(i) ad J has three eigenspaces in $\mathfrak{G}_{\boldsymbol{C}}$: $\mathfrak{K}_{\boldsymbol{C}}$ (the complexification of a real subspace $\mathfrak{K}$), $\mathfrak{p}_+$, and $\mathfrak{p}_-$ with eigenvalues 0, $+2i$, $-2i$,

(ii) $\mathfrak{K}$ is the Lie algebra of a maximal compact subgroup $\mathscr{K}$ in $\mathscr{G}$.

DEFINITION. Let $(\mathscr{G}, J)$ be a Hermitian symmetric pair. A faithful representation

$$\rho : \mathscr{G} \to \mathrm{GL}(2g)_{\boldsymbol{R}}$$

is *of abelian type* if

(i) $\rho(\mathscr{G})$ is contained in $\mathrm{Sp}(2g)_{\boldsymbol{R}}$ and is an algebraic subgroup defined over $\boldsymbol{Q}$,

(ii) $d\rho(J)$ is conjugate under $\mathrm{Sp}(2g)_{\boldsymbol{R}}$ to

$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix},$$

which is the "complex structure" in $\mathrm{Sp}(2g)_{\boldsymbol{R}}$.

(ii) is equivalent to asserting that

$$(B, -B\, d\rho(J))$$

form a "symplectic pair" in Kuga's sense, where

$$B = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix};$$

also, (ii) is condition (H_2) of Satake.

(iii) J is not contained in the Lie algebra of any *normal* subgroup $\mathscr{G}_0 \subset \mathscr{G}$ such that $\rho(\mathscr{G}_0)$ is defined over $\boldsymbol{Q}$.

An immediate consequence of this definition is that if we exponentiate J in $\mathscr{G}$ we obtain a homomorphism:

$$\phi: T \to \mathscr{G} \qquad \text{where} \qquad \phi(e^{i\theta}) = \exp(\theta J).$$

In fact, $(\rho(\mathscr{G}), \rho \circ \phi)$ is a Hodge type, and every Hodge type arises from an abelian representation of a symmetric pair.

Now suppose $\mathscr{G}, J$ and ρ are given. Let $\phi: T \to \mathscr{G}$ denote the above homomorphism. Let K be the compact subgroup of $\mathscr{G}$ which centralizes $\phi(T)$, and let K' be the compact subgroup of $\mathrm{Sp}(2g)_{\boldsymbol{R}}$ which centralizes $\rho(\phi(T))$. Then ρ induces a holomorphic map of symmetric domains

$$\mathscr{G}/K \xrightarrow{\tau} \mathrm{Sp}(2g)_{\boldsymbol{R}}/K'.$$

Via τ, the standard family of abelian varieties on Siegel's upper $\frac{1}{2}$-plane induces a family over $\mathscr{G}/K$: call it

$$\begin{array}{c} \mathfrak{X}(\mathscr{G}, J, \rho) \\ \big\downarrow{\scriptstyle \pi} \\ \mathscr{G}/K. \end{array}$$

Since ρ is defined over $\boldsymbol{Q}$, ρ maps all small enough arithmetic subgroups Γ of $\mathscr{G}$ into $\mathrm{Sp}(2g)_{\boldsymbol{Z}}$, and hence the action of such Γ on $\mathscr{G}/K$ lifts to an action on the family $\mathfrak{X}(\mathscr{G}, J, \rho)$.

PROPOSITION 2. *The abelian variety $\pi^{-1}(x)$ in the family $\mathfrak{X}(\mathscr{G}, J, \rho)$ is isogenous to $A = \boldsymbol{R}^{2g}/\boldsymbol{Z}^{2g}$, with complex structure defined by $\rho(g\phi(i)g^{-1})$ where $g \in \mathscr{G}$ represents $x \in \mathscr{G}/K$.*

COROLLARY. *An abelian variety A is isogenous to one in the family $\mathfrak{X}(\mathscr{G}, J, \rho)$ if and only if A has Hodge type equal to or finer than $(\rho(\mathscr{G}), \rho \circ \phi)$.*

PROPOSITION 3. *The families $\mathfrak{X}(\mathcal{G}, J, \rho)$ include all the families associated by Kuga to symplectic representations $\rho: G \to \mathrm{Sp}(2g)$ of semi-simple groups G defined over Q, in the case when $G_{\boldsymbol{R}}$ has no compact factors.*

4. **The conjecture.** The most intriguing possibility suggested by this theory is is an arithmetic conjecture. Serre [Colloque de Clermont-Ferrand, *Groupes de Lie l-adiques attachés aux courbes elliptiques*] has defined l-adic Lie algebras acting on $H_1(A, \boldsymbol{Q}_l)$, for any abelian variety A, which are essentially the Lie algebras of the Galois group of the extension obtained by adjoining all points of order l^ν to some smallest field of definition of A. Call these $\mathfrak{G}_l$. Let $\mathrm{Lg}(A)$ be the Lie algebra of $\mathrm{Hg}(A)$. It is a sub-Lie-algebra of $\mathrm{Sl}[H_1(A, \boldsymbol{Q})]$. Then one may ask whether:

$$\mathfrak{G}_l \cap \mathrm{Sl}[H_1(A, \boldsymbol{Q}_l)] = \mathrm{Lg}(A) \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l.$$

If $\dim A = 1$, and A is defined over $\boldsymbol{Q}$, Serre has verified this. For A of CM-type, this result is apparently proven in Shimura-Tamiyama, *Complex multiplication of abelian varieties.*