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THE BOUNDARY OF MODULI SCHEMES

by David Mumford

1^o Discussion

To begin with, what is a variety of moduli ? Start with the set of all non-singular complete varieties of dimension n and arithmetic genus p . For each isomorphism class of these, take one point : then try to put these points together in a variety. There are some more requirements : a "nearby" pair of varieties V_1, V_2 should correspond to a "nearby" pair of points : e. g.

Let \mathcal{S} = set of isomorphism classes of V 's

$U \subset \mathcal{S}$ is "open", if for all families of varieties of the given type, varieties of type U occur over an open set in the parameter space.

Another requirement is that for all families

$$\pi: \mathcal{V} \longrightarrow S$$

suppose you map S to \mathcal{S} by assigning to each $s \in S$ the class of the fibre $\pi^{-1}(s)$: then this map should be algebraic.

The problem, in this raw form, has been modified bit by bit so as to make it more plausible :

(I.) Instead of classifying "bare" varieties V , one seeks to classify pairs (V, \mathcal{D}) where \mathcal{D} is a numerical equivalence class of very ample divisors on V .

(II) Then break up the set \mathcal{S} via the Hilbert polynomials of the divisors in \mathcal{D} : viz. for every P , let $\mathcal{S}^P = \text{isom. classes of } (V, \mathcal{D})$ such that for all $D \in \mathcal{D}$

$$P(n) = \chi(\underline{\mathcal{O}}_V(nD)) .$$

Now we are close to a good problem :

<p>for all $D \in \mathcal{D}$</p> <p>for all bases of $H^0(V, \underline{\mathcal{O}}_V(D))$ you get a canonical immersion</p> $V \subset \mathbb{P}_n \quad (n = \dim H^0(V, \underline{\mathcal{O}}_V(D)) - 1)$ <p>s.t. hyperplane sections are linearly equivalent to D .</p>
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i.e. $\mathcal{S}^P \cong \frac{\text{certain set of subvarieties } V \text{ of } \mathbb{P}_n}{\text{certain equivalence relation, especially projective equivalence}}$

(III) . Why insist that V be non-singular ? The only reason appears to be that over \mathbb{C} families of non-singular varieties are locally differentiably trivial : so one can view them as families of complex structures on a fixed differentiable manifold, (or, as in the Bers-Ahlfors approach, on a fixed topological manifold) . Algebraically, there is no point : let's let V be any complete variety at all, maybe even reducible and assume that \mathcal{D} is a class of Cartier divisors.

To go further, let's stop and ask what problems arise : first we should take a broad look at the topology which we are getting by throwing in all varieties - typically it will be very un-separated ; second we should try to find open subsets $U \subset \mathcal{S}^P$ such that, in their induced topology, they are separated, and "compact" if possible.

[This means that if U could be given the structure of a moduli variety, it would turn out complete ; and it also means, directly, that if $(V, \mathcal{D}) \in U$, and we specialize the groundfield, then we can find a specialization $(\bar{V}, \bar{\mathcal{D}})$ of (V, \mathcal{D}) also in U .]

Thirdly, we will finally have to find out if U can be made into a variety.

(IV.) We understand the last problem better when we realize that, e.g. via chow coordinates, almost all of U is bound to come out as a variety. We saw that \mathcal{S}^P was a quotient of a piece ~~\mathcal{H}~~ of the chow variety by an algebraic equivalence relation. Such quotients always exist birationally, i.e. for a small enough Zariski-open subsets $U^* \subset \mathcal{H}$, $[U^*/\text{modulo equivalence relation}]$ will be a good variety. So the 3rd problem is like the first two :

|| The only problem is to pick the "boundary" components shrewdly,
|| i.e. to decide which non-generic varieties to allow .

there again, it would prejudice the issue to think that we should necessarily use all and/or only non-singular varieties. And the choice should be made by a) checking the topology and b) checking its "algebraizability".

(V.) A final step in setting up the problem reasonably is to realize that all the same ~~questions~~ occur equally well for a much more general class of problems : viz. that of forming quotients of varieties by algebraic equivalence relations. Only by realizing this can we hope to find simple enough examples to study first so as to get the right feeling. Especially, the hard equivalence relations are the non-compact one's ; and in the case of moduli, this occurs principally in forming :

$$\mathcal{H} / \{ \text{Projective equivalence of } V\text{'s in } \mathbb{P}_n \}$$

i.e. in forming an orbit space by $\text{PGL}(n)$.

2° Present State of the Theory

very good (i) analogous problem in classifying vector bundles on a fixed curve
pretty good(ii) moduli of curves (canonically polarized)
half good (iii) moduli of polarized abelian varieties
no good (iv) moduli of surfaces of general type

3^o An Example

Rather than analyze an actual moduli problem, I want to take one of the simplest non-trivial orbit space problems, in which all the features of the conjectured results occur :

$G = \text{PGL}(1)$ acting on \mathbb{P}_n , where $\mathbb{P}_n = n^{\text{th}}$ symmetric product of \mathbb{P}_1 , i.e. $\text{PGL}(1)$ acting on the set of 0-cycles of degree n .

(= theory of binary quantics) .

a) jump phenomenon

look at $\mathbb{P}_2/\text{PGL}(1)$. There are 2 orbits : $\{P+Q \mid P \neq Q\}$ and $\{2P\}$. Therefore, get 2 pts. x, y where x is open but not closed, y is closed but not open :



This occurs in all moduli problems, and one always must exclude some points to avoid this.

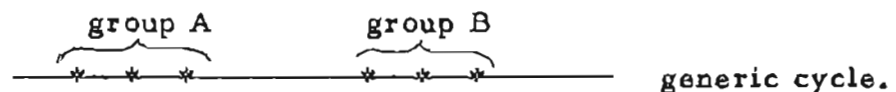
In \mathbb{P}_n , exclude the 0-cycles

$$kP + (n-k)Q$$

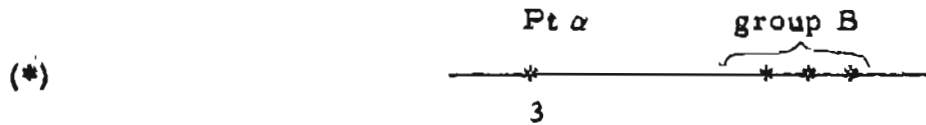
whose isotropy group is infinite.

b) further non-separation

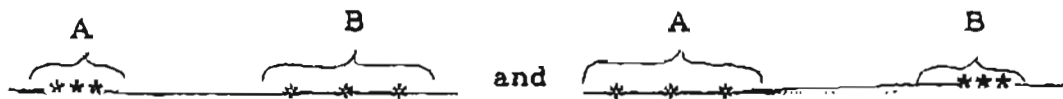
take $n = 6$



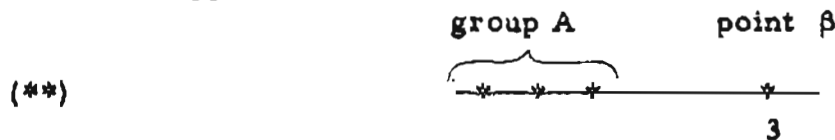
Let all points in group A come together ; you get in the limit :



But suppose, as group A collapses to α , you apply a one-parameter subgroup $G_m \subset \text{PGL}(1)$, moving points away from α to β . Then the following are projectively equivalent:



the latter approaches :



But the 0-cycles (*) and (**) are probably not projectively equivalent.

c) the unitary retraction : to avoid these bad things, define

$$\mathcal{K} \subset \mathbb{P}_n$$

$\mathcal{K} =$ Set of 0-cycles $\sum_{i=1}^n P_i$, such that, putting the P_i on the Gauss sphere, and embedding the Gauss sphere in \mathbb{R}^3 as $x^2 + y^2 + z^2 = 1$, then the vector sum of the P_i in \mathbb{R}^3 is $(0, 0, 0)$.

One checks, if $x, y \in \mathcal{K}$, then x, y are equivalent under $\mathrm{PGL}(1)$ if and only if they are equivalent under the maximal compact subgroup

$$K = \mathrm{SO}(3; \mathbb{R}) \subset \mathrm{PGL}(1, \mathbb{C}) = G.$$

But \mathcal{K} is compact, therefore \mathcal{K}/K is compact and separated. And

$$\mathcal{K} \cdot \mathrm{PGL}(1) = \left\{ \mathcal{O}_L \mid \begin{array}{l} \text{no point } Q \text{ occurs in } \mathcal{O}_L \text{ with} \\ \text{multiplicity } > n/2; \text{ and if } Q \text{ occurs} \\ \text{with multiplicity } n/2, \text{ then} \\ \mathcal{O}_L = \frac{n}{2} (Q + Q') \end{array} \right\}.$$

d) stability restriction : $\mathcal{K} \cdot \mathrm{PGL}(1)$ contains a Zariski-open set

$$U_{\text{stable}} = \left\{ \mathcal{O}_L \mid \begin{array}{l} \text{no point } Q \text{ occurs in } \mathcal{O}_L \text{ with} \\ \text{multiplicity } \geq n/2 \end{array} \right\}$$

So U_{stable}/G has separated topology, and is compact if n is odd. It is also a variety by virtue of a general theorem of mine.

e) semi-stability : when n is even, things are less clean.

\mathcal{K} showed that there was a natural compactification of U_{stable}/G by adding a single point representing the cycles $n/2(Q+Q')$. In fact, there is a complete variety \overline{V}_n , with point ∞ and diagram of algebraic maps:

$$\begin{array}{ccc}
 U_{\text{semi-stable}} & \xrightarrow{\quad} & \overline{V}_n \\
 U_{\text{stable}} \cup & \xrightarrow{\quad} & U_{\text{stable}/G} = \overline{V}_n \cup (\infty)
 \end{array}$$

where

$$U_{\text{semi-stable}} = \{ \mathcal{O} \mid \text{no point } Q \text{ occurs in } \mathcal{O} \text{ with multiplicity } > n/2 \} .$$

FURTHER COMMENTS ON BOUNDARY POINTS

by

David B. Mumford

In these notes, I shall describe some joint work of A. Mayer and myself as well as some related results, summarizing further comments made in my lecture and a 2nd lecture by Mayer. During the institute, lectures were also given by H. Rauch and L. Ehrenpreis discussing various aspects of the Torelli and Teichmüller covering spaces of the moduli scheme for curves of genus g (cf. the notes of Ehrenpreis). The ground field will be assumed to be the complex numbers in our discussion. One word of apology: the full proofs of many of our results have not been written down, so strictly speaking, much of what follows should be taken as conjectures not theorems.

§1. Compact moduli spaces for vector bundles over curves.

This theory has been worked out by Seshadri, Narasimhan, and myself.

Let E be a vector bundle of rank r over a curve C .

Definitions:

- i) E is regular if the only endomorphisms of E are multiples of the identity.
- ii) E is stable if, for all sub-bundles $F \subset E$, $\deg[c_1(F)] < \frac{\text{rank}(F)}{\text{rank}(E)} \cdot \deg[c_1(E)]$.
- iii) E is semi-stable if, for all sub-bundles $F \subset E$, $\deg[c_1(F)] \leq \frac{\text{rank}(F)}{\text{rank}(E)} \cdot \deg[c_1(E)]$.
- iv) E is retractable if it is a direct sum of stable bundles.

If $\deg[c_1(E)] = 0$, E is retractable if and only if E admits a hermitian structure with curvature form 0.

To obtain a modulus space for vector bundles with given rank and $\deg(c_1)$, first one must throw out irregular bundles since they give rise to jump phenomenon, i.e., constant families of bundles, which suddenly jump to another bundle (cf. my lecture notes, "Curves on an algebraic surface", Lecture 7, §4). In the remaining class of bundles, the topology is still un-separated; but in the set of retractable bundles the topology is both compact and separated, since this set of bundles is isomorphic to the set of unitary representations of π_1 of the base curve (for $\deg[c_1(E)] = 0$; otherwise the argument can be modified). This set turns out to contain the open set of stable bundles, and to be contained

in the open set of semi-stable bundles (it is not open itself). One finds that the stable bundles are classified by the points of a non-singular variety V , and that V is an open subset of a compact variety \overline{V} . The set of points of \overline{V} is isomorphic to the (non-algebraic) set of retractable bundles, and there is even a natural map from the set of all semi-stable bundles to \overline{V} ; but non-isomorphic bundles no longer correspond to distinct points:

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{regular} \\ \text{bundles} \end{array} \right\} & \supset & \left\{ \begin{array}{c} \text{stable} \\ \text{bundles} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{points of} \\ V \end{array} \right\} \\ & & \cap \\ & & \left\{ \begin{array}{c} \text{retractable} \\ \text{bundles} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{points of} \\ \overline{V} \end{array} \right\} \\ & & \cap \\ & & \left\{ \begin{array}{c} \text{semi-stable} \\ \text{bundles} \end{array} \right\} \nearrow \end{array}$$

§2. Compact moduli spaces for abelian varieties: Satake

Let V_n denote the moduli scheme for principally polarized abelian varieties of dimension n . That is,

$$V_n \cong \mathfrak{H}_n / \Gamma_n \quad (\text{as analytic space})$$

where \mathfrak{H}_n is the Siegel upper $\frac{1}{2}$ -plane of type n , and Γ_n is the modular group acting on \mathfrak{H}_n . V_n has even a canonical structure of algebraic variety over \mathbb{Q} , due to its interpretation as a moduli scheme*. V_n carries a canonical class of ample invertible sheaves $\mathcal{L}(i)$ defined for all sufficiently large i , and such that

$$\mathcal{L}(i) \otimes \mathcal{L}(j) = \mathcal{L}(i+j)$$

when this makes sense. Therefore one has the graded ring

$$R_n = \bigoplus_{i \geq i_0} \Gamma(V_n, \mathcal{L}(i))$$

which is known to be isomorphic to the ring of modular forms on \mathfrak{H}_n with respect to Γ_n , if $n \geq 2$.

*cf. Baily's work, or my "Geometric Invariant Theory".

The Satake compactification of V_n is then the open immersion:

$$V_n \subset \text{Proj}(R_n) = V_n^*.$$

It turns out that there is a canonical isomorphism of $V_n^* - V_n$ and V_{n-1}^* , so that set-theoretically:

$$V_n^* = V_n \cup V_{n-1} \cup \dots \cup V_1 \cup V_0.$$

(V_0 is a single point). This amazing equation suggests that this compact variety, which is defined only as a kind of "minimal model", should have an interpretation as a moduli space. In fact, consider all commutative group schemes X connected and of finite type over \mathbb{C} .

Definition: X is stable if X is an abelian variety.

X is semi-stable if X is an extension of an abelian variety by multiplicative groups $(\mathbb{G}_m)^r$.

X is retractable if X is the product of an abelian variety by multiplicative groups.

Exactly as before, A. Mayer and I have proven:

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{Stable } X \text{ with} \\ \text{polarization} \end{array} \right\} & \cong & \left\{ \begin{array}{c} \text{points of} \\ V_n \end{array} \right\} \\ \cap & & \cap \\ \left\{ \begin{array}{c} \text{retractable } X \\ \text{with polarization} \end{array} \right\} & \cong & \left\{ \begin{array}{c} \text{points of} \\ V_n^* \end{array} \right\} \\ \cap & & \\ \left\{ \begin{array}{c} \text{semi-stable } X \\ \text{with polarization} \end{array} \right\} & \nearrow & \end{array}$$

Explanations

¹ A polarization of X may be taken to mean a divisor D on X , determined up to algebraic equivalence, such that if

$$\pi : X \rightarrow X_0$$

is the projection of X onto its abelian part, and if $D = \pi^*(D_0)$ (recall that $\text{Pic}(X) \cong \text{Pic}(X_0)$), then D_0 is ample on X_0 and

$$\begin{cases} (D_0^{n_0}) = n_0! \\ n_0 = \dim X_0. \end{cases}$$

2° A family of these objects is a morphism

$$f: X \rightarrow S$$

with the structure of group scheme (i.e., a "multiplication" $\mu: X \times_S X \rightarrow X$, etc.) and a family of Cartier divisors \mathcal{D} on X determined up to algebraic equivalence, and replacements

$$\mathcal{D}' = \mathcal{D} + f^*(\xi)$$

for any Cartier divisors ξ on S , and inducing a polarization of each fibre $f^{-1}(s)$. With this definition, stable and semi-stable X 's form open sets, but retractable X 's do not.

3° The meaning of the arrows in the diagram is this: let $f: X \rightarrow S$ be a family of semi-stable objects where S is a normal algebraic variety. Map S to V_n^* by assigning to each $s \in S$ the point of V_{n_0} corresponding, in the classical way, to the abelian part of $f^{-1}(s)$. ($n_0 = \dim$ of this abelian part). Then this is a morphism.

This last result is proven by reducing to the case where S is a curve. Then one passes to the corresponding analytic set-up, and replaces S by a disc $\{z \mid |z| < 1\}$ where all fibres of f are diffeomorphic except for $f^{-1}(0)$. Next one introduces the invariant and vanishing cycles on the general fibre, so as to put the period matrix $\Omega_{ij}(z)$ of the abelian part of $f^{-1}(z)$ in a normalized form. One then computes (using very helpful tricks of Kodaira):

$$\Omega_{ij}(z) = \frac{1}{2\pi i} \log z \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A(z) & B(z) \\ {}^t B(z) & C(z) \end{pmatrix}$$

where S is integral, positive definite and symmetric, and is obtained from the monodromy substitution for the cycle $|z| = 1$; where A, B, C are holomorphic in z at $z = 0$; and where $C(0)$ is the period matrix of the abelian part of $f^{-1}(0)$. This implies that $\Omega_{ij}(z) \rightarrow C(0)$ in Satake's topology, when $z \rightarrow 0$.

§ 3. Compact moduli spaces for curves

Let M_g denote the moduli scheme for curves of genus g . Let

$$\Theta : M_g \rightarrow V_g^-$$

be the morphism which assigns to a curve its jacobian variety with its theta-polarization. From the work of Baily, Matsusaka, and Hoyt, it is known that Θ is an isomorphism of M_g with a locally closed subvariety of V_g^- , which we also denote M_g^u . The simplest approach to compactifying M_g^u is to use its closure M_g^* in V_g^* . This breaks up into two pieces

$$M_g' = (M_g^* \cap V_g) - M_g,$$

$$M_g'' = M_g^* - (M_g^* \cap V_g).$$

Matsusaka and Hoyt showed that M_g' is exactly the set of products of lower dimensional jacobian varieties. We have proven that $M_g'' = M_{g-1}^*$, so that

$$M_g^* = M_g \cup M_g' \cup M_{g-1} \cup M_{g-1}' \cup \dots \cup M_0$$

($M_0 = V_0^-$ is a single point).

The proof is based on two lemmas, and on the results of § 2:

Lemma A: Let C be a curve and let $f : \mathcal{K} \rightarrow C$ be a family of curves of arithmetic genus g [i.e., f is proper and flat and its fibres $f^{-1}(P)$ are connected curves of arithmetic genus g]. Let $P_0 \in C$ and assume that $f^{-1}(P)$ is non-singular if $P \neq P_0$. Then there exists a diagram:

$$\begin{array}{ccc} \mathcal{K}' & \xrightarrow{\pi'} & \mathcal{K} \\ f' \downarrow & & \downarrow f \\ C' & \xrightarrow{\pi} & C \end{array}$$

where

- 1) C' is a curve and π is a finite morphism totally ramified over P_0 : let $P_0' = \pi^{-1}(P_0)$,
- 2) f' is a family of curves over C' ,
- 3) $X' = f'^{-1}(P_0')$ is just the induced family of curves over $C' - P_0'$, i.e.

$$(C' - P_0') \times_C X = X' - f'^{-1}(P_0'),$$

- 4) $f'^{-1}(P_0')$ is reduced and has only ordinary double points.

Lemma B: Let C be a curve and let

$$f: X \rightarrow C$$

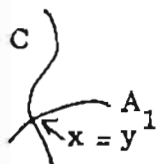
be a family of curves of arithmetic genus g such that each curve $f^{-1}(P)$ is reduced and has only ordinary double points. Then the set of generalized jacobian varieties of the curves $f^{-1}(P)$ forms a family of polarized semi-stable group varieties over C .

These lemmas give the inclusion $M_g'' \subset M_g^*$ directly; Lemma B and an easy construction of some actual families give the converse $M_g'' \supset M_g^*$.

Unfortunately, M_g^* is not a reasonable moduli space for curves: for example, let a point of M_g' correspond to

$$A_1 \times A_{g-1}$$

where A_1 is an elliptic curve, and A_{g-1} is the jacobian of a curve C of genus $g-1$. Let $x \in A_1$ and $y \in C$ be any points. Then $A_1 \times A_{g-1}$ is the generalized jacobian variety of the curve:



with an ordinary double point. In other words, the jacobian is independent of which y is chosen: i.e., Torelli's theorem is false for reducible curves. It is clearly necessary to blow up M_g' . This phenomenon is closely related to the fact, discovered by Bers and Ehrenpreis that the generic point of M_g' is not only singular on M_g^* : it is not even

"almost non-singular" (= "Jungian" = "V-manifold"). In fact, Lemma A suggests

Definition:

A curve C of arithmetic genus g is stable if C is reduced and connected, has only ordinary double points, and has only a finite group of automorphisms.

It appears that the set of all stable curves is open and compact and is naturally isomorphic to the set of points of a compact analytic space with almost non-singular points: \tilde{M}_g^* . It is still unknown whether \tilde{M}_g^* is a projective algebraic variety, although it is a \mathbb{Q} -variety. There is a proper holomorphic map

$$\tilde{M}_g^* \longrightarrow M_g^*$$

which is an isomorphism over the open subset M_g . One of the remarkable features of this case is that there are no semi-stable but not stable curves.

§ 4. Compact moduli spaces for abelian varieties: blown up

The preceding construction suggests the possibility of blowing up V_n^* so as to obtain a \tilde{V}_n^* which corresponds to a moduli problem with a larger set of stable objects. We would like the stable points of \tilde{V}_n^* to correspond to polarized compactifications of commutative group schemes X . One approach is to compactify the generalized jacobian varieties of curves C . Say C is irreducible and reduced: let J be the generalized jacobian of C . Then one has an isomorphism:

$$\left\{ \begin{array}{c} \text{points of} \\ J \end{array} \right\} = \left\{ \begin{array}{c} \text{invertible sheaves } L \text{ on } C \\ \text{such that } \chi(L) = \chi(\mathcal{O}_C) \end{array} \right\}.$$

We can prove that there is a projective scheme J^* containing J as an open subset, and on which J acts, plus a natural isomorphism

$$\begin{aligned} \left\{ \begin{array}{c} \text{points of} \\ J \end{array} \right\} &\cong \left\{ \begin{array}{c} \text{invertible sheaves } L \text{ on } C \\ \text{such that } \chi(L) = \chi(\mathcal{O}_C) \end{array} \right\} \\ \bigcap & \\ \left\{ \begin{array}{c} \text{points of} \\ J^* \end{array} \right\} &\cong \left\{ \begin{array}{c} \text{rank 1, torsion-free sheaves } \mathcal{F} \text{ on } C \\ \text{such that } \chi(\mathcal{F}) = \chi(\mathcal{O}_C) \end{array} \right\}. \end{aligned}$$

Using this, we find an interesting \tilde{V}_2^* , in which only one point is still mysterious: that is the point which is the image under Θ of the curve of genus 2 depicted below:

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