Some Aspects of the Problem of Moduli

by D. Mumford

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I. The first aspect which I wish to discuss is the question of how to make precise the heuristic concept of moduli. For example, suppose one is concerned with curves of genus g: then, for every algebraically closed field Ω, let $\mathfrak{M}_g(\Omega)$ be the set of curves of genus g, defined over Ω, up to isomorphism. Since the moduli scheme M_g is to classify curves, one asks at least that there be given an isomorphism between the set of Ω-rational points of M_g and $\mathfrak{M}_g(\Omega)$. This obviously does not determine M_g, however. A stronger demand is to ask for a collection of isomorphisms between the set of R-valued points of M_g, and the set of curves of genus g over R, for <u>every commutative ring</u> R; here a curve over R means a scheme, simple and proper over <u>Spec</u>(R), whose geometric fibres are curves of genus g. Moreover, these isomorphisms should be functorial in R. Then, in fact, this determines M_g, if it exists. An essentially equivalent demand is to ask that there exist a "Universal Family" of curves over M_g itself. Such an M_g I call a fine moduli scheme; unfortunately, it does not exist unless the classificational problem is slightly modified (via a "higher level structure"). For higher dimensions, to find suitable modifications to "eliminate the automorphisms" is an interesting problem.

In any case, one can compromise for a coarse moduli scheme: here one merely asks for some collection of maps, from the sets of curves over R to the sets of R-valued points of M_g, which are (i) functorial in R, (ii) isomorphisms when R is an algebraically closed field. Finally, to determine M_g completely, one should ask that it satisfy a universal mapping property with respect to all other solutions of the first two demands.

II. The next aspect we consider is that of the qualitative properties of the sought-for moduli scheme: especially, whether it is a true scheme, or only a pre-scheme; and whether it is of finite type over the integers. But, in fact, examples due to Kodaira, Nagata, Nishi, and others indicate the absense of both of these properties in the general case of classifying higher dimensional varieties. To remedy this difficulty, the simplest solution seems to be to modify the problem: instead of classifying varieties, one seeks to classify polarized varieties. By a polarized variety, we mean a variety V together with a Cartier divisor class D, determined up to algebraic equivalence and torsion, such that nD is induced by a projective embedding of V, if $n \gg 0$.

For this classificational problem, Matsusaka and the writer have shown that the moduli scheme should be a true scheme, if the varieties are assumed non-singular, and not birationally ruled. Moreover, note that a Hilbert polynomial $P(n) = \chi(\underline{\mathcal{O}}_V(nD))$ can be attached to any polarized variety, and that it remains constant in flat families of such polarized varieties. Then we have also shown that, for non-singular surfaces, the moduli scheme of, polarized surfaces with fixed Hilbert Polynomial should be of finite type. Whether the same is true in dimension 3 is a very intriguing question. Another difficult problem is to ascertain how essential is the role of the non-singularity assumption in these matters. In the complex analytic case, non-singular families recommend themselves as being differentiably trivial, so that they can be visualized as families of complex structures on a fixed manifold. In the algebraic case, however, there seems to be no compelling reason for thinking that this is a reasonable assumption.

III. Beyond the qualitative problems already discussed, there looms the big question of whether, although possessing all good local and global properties, the moduli scheme may fail to exist for more subtle reasons. One may put the problem this way: the "moduli scheme" may be formally described as the quotient of a scheme by some topologically beautiful equivalence relation but it may be impossible to give a scheme realizing this quotient. For instance, it is sometimes impossible to "blow down" certain subvarieties, or to "divide" some variety by the action of some group. In this case, there would be only an open subset U of stable polarized varieties which could be realized as a scheme.

This problem appears to be closely connected with the local projective differential geometry of embedded varieties $V \subset P_n$. To illustrate, suppose V is a non-singular curve, and that the embedding is determined by a complete linear system on V of high degree. Then the Weierstrass gap theorem, and the Frenet-Serret equations give a very explicit picture of this embedded curve. This enables us to do two things: In the first place, you can look at the set of $x \in V$ where the Frenet-Serret equations break down. I call these points of Hyper-Osculation, and with convenient multiplicities, they can be added together to give a divisor on V. This possesses two key properties: (i) as V and the embedding vary continuously, this divisor varies continuously, (ii) the maximum multiplicity with which any x occurs in this divisor is bounded by g^2 (g=the genus of V). This being so, projective invariants of V can be constructed in a highly explicit fashion out of determinants in the coordinates of these points. This is tantamount to constructing the moduli scheme for curves. In the second place, the very explicit expression of V gives directly information on the Chow form of V: especially on the monomials which occur in the Chow form with non-zero coefficient and are extremal in the convex hull of all monomials of fixed degree with this property. This, too, leads to projective invariants of space curves, hence to moduli. In this connexion, the difficulty in the surface case appears to be lack of very much information on the local projective differential geometry of surfaces in P_n.

IV. Riemann originally asked for $3g-3$ complex numbers, called moduli, to be attached to each curve over the complex numbers. One interpretation of this assertion is to ask, not only for a construction of M_g but for a projective embedding $M_g \subset P_N$. This leads to the fourth aspect: to study the Picard group of M_g. One interesting point in this connexion is that it is possible to define the Picard group of the moduli problem itself without reference to the moduli scheme. Namely, by a line bundle on the moduli problem we shall mean, a collection of line bundles, one on each scheme S for each family of curves over S; plus, for each morphism between families, a corresponding morphism between line bundles. Heuristically, such line bundles arise from attaching canonically one dimensional vector spaces over Ω to each curve over Ω.

I can prove that the group of line bundles on the moduli problem, i.e. the Picard group, is finitely generated; and that, up to torsion, there is exactly a subgroup isomorphic to $\mathbf{Z}$ of line bundles which extend to line bundles on the whole moduli problem of principally polarized abelian varieties (via the Jacobian). I conjecture that the group itself is $\mathbf{Z}$, but in this connexion I can give only some curious relations. For example, to any curve C, we can attach two 1-dimensional vector spaces: a) $\Lambda^{g} H^0(C, \Omega)$, where Ω is the sheaf of differentials on C, and b) $\Lambda^{3g-3} H^0(C, (\Omega)^2)$, where $(\Omega)^2$ is the sheaf of quadratic differentials. These extend naturally to line bundles L_1 and L_2 on the whole moduli problem. Then, up to torsion:

$$L_2 \cong (L_1)^{13} .$$