ally negative component of D (see Theorem 7.7), then, under the assumption that $\dim |hD| > 0$ for some h, there exists an integer m and there exist m effective cycles $\bar{B}_1, \bar{B}_2, \cdots, \bar{B}_m$ (with rational coefficients) such that

$$B_n = n\mathcal{E} + \bar{B}_{\nu(n)} ,$$

where $\nu(n)$ is one of the integers $1, 2, \cdots, m$ (Theorem 8.1; Theorem 10.1; Theorem 11.4). Thus, under the assumption that $\dim |hD| > 0$ for some h, B_n is bounded (from above) if and only if D is arithmetically effective.[9]

(7) If the quadratic form φ_D is of type $(0, t)$, the function $\nu(n)$ which occurs in (6) is periodic (Theorem 11.4).

(8) The ring $R^*[D]$ (introduced in § 2) is finitely generated over k if and only if one of the following conditions is satisfied:

(8a) φ_D is of type $(1, t)$ and some multiple $|h(D - \mathcal{E})|$ ($\mathcal{E} =$ arithmetically negative component of D) has no fixed components;

(8b) φ_D is of type $(0, t)$ (Theorem 10.6; Proposition 11.5).
In the case (8b), $R^*[D]$ has transcendence degree ≤ 1 over k.

APPENDIX

THE CANONICAL RING OF AN ALGEBRAIC SURFACE

BY DAVID MUMFORD

In this appendix we wish to examine how the general theory developed by O. Zariski applies to the canonical divisor class. To be precise, suppose F is a non-singular algebraic surface over an algebraically closed field k, which

(a) is not birationally equivalent to a ruled surface, and

(b) is minimal [11].

Moreover let K be the canonical divisor class. We set

$$R = \bigoplus_{n=0}^{\infty} H^0(\mathfrak{o}_F(nK)) ,$$

and we call R the canonical ring of the surface ($R = R^*[K]$ in Zariski's notation). There are three essentially different cases to consider according as (K^2) is negative, zero, or positive. We assert:

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[9] The assumption that $\dim |hD| > 0$ for some h is necessary. Thus, it is possible to have a prime cycle E such that $(E^2) = 0$ (and which is therefore arithmetically effective, whence $\mathcal{E} = 0$) and such that $\dim |nE| = 0$ or all n (whence $B_n = nE$, and B_n is not bounded). To obtain such a cycle E, we use the construction of §2, with the following modifications: we take for E' a generic plane section of F', we take for $\mathfrak{A}$ the divisor class determined on E' by $|E'|$ (i.e., we take for h the integer 0) and we determine $P_1', P_2', \cdots, P_m'$ (where $m = (E'^2)$) by the condition (4). Then it is immediate that the proper transform E of E' satisfies the desired conditions.

THEOREM. (i) *Under the above hypotheses, $(K^2) < 0$ is impossible.*

(ii) *If $(K^2) = 0$, then for some n either $nK \equiv 0$, or $|nK|$ is a linear system without base points, composite with a pencil. Therefore R is a finitely generated ring of dimension 1 or 2.*

(iii) *If $(K^2) > 0$, then for sufficiently large n, $|nK|$ is a linear system without base points. Therefore R is a finitely generated ring of dimension 3. Moreover $s(nK) = \dim H^1(\mathfrak{o}_F(nK)) = 0$ for sufficiently large n.*

Only the proof of (iii) will be given in this Appendix, since the proof of (i) and (ii) is rather long and will be published elsewhere. If the characteristic is 0, the latter proof depends chiefly upon Enriques' theorem: *if F is a relatively minimal non-singular algebraic surface, and $|nK|$ is empty for all n, then F is ruled.* The first complete proof of this in characteristic 0 (and of its refinement: if $|12K|$ is empty, then F is ruled) was obtained several years ago by K. Kodaira (unpublished). In characteristic p, new difficulties arise, but Enriques' result and parts (i) and (ii) of the theorem can still be proved.

We shall now establish (iii). Notice first that by the Riemann-Roch theorem, either $|nK|$ or $|-nK|$ is non-empty for large n. The latter case is impossible. For suppose q is the irregularity of F ($=$ dimension of the Picard variety), and ρ is the base number ($=$ rank of the Neron-Severi group). Then by Noether's formula for $p_a(F)$ and by Igusa's inequality,[10] we see that

$$12\big(p_a(F) + 1\big) = (K^2) - \deg(c_2) > 2 - 4q + \rho \ .$$

But since $p_g(F) = 0$, it follows that $p_a(F) = -q$.[11] Therefore:

$$8(1 - q) \geqq \rho - 1 \ .$$

But if $q = 0$, then F is rational, by Castelnuovo's criterion[12] which we have excluded; and if $q = 1$, then the Albanese map is a regular map onto a curve, and $\rho \geqq 2$. Therefore this last inequality cannot be fulfilled.

Therefore $|nK|$ is at least non-empty for sufficiently large n. Let D be any irreducible curve on F. Suppose $(D \cdot K) \leqq 0$. Then by Hodge's index theorem, since $(K^2) > 0$, it follows that $(D^2) < 0$. But also $-2 \leqq 2p_a(D) - 2 = (D^2) + (D \cdot K)$. Therefore $p_a(D) = 0$; i.e., D is a non-singular rational curve, and (D^2) equals -1 or -2. In the first case, D would be

[10] See J. I. Igusa, *Betti and Picard numbers of abstract algebraic surfaces*, Proc. Nat. Acad. Sci. U.S.A., 46 (1960), p. 724

[11] See Y. Nakai, *On the characteristic linear systems of algebraic families*, Ill. J. Math., 1 (1957), p. 552.

[12] See O. Zariski, *On Castelnuovo's criterion of rationality $p_a = p_2 = 0$ of an algebraic surface*, Ill. J. of Math., 2 (1958), p. 303.

exceptional of the first kind and F would not be minimal [11]. Therefore, we conclude:

($*$) *If D is an irreducible curve, and $(D \cdot K) \leq 0$, then D is a nonsingular rational curve, $(D^2) = -2$, and $(D \cdot K) = 0$.*

Notice that there can be at most a finite number of such irreducible curves D. In fact, by the Riemann-Roch theorem, there is an m such that $\dim |mK| \geqq 2$. Then every curve D such that $(D \cdot K) = 0$ is either a fixed component of the linear system $|mK|$ or else is disjoint from every divisor of $|mK|$. In either case, there is only a finite set of such irreducible curves.

Let $E_1, E_2, \cdots, E_n$ be the set of all irreducible curves D such that $(D \cdot K) = 0$. Then by a very beautiful theorem of M. Artin,[13] which is the central point of this proof, there is a normal surface F^*, and a regular birational map $f \colon F \to F^*$, with the following five properties:

(i) f is biregular on $F - \bigcup E_i$,

(ii) f maps each E_i to one point,

(iii) the canonical divisor K^* on F^* is a Cartier divisor,

(iv) $f^{-1}(K^*) = K$,

(v) $p_a(F) = p_a(F^*)$.

By (iv), the linear systems $|nK|$ (on F), and $|nK^*|$ (on F^*) are canonically isomorphic. The proof that for sufficiently large n, $|nK^*|$ has no base points proceeds in three steps:

Step I. For all sufficiently large n, $|nK|$ (and hence $|nK^*|$) is non-empty. This is a corollary of the Riemann-Roch theorem.

Step II. For all sufficiently large n, $|nK^*|$ has no fixed components. For let k and l be relatively prime integers such that $|kK|$ and $|lK|$ are non-empty. Then by Theorem 9.1 above, for all sufficiently large n, the only fixed components of $|nkK|$ and $|nlK|$ are the irreducible curves E_i. But since all sufficiently large integers are of the form $nk + n'l$, for "sufficiently" large n and n', it follows that the only fixed components of $|nK|$ for sufficiently large n are the curves E_i. Hence by (ii) and (iv), the corresponding linear system $|nK^*|$ has no fixed components.

Step III. For sufficiently large n, $|nK|$ has no base points at all. The proof of this depends on a slight extension of Theorem 6.2 above. Namely, notice that this theorem, together with the proof of that theorem (§6), are equally valid whenever (in the notation of that theorem) V is a normal surface, and D is a Cartier divisor. Now let k and l be relatively prime integers such that $|kK^*|$ and $|lK^*|$ have no fixed components. By (iii),

[13] See M. Artin, *Some numerical criteria for contractability of curves on an algebraic surface*, Amer. J. Math., forthcoming, Th. (2.7).

and this extension of Theorem 6.2, for all sufficiently large n, the linear systems $|nkK^*|$ and $|nlK^*|$ have no base points. Hence just as before, for all sufficiently large n, $|nK^*|$ (and hence $|nK|$) has no base points.

The result on the superabundance follows from (v) and Theorem 6.5 above, once one observes that for sufficiently large n, the linear system $|nK|$ must define a regular map of F into projective space, with image F^*.

Finally, one sees that R is finitely generated as follows: Let k and l be relatively prime integers such that $|kK|$ and $|lK|$ have no base points. Then by Theorem 6.5, the rings $R^*[kK]$, and $R^*[lK]$ are finitely generated. But these two rings together generate a 'subring of R that contains all but a finite number of its homogeneous components. As each component of R is a finite dimensional vector space, R itself is therefore finitely generated.   q.e.d.

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