PATHOLOGIES OF MODULAR ALGEBRAIC SURFACES.*

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The purpose of this note is to present two counterexamples to conjectures about the geometry of algebraic surfaces, which are, on the contrary, true in characteristic zero. Thus if F is a non-singular algebraic surface defined over an algebraically closed field k, one can form the vector spaces

$$H_{0,1} = H^1(F, \Omega^0),$$
$$H_{1,0} = H^0(F, \Omega^1),$$

where Ω^i is the sheaf of regular i-forms. One also has the vector space

$$A_0 = \text{cotangent space to the Albanese Variety } A \text{ of } F \text{ at the origin } 0.$$

Among these vector spaces various maps may be defined regardless of the ground field. First, the canonical map

$$\phi : F \to A$$

induces a homomorphism

$$\phi^* : A_0 \to H_{1,0},$$

which Igusa [4] has shown to be one-one. Secondly, if an embedding of F in projective space is fixed, and hence a canonical element h in $H^1(F, \Omega^1)$, then cup product and Serre duality induces a pairing

$$\mathfrak{H} : H_{0,1} \times H_{1,0} \to k.$$

(See Kodaira [6] and Serre [9]).

Now in the classical case, the usual constructive existence proof of the Albanese variety A shows immediately that ϕ^* is onto (see Weil [12]); while the famous topological results of Lefschetz on the embeddings of varieties in projective space (see Wallace [11], and Kodaira [6]), allow one to conclude that the pairing $\mathfrak{H}$ is non-degenerate. One deduces the "fundamental equalities" of Italian surface theory:

$$\dim A_0 = \dim H_{1,0} = \dim H_{0,1},$$

* Received January 4, 1961.

proven first by Poincaré by his method of normal functions in 1910, (Poincaré [8]). A corollary of ϕ^* being onto, combined with the easy proof (valid in all characteristics—see Koizumi [7]) that regular differentials on an Abelian variety are closed, shows that the differentials in $H_{1,0}$ are all closed. This has been generalized by Hodge, and has led ultimately to the result: analytic regular differentials on a Kähler manifold are closed (see Weil [12]).

The question is what extends to the modular, or non-zero characteristic case. Igusa showed, first of all, that ϕ^* is not always onto [5]. Serre showed next that the pairing $\mathfrak{H}$ is not always non-degenerate, since it can happen that $H_{1,0} = (0)$, while $H_{0,1} \neq (0)$, (see [10]). There remain the questions:

(a) Are all differentials in $H_{1,0}$ closed?

(b) Does the pairing $\mathfrak{H}$ have the property that if x in $H_{1,0}$ is such that $\mathfrak{H}(x,y) = 0$ for all y in $H_{0,1}$, then $x = 0$?

The answer to both is no.

I.

To answer (a), we prove (assuming the characteristic $\neq 0$):

THEOREM. *Let F be a non-singular algebraic surface, ω any simple differential on F (i. e. 1-form); then there exists a non-singular surface F^* and a regular map $\phi\colon F^* \to F$ which is separable and algebraic, such that $\phi^*(\omega)$ is regular on F^*.*

Proof. Note first that it is enough to prove:

(#) For all P in F, there exists an open U containing P, a surface F^* and a regular map $\phi\colon F^* \to F$ separable and algebraic such that $\phi^{-1}(U)$ is non-singular, and $\phi^*(\omega)$ is regular on $\phi^{-1}(U)$.

If this is proven, then we can find a finite set U_i^* of open sets covering F and of surfaces F_i^*, and maps $\phi_i\colon F_i^* \to F$ with the above properties. Then let F^* be some non-singular model of the function field of any compositum of $k(F_i^*)$ (compatible with the identification of the common subfields $k(F)$), that dominates the models F_i^* (such exists by Abhyankar [1] and Zariski [13]). Let ϕ be the map from F^* to F. Then for all P in F^*, say $\phi(P)$ in U_i. Then $\phi_i^*(\omega)$ is regular at $\phi_i^{-1}(\phi(P))$, hence $\phi^*(\omega)$ is regular at P.

But in fact it is enough to prove:

(##) Same as (#) except for differentials $\omega = A\,dx$, x a uniformizing parameter at P.

For if at P, $\omega = A\,dx + B\,dy$, then pick $F_1{}^*$ that does the trick for $A\,dx$ and then $F_2{}^*$ that does the trick for $B\,dy$, and take a suitable non-singularization of their join as in (#).

But now if $\omega = (A_0/A_1)\,dx$, A_0 and A_1 regular at P, and $A_1 = 0$ at P, consider the extension

$$Z^p + (A_1{}^p \cdot Z) + (x) = 0, \text{ where } p \text{ is the characteristic.}$$

It follows

$$dx = -A_1{}^p dZ.$$

Now in the normalization F^* of F in this field extension, there is a unique point P^* above P. Moreover, it is simple since if x, y are uniformizing parameters at P, then Z, y are uniformizing parameters at P^*. Finally,

$$\omega = (A_0/A_1)\,dx = A_0A_1{}^{p-1}(dx/A_1{}^p) = -A_0A_1{}^{p-1}\,dZ.$$

Hence ω is regular at P^*, hence in an open set about P^*. QED

COROLLARY. *There exist algebraic surfaces, non-singular and with a simple regular differential ω that is not closed.*

Proof. Take in the above theorem $F = P^2$, the projective plane, and $\omega = x\,dy$. Then the differential $\phi^*(\omega)$ on the covering F^* given by the theorem satisfies the corollary. One must merely note that $d\phi^*(\omega) = \phi^*(d\omega) \neq 0$ as the covering is separable. QED

II.

To answer (b), consider the famous Enriques surface E, which is, in any characteristic, the normalization of the sextic surface E_0 in P^3:

$$0 = x^2y^2z^2 + x^2y^2 + x^2z^2 + y^2z^2 + xyzf_2(x, y, z),$$

where f_2 is a general polynomial of second degree. Its normalization is non-singular and, regardless of characteristic, can be constructed as the join of the graphs of the following set of maps:

$$\phi_1 \colon E_0 \to P^1, \text{ given by } xy/z,$$
$$\phi_2 \colon E_0 \to P^1, \text{ given by } xz/y,$$
$$\phi_3 \colon E_0 \to P^1, \text{ given by } yz/x,$$

as the reader may with some pains verify. It follows that the surface E in characteristic 2 is a specialization of E in characteristic 0, hence has the same p_a (see Hironaka [3]), which has long been known to be 0 (for a modern treatment, see Artin's thesis [2]). On the other hand, the classical theory of adjoint surfaces tells us that canonical divisors on E, as divisors on E_0, must be cut out by $(6-4)=2$nd order forms in P^3 passing through the double lines (see Zariski [14]); in this case, we have a tetrahedron of double lines and such are contained in no quadric. Hence $p_g=0$, hence $H^{0,1}=(0)$.

Now in the remarkable case of characteristic 2, it is also the case that $H^{1,0}\neq(0)$. In fact, let t be a coordinate on P^1; then:

$$\phi_1{}^*(dt)=\phi_2{}^*(dt)=\phi_3{}^*(dt)=d(xyz)$$

is immediately seen to be a regular differential on E.

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