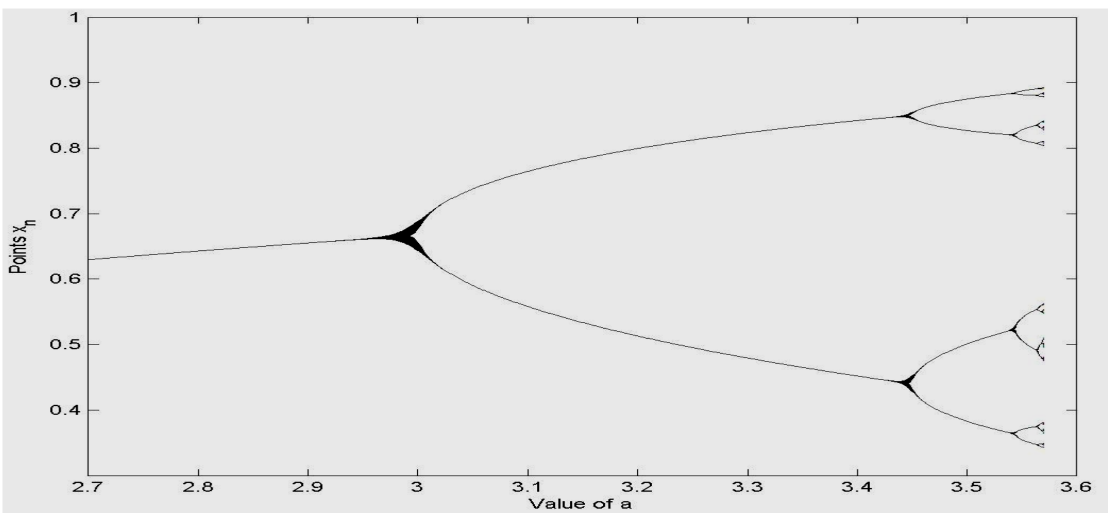
AM18, HW #8 (last one!), due Monday April 17
Strange attractors and their dimension

Recall the logistic map:

$$
\begin{aligned}
x_1 &= 0.5 \\
x_2 &= ax_1(1-x_1) \\
x_3 &= ax_2(1-x_2) \\
&\cdots\cdots\cdots \\
x_{n+1} &= ax_n(1-x_n) \\
&\cdots\cdots\cdots
\end{aligned}
$$

which generates an infinitely long sequence of numbers between 0 and 1. Here a is a fixed parameter between 0 and 4. It models, for example, the way a population size (normalized to maximum of 1) changes from year to year, if its rate of growth when small is a, but which is limited by the environment to some max – which we take to be 1 by division. The interest lies in what happens after a while which is the law's 'attractor', the set of values to which it tends after a long time.



You are to study the case:

$$a = 3.56994567$$

Why this odd number? It happens to be the smallest number at which something unexpected happens. For any smaller number, the sequence x_n will tend either(a) to one limiting number – the stable population size – or (b) to a cycle of 2 – alternate boom and bust years – or (c) to a cycle of 4, or (d) to a cycle of 8, etc. But at this value, the population jumps around an infinite set of values *but by no means all possible values*. The set of values is its attractor and it is a 'Cantor' type set (as in class). You can see all this in the figure which shows the 'attractors' for a in the range 2.7 to 3.56994567. The goal of this assignment is to estimate its dimension.

Step I: generate at least the first 1000 values x_n. You should also make a plot of points on a line at positions x_n to see the attractor (though, being one dimensional, it's not so startling).

Step II: Next work out the number of distinct intervals of the type

$[0,1/d],\ [1/d,2/d],\ [2/d,3/d],\ \cdots,\ [(d-1)/d,1]$
which contain at least one of the numbers in the sequence x_n. Do this for
$1/d = 1/2,\ 1/4,\ 1/8,\ 1/16,\ 1/32,\ 1/64,\ 1/128,\ 1/256$, i.e. $d = 2^m$, for $m = 1,\ldots,8$.
The way to work this out is to multiply the sequence x_n by d and take its 'floor' mod 1 (this means round down to the nearest integer and is done in Excel by FLOOR(d**ref*, 1) where *ref* is the range of numbers to be worked out. Then Data→Filter→AdvancedFilter allows you to find out how many distinct rounded numbers you get. Click "Unique Records Only", ask for a new range to put the results in and make sure the column has a header – for some unknown reason, Excel requires this. This should give the list of distinct intervals containing a point in the sequence.)
Step III: Once you have the numbers $k_1, k_2, \cdots, k_8$ of the filled intervals at these scales, you should plot $\log(k_m)$ against $\log(2^m)$ and fit this to a straight line, (but assuming that the first few numbers may not fit well). It's OK to do this on a piece of paper. This slope is the dimension of the attractor.
Step IV: Explain what this is doing in words.