AM18, HW #7 , due Monday April 10.
Planets in a binary star system

We suppose we are on a planet orbiting a binary star system, say a green star and an orange star of equal mass. Life is fun: there are two sunrises and two sunsets, very colorful. But the earth's motion is now much less predictable. In fact, it is dangerously unstable. Let's set this up like this:

1. Let the two suns have mass *M*, let them be at a distance 2*d*, moving in a circle of radius *d* around each other:

$$(x_{S1}, y_{S1}) = d \cdot (\cos(at), \sin(at)),\ (x_{S2}, y_{S2}) = -d \cdot (\cos(at), \sin(at)),$$

This makes both of them have constant velocity *da* and constant acceleration, each towards the other, of size da^2, so this will be produced by the law of gravity provided that:

$$\frac{GM^2}{4d^2} = \text{force} = \text{mass} \times \text{accel.} = Mda^2,\ \text{or}\ GM = 4d^3a^2$$

2. Next, suppose that a planet of much smaller mass *m* circles both of them, starting at a distance *e*. Let's give it the same velocity it would have if it were circling a single sun of mass 2*M* at distance *e*. If it were so simple, we'd have

$$(x_P, y_P) = e \cdot (\cos(bt), \sin(bt)),$$

making the velocity *eb* and the acceleration eb^2. Again, the law of gravity puts a constraint on this:

$$\frac{GMm}{e^2} = \text{force} = \text{mass} \times \text{accel.} = meb^2,\ \text{so}\ GM = e^3b^2\,.$$

So let's require that *a,b,d,e* satisfy $4d^3a^2 = e^3b^2$. But because there are two suns, the planet won't move so simply.

3. Set this up in Excel like this: make the columns represent

A. *time,*
B. *x position of sun 1, (minus this is then the x position of sun 2)*
C. *y position of sun 1, (minus this is then the y position of sun 2)*
D. *x position of planet,*
E. *y position of planet,*
F. *distance from sun 1 to planet and*
G. *distance of sun 2 to planet.*

Then we can calculate columns *A,B,C* using the formulas above. I found it convenient to take $a = 2\pi$ so that the sun's period is *t* = 1 and a time sampling of $\Delta t = 1/100$, giving 100 samples around each orbit of the two suns. We can fix the units of space by taking *d* = 1 (so now $GM = 16\pi^2$). Note that columns *F,G* can be calculated explicitly from columns *B,C,D,E*, and to propagate columns *D,E*, we use Newton's law:

$$\ddot{x} = GM\left(\frac{x_{S1}-x}{r_1^3} + \frac{x_{S2}-x}{r_2^3}\right),$$
$$\ddot{y} = GM\left(\frac{y_{S1}-y}{r_1^3} + \frac{y_{S2}-y}{r_2^3}\right),$$
$$r_1 = \sqrt{(x_{S1}-x)^2 + (y_{S1}-y)^2},$$
$$r_2 = \sqrt{(x_{S2}-x)^2 + (y_{S2}-y)^2}$$

We can make time discrete using the same method as described in Chapter 8 of the notes, with second differences allowing us to move ahead one step in time.

4. As a warmup, take e very large and start the planet off at $(e,0)$ with velocity $(0,be)$ and check that the planet orbits nicely in something that really looks like a circle. You'd better take $\Delta t = 1/100$ or it will take forever for the planet to get around the distant suns. Then increase the velocity and you should find the planet going in a large ellipse. A good way to track the results is to plot x and y against each other and also $\|(x,y)\|$ and atan2(x,y) against time (the function 'atan2' coverts a point (x,y) in the plane to its angle in polar coordinates).

5. Then, try $e = 4$, $\Delta t = 1/100$ and compute the orbit for $0 \le t \le 8$. This is the fun case. What do you find? If a 'year' is the time for the planet to circle both suns, how many times each year will the 2 suns line up – causing consternation among the inhabitants of this planet. There will be orange *eclipses* and green ones when the orange or green sun comes in front of the other. You may also want to plot the angle between the 2 suns as seen from the planet: how far apart do they get in the sky and what does this mean for twilight? (To work out this angle ϕ, look at the triangle formed by the planet and 2 suns. Its area can be expressed by the formula above in terms of the positions of the planet and the suns and can also be written as $(1/2) r_1 r_2 \sin\phi$.) The ancient astronomers of this planet would have been confounded by the complexity of the motion of the two suns.

6. Finally try $e = 3.5$, $\Delta t = 1/100$. Try fiddling with the initial velocity – can you make the planet behave and stay near the suns? I found huge instability. The simulations tend to be inaccurate for this time step but the overall effect seems the same even if the time step is much smaller.