AM18 Problem Set #6 Due Monday, March 13

1. The story of the complex numbers began when people asked: why shouldn't -1 have a square root. So people just created it by fiat. Hmm: how a square root of $i = \sqrt{-1}$? Show that there is a complex number $a + ib$ whose square is i, so we don't need to seek further.

2. Differential equations are not just physics. They are also handy for biology. Let's study foxes and rabbits. An influential model is the so-called predator-prey model, which models 2 species, one eating the other, whose population cycles over a period of years. When there a lot of rabbits, there is plenty for the foxes to eat and they multiply rapidly. But then the rabbit population is depleted and, as a result, the foxes starve. When the foxes are nearly gone, the rabbits flourish again. And so 'the cycle of nature repeats' as nature specials on public TV have told us so often. OK, let's do math. Instead of forces, we now have birth and death. Let $r(t)$ be the population of rabits and $f(t)$ that of foxes. If A is the reproduction rate of each rabbit, then r should increase at a rate Ar due to births. If B is the probability of one fox eating one rabbit in some unit of time, then the product Bfr is the death rate due to rabbits being eaten (either more foxes or more rabbbits result in more fox-eating-rabbit meals). Thus:

$$\frac{dr}{dt} = Ar - Bfr.$$

Now let C be the excess of the death rate of foxes over their birth rate when there are no rabbits to eat and let D be the increase in births minus deaths of foxes for each rabbit in the population. Then:

$$\frac{df}{dt} = -Cf + Dfr.$$

(Were clearly simplifying a lot, e.g. ignoring the natural deaths of rabbits from old age.) These equations were first written down by Lotka in 1910 and further studied by Volterra, hence are also called the Lotka-Volterra equations. There are 4 constants A, B, C, D in the equations, but by changing the units in which the populations r, f and time t are measured, it turns out that the results only depend on A/C. Take $A = 2, B = C = D = 1$ and first, by hand, make a rough phase-plot with arrows as in Chapter 7. That is, make a plane in which the 2 coordinates are f and r, the fox and rabbit populations. We are only interested in the positive quadrant $f > 0, r > 0$. Then draw arrows showing, for a good sample of points (f, r) in what direction and how fast the point is moving after a small amount of time. Try to guess what happens to $r(t), f(t)$ from the phase-plane plot. Then solve the differential equation approximately in Excel, starting at $r = f = .5$ using some small time step Δt. Experiment with various Δt until the result stabilizes: what is the system doing? What would you guess about the long term behavior of solutions $f(t), r(t)$?