AM18 Problem Set #4 Due Wednesday, Febuary 22

1. Suppose Marco Polo, on his return from China in 1295, had brought with him a marvelous oriental invention in which a Tibetan prayer wheel, a cylinder with many slots which was spun around a brilliant light of burning phosphorus. The Chinese had been delighted with its rapidly flickering light and the marvelous illusions it made when dancers were seen by its light. We think of this device as an early precursor to a high frequency strobe light! (All this is not true, by the way.)

   Your job is to write three paragraphs as nearly as possible *in the style of Oresme, Galileo and Newton* respectively in which they mention this device and make use of it to develop their ideas/theories. Specifically, Oresme might have related this to his theory of graphing. Galileo might have made use of it in his theories both of falling bodies and pendula and might even have related it to his interest in musical instruments! And Newton, with his talk of small increments and 'ultimate ratios' might have used this in a 'scholium' to illustrate these ideas or might have related it to the study of waves.

2. We can use simple harmonic motion as an opportunity to explore how well computers can approximate the exact solutions to differential equations. In the computer, derivatives like dx/dt are replaced by the approximation $\Delta x/\Delta t$ and the best we can hope for is that we get values for $x(t)$ close to the correct ones. Newton's law is:

$$\frac{d^2x}{dt^2} = \frac{1}{m}F(x).$$

   The compuyter approximates it with:

$$\frac{x((k+1)\cdot\Delta t) - 2x(k\cdot\Delta t) + x((k-1)\cdot\Delta t)}{(\Delta t)^2} = \frac{1}{m}F(x(k\cdot\Delta t))$$

   where $x(k\cdot\Delta t)$ is the position of the spring weight at time $t = k\cdot\Delta t$.

   For a spring $F(x) = -x$, a restoring force which always accelerates the weight back to its rest position $x = 0$. If $x_k = x(k\cdot\delta t)$, then law of motion, approximated by discrete time, becomes:

$$x_{k+1} = 2x_k - x_{k-1} - (\Delta t)^2 x_k.$$

   Solve this in Excel starting with $x_0 = 0, x_1 = \Delta t$. Note that each successive $x_2, x_3, \cdots$ is computed from the previous ones by the same formula, so you only have to enter this formula in one cell and *drag*. Carry this out for $\Delta t = .5, .25, .1$ and $.05$ and for k up, say, 200 (giving $t = 200\Delta t$). Excel tips: you can select and fill *without dragging* if you prefer by using the 'name box' on the top left to select the cells and then `Edit-->Fill-->Series` or cntrl-D (fill downwards).

Plot the sequence of x_k against $t = k \cdot \Delta t$. What is the *exact* solution to the differential equation? Plot this on the same graph as the solution gotten by "finite differences". (N.b. Excel knows about sin and cos and that you can stretch/resize the plot to see what it is showing best.) For each Δt, what is the maximum error?

Finally make plots of position against velocity instead of position against time, i.e. plot x_k against $(x_{k+1} - x_{k-1})/(2\Delta t)$.