Scalar conservation laws with random data and the equation free method.

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Two goals

(1) To communicate interesting (and simple!) exact solutions in Burgers turbulence, and the kinetic theory of shock clustering.

(2) To illustrate the utility of these exact solutions as a useful test case for numerical schemes for complex stochastic systems. In this talk, the scheme is the "equation free" method developed by Yannis Kevrekidis and his collaborators.

Burgers model (1930s)

Consider the scalar conservation law

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0, \quad x \in \mathbb{R}, t > 0.$$

with white noise as initial data.

Main question: How does the equation evolve white noise or other random data?

The entropy solution emerging from white noise.

Shocks form instantaneously. They then collide and cluster.

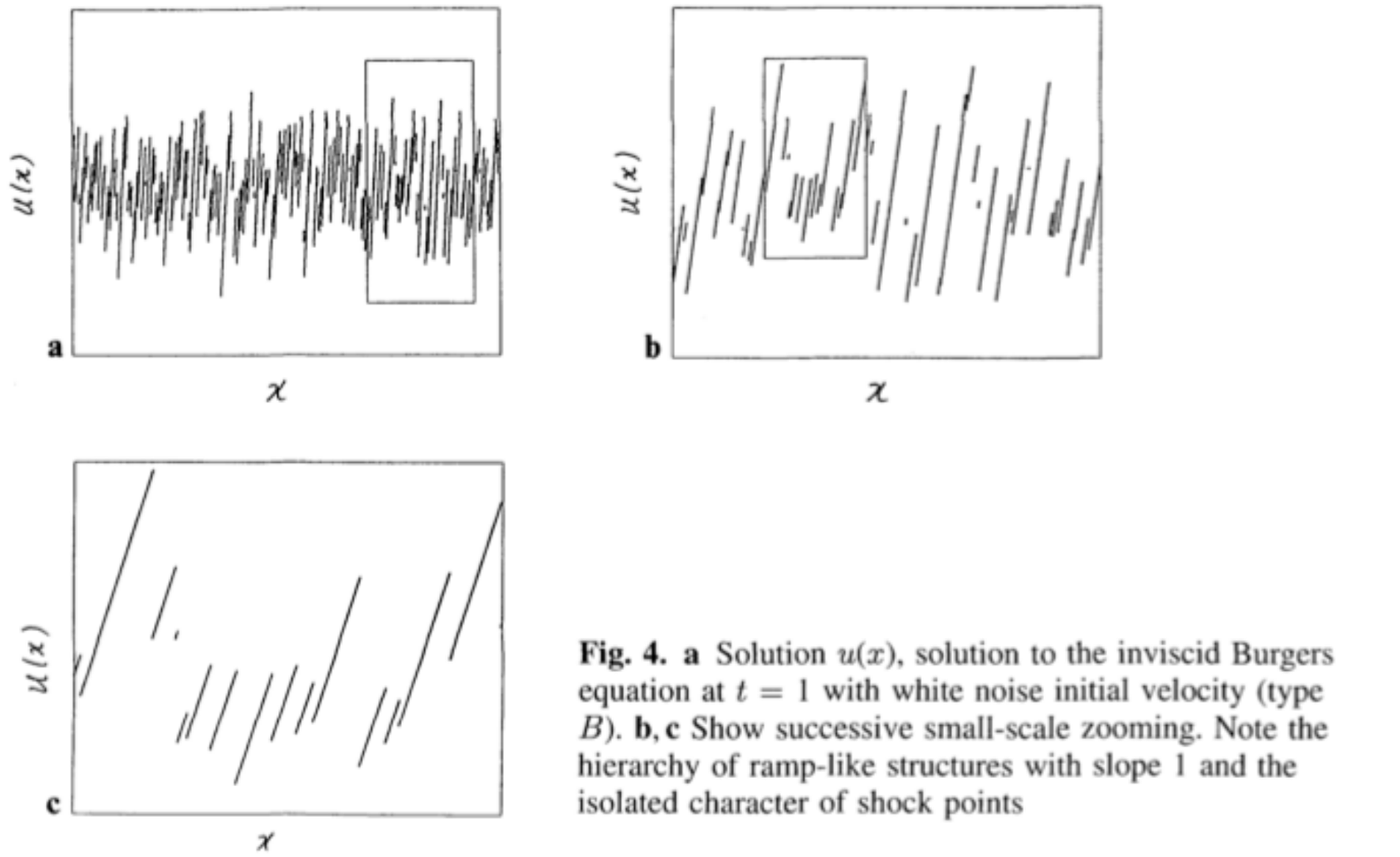


Fig. 4. a Solution $u(x)$, solution to the inviscid Burgers equation at $t = 1$ with white noise initial velocity (type B). **b, c** Show successive small-scale zooming. Note the hierarchy of ramp-like structures with slope 1 and the isolated character of shock points

She, Aurell, Frisch, Comm. Math. Phys. 148,(1992),
Sinai, Comm. Math. Phys. 148, (1992).

This problem is exactly solvable!

Here an exact solution means that we can explicitly describe the statistics of the solution.

Recall that the existence of a unique entropy solution to the PDE was established in the 1950s by Hopf, Lax and Oleinik. So the point is that the law (i.e. probability distribution of the solution) can be described exactly for some classes of random initial data.

Example 1: Solution with white noise (Groeneboom, 1985).

Example 2: Levy process initial data (Carraro-Duchon, Bertoin, 1996).

Groeneboom (1985): The exact solution with white noise.

It is enough to determine the solution at time 1. Then:

(1) The solution u(x,1) is a Markov process in x.

(2) The generator of u(x,1) is the integro-differential operator:

$$\mathcal{A}\varphi(u) = \varphi'(u) + \int_{-\infty}^{u} n_*(u,v)\,(\varphi(v) - \varphi(u))\,dv$$

with a jump density n_* given explicitly on the next slide.

Both (1) and (2) are surprising :

(1) because it is "structural": white noise --> Markov ?!

(2) because it is so explicit.

The exact solution with white noise (contd.)

The jump density n_* depends on two positive function J and K

$$n_*(u,v) = \frac{J(v)}{J(u)} K(u-v), \quad u > v.$$

The Laplace transforms of J and K are given in terms of Airy functions.

$$j(q) = \frac{1}{\mathrm{Ai}(q)}, \quad k(q) = -2\frac{d^2}{dq^2}\log(\mathrm{Ai}(q).$$

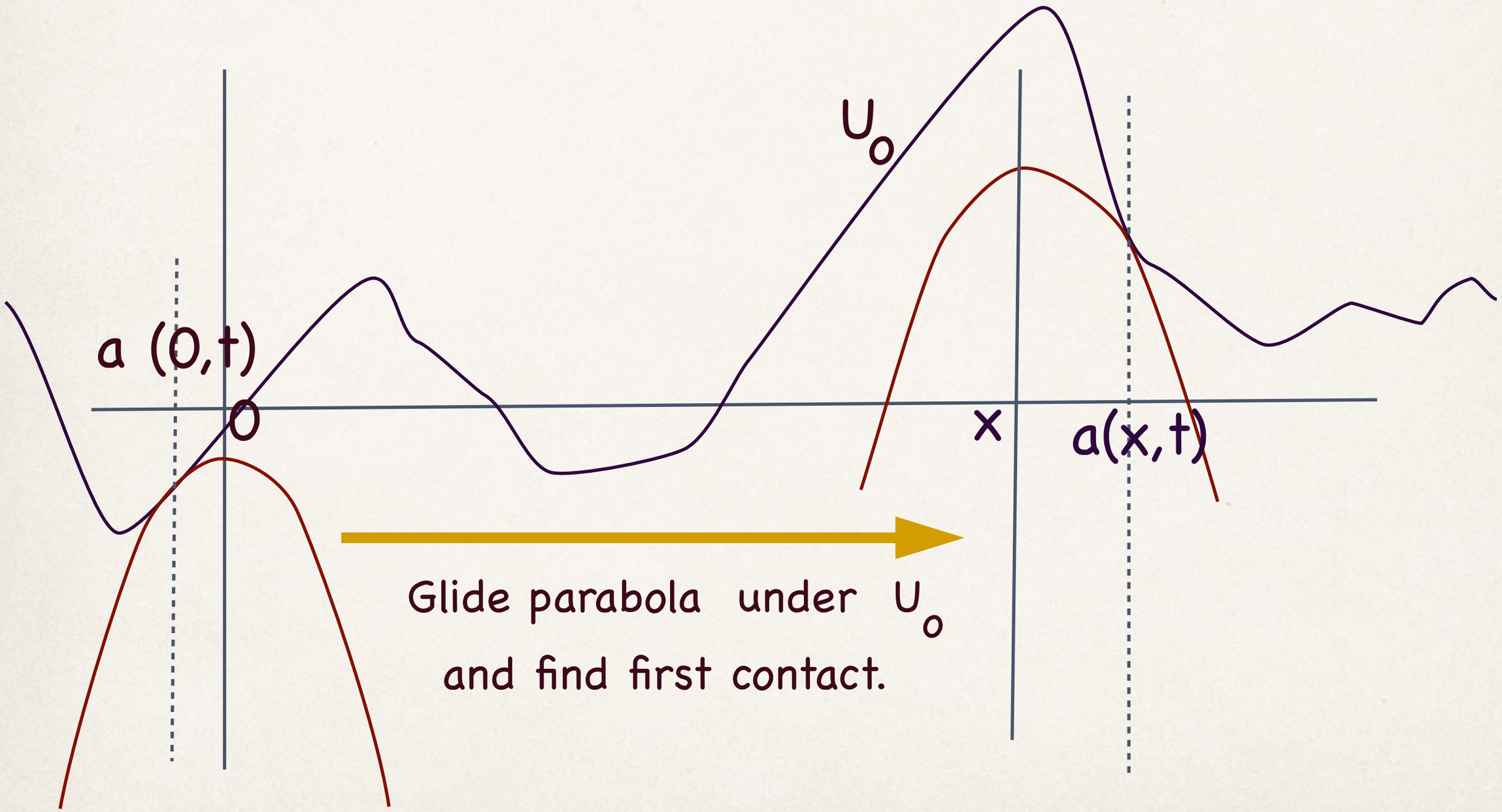
The unique entropy solution is given by a variational principle.

$$u(x,t) = \frac{x - a(x,t)}{t}$$

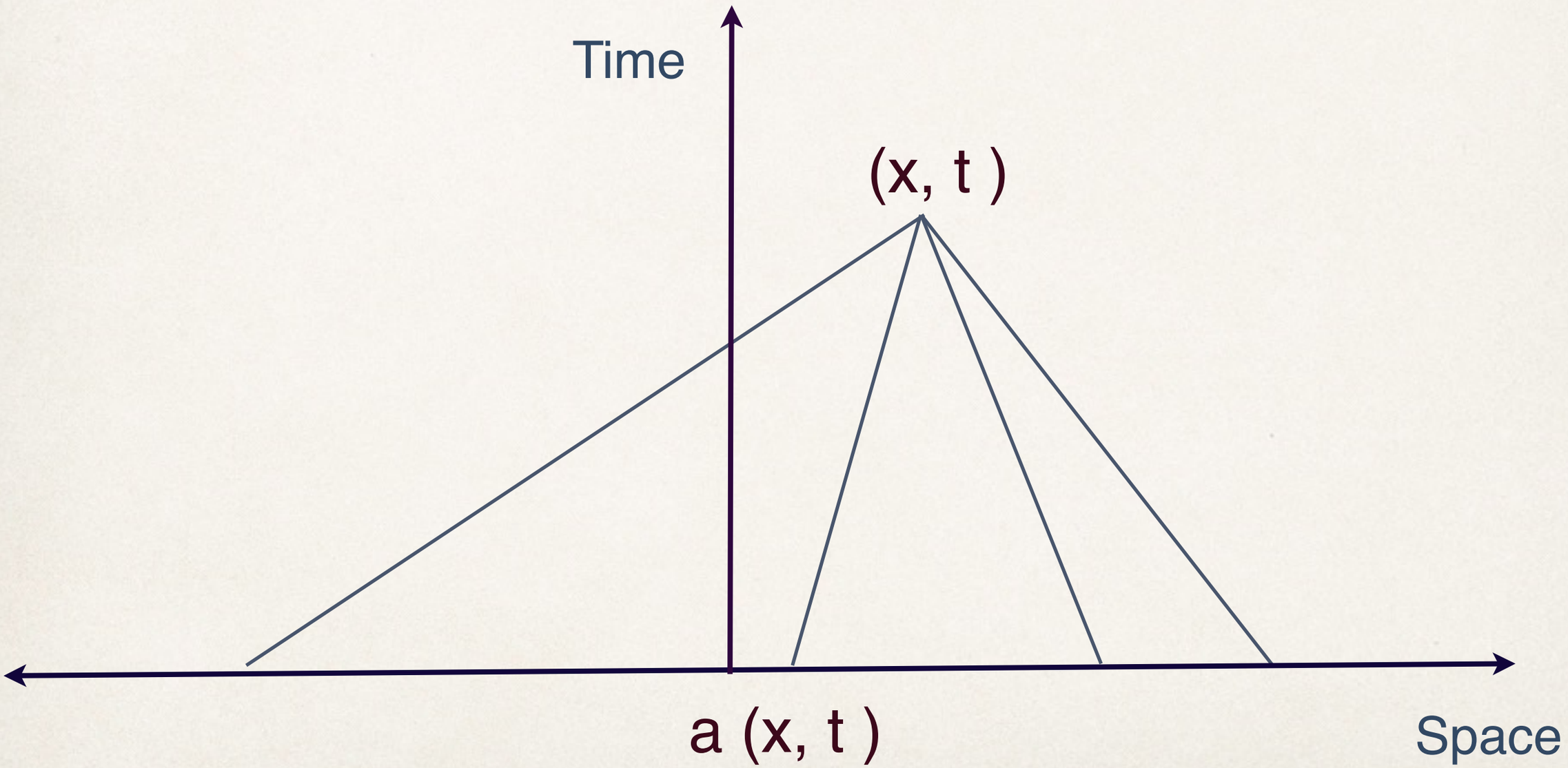
$$a(x,t) = \operatorname{argmin}_y^+ \left\{ U_0(y) + \frac{(x-y)^2}{2t} \right\}$$

$$U_0(y) = \int_0^y u_0(s)\, ds$$

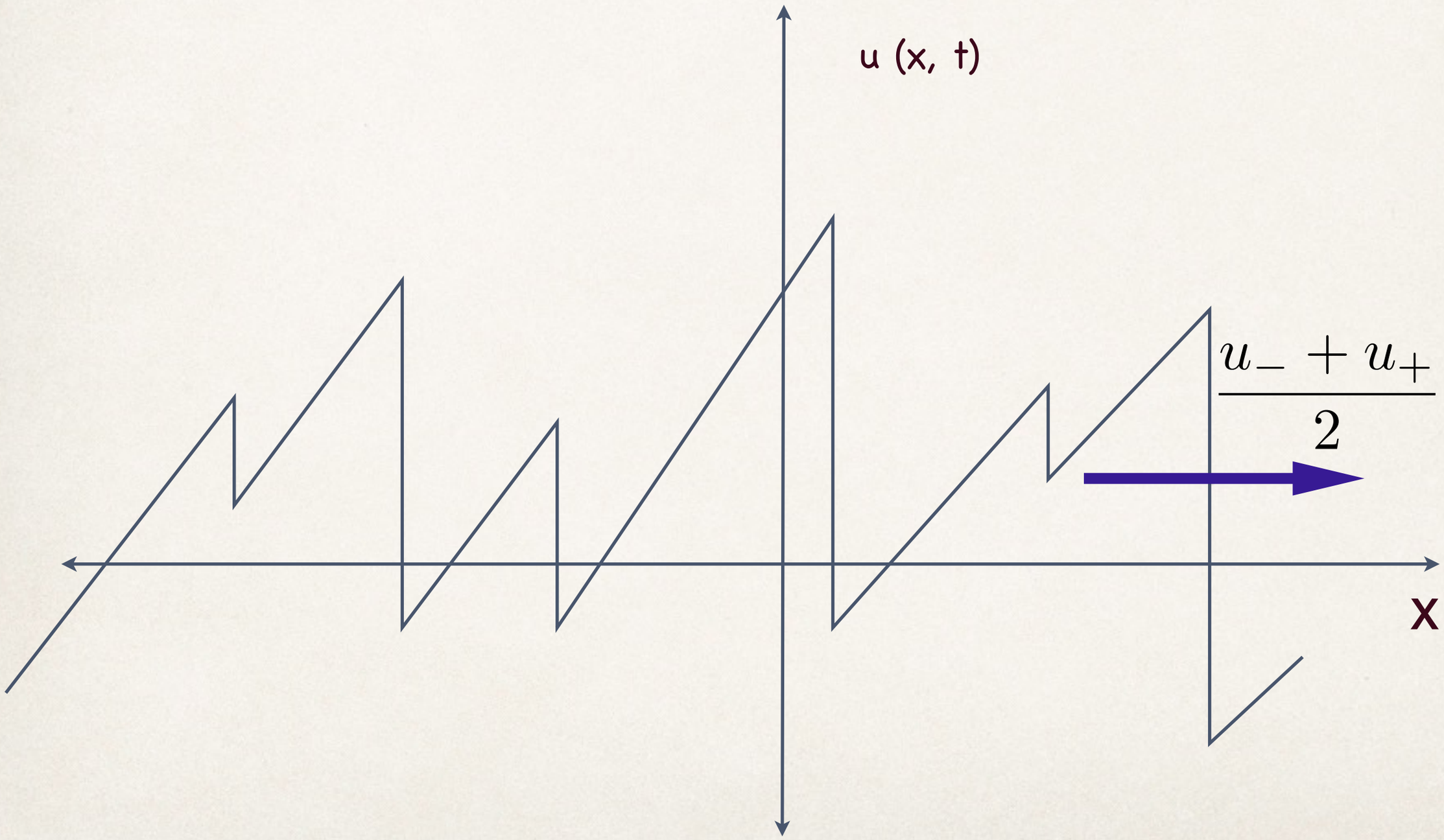
u(x,t) is the velocity field, U is called the potential, and a(x,t) the inverse Lagrangian function. The variational principle is a geometric recipe that uses the initial potential.



Roughly, a(x,t) gives the `correct' characteristic through the point (x,t) in space-time.



u(x,t) is of bounded variation. Jumps in inverse Lagrangian, a, give rise to shocks in u. These correspond to `double-touches' in the geometric principle.



The basics of Levy processes

(a) The simplest Levy process is the counting function for a Poisson process.

(b) The next simplest Levy process is a compound Poisson process.

(c) Limits of these processes give Brownian motion.

Probabilists are most interested in diffusions. But since solutions to conservation laws are BV, we will focus on jump processes.

The Laplace exponent: definition.

In general, in order to understand the statistics of a process, we must understand the joint distribution at n-points. For Levy processes, all of this information collapses into one function -- the Laplace exponent.

This is simplest to explain under the assumption that all jumps are downward.

$$\mathbb{E}\left(e^{qu(x)}\right) = e^{-x\psi(q)}, \quad q \in \mathbb{C}_+, \quad x \geq 0.$$

$$\psi(q) = \int_0^\infty \left(e^{-qs} - 1 + qs\right) \Lambda(ds).$$

The Laplace exponent: examples.

(a) Poisson process of rate λ in space, and jumps of unit size downwards.

$$\mathbb{E}\left(e^{qu(x)}\right) = e^{\lambda x(1-e^{-q})}.$$

(b) Compound Poisson process with independent downward jumps with pdf f.

$$\mathbb{E}\left(e^{qu(x)}\right) = e^{\lambda x \int_0^\infty (1-e^{-qu})f(u)du}.$$

(c) Brownian motion.

$$\mathbb{E}\left(e^{qu(x)}\right) = e^{xq^2}.$$

Simplest case: Burgers with monotone, compound Poisson initial data

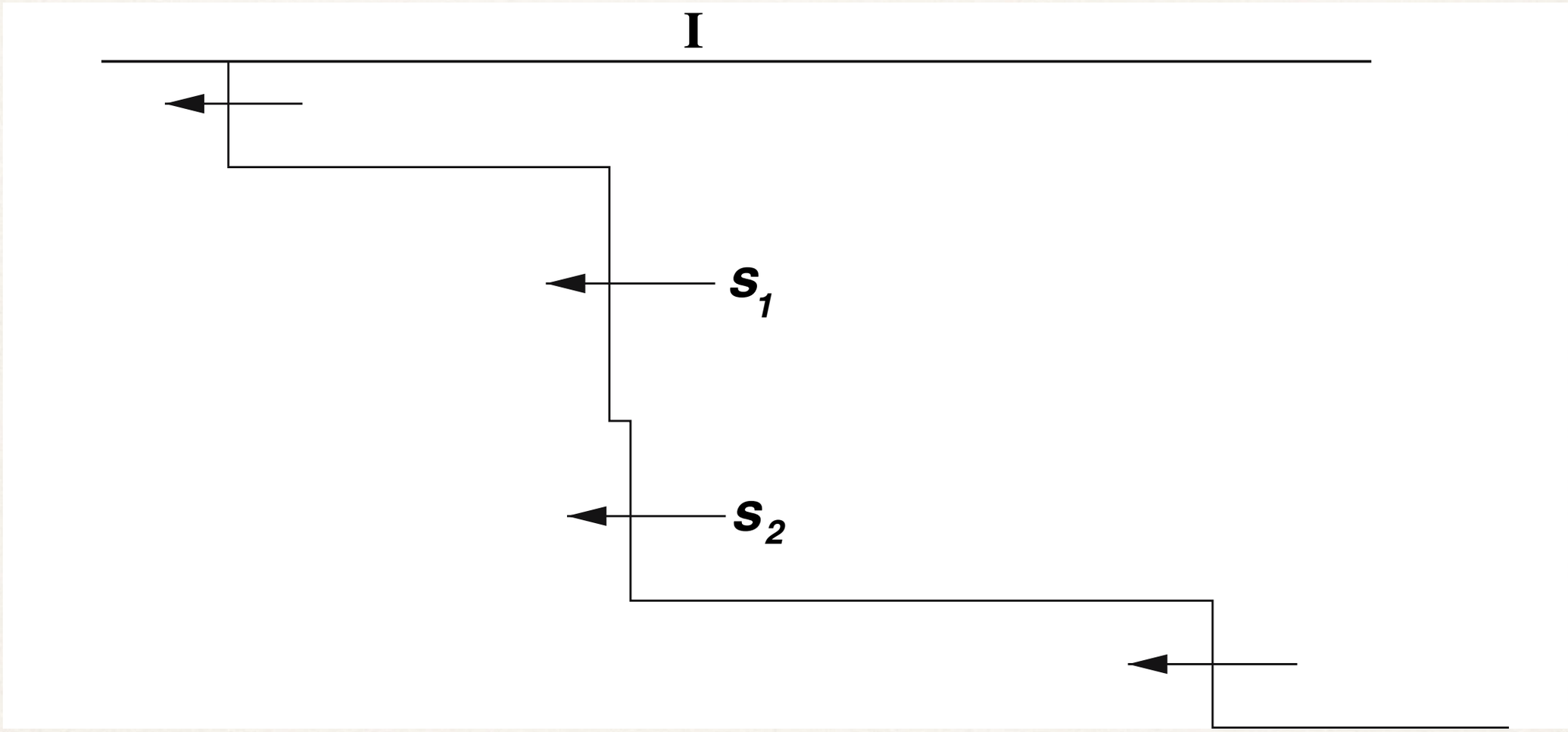
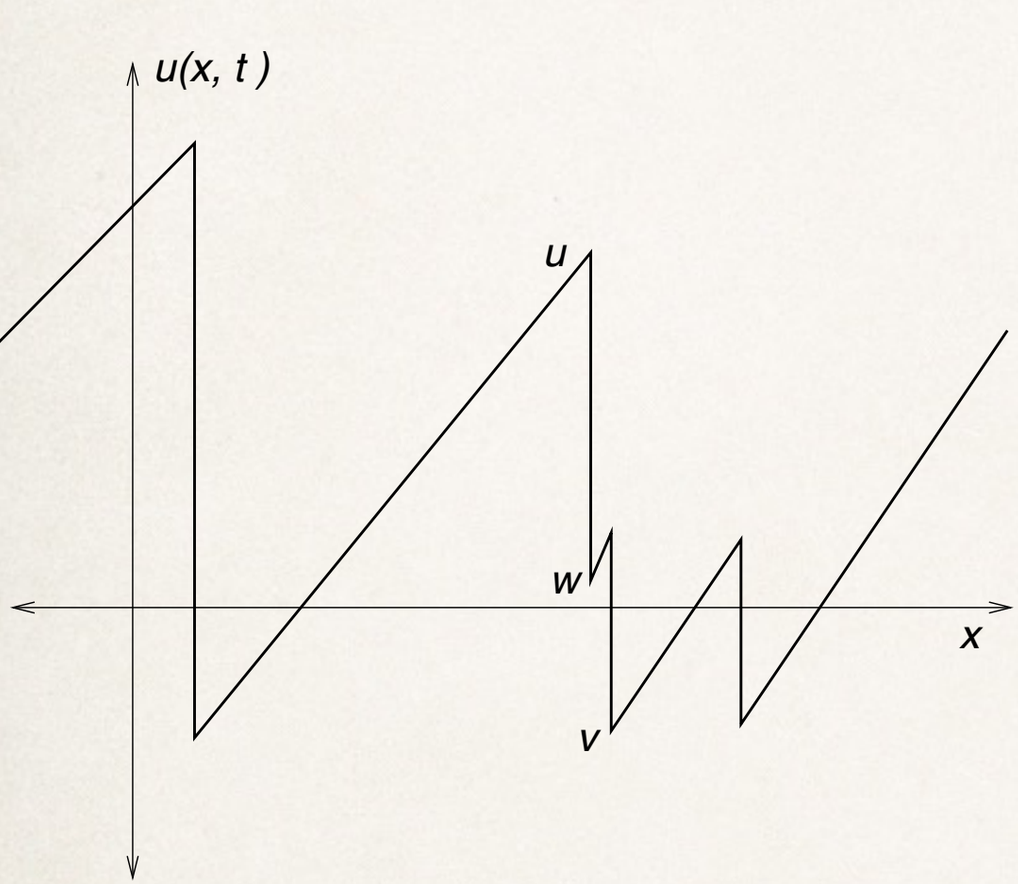
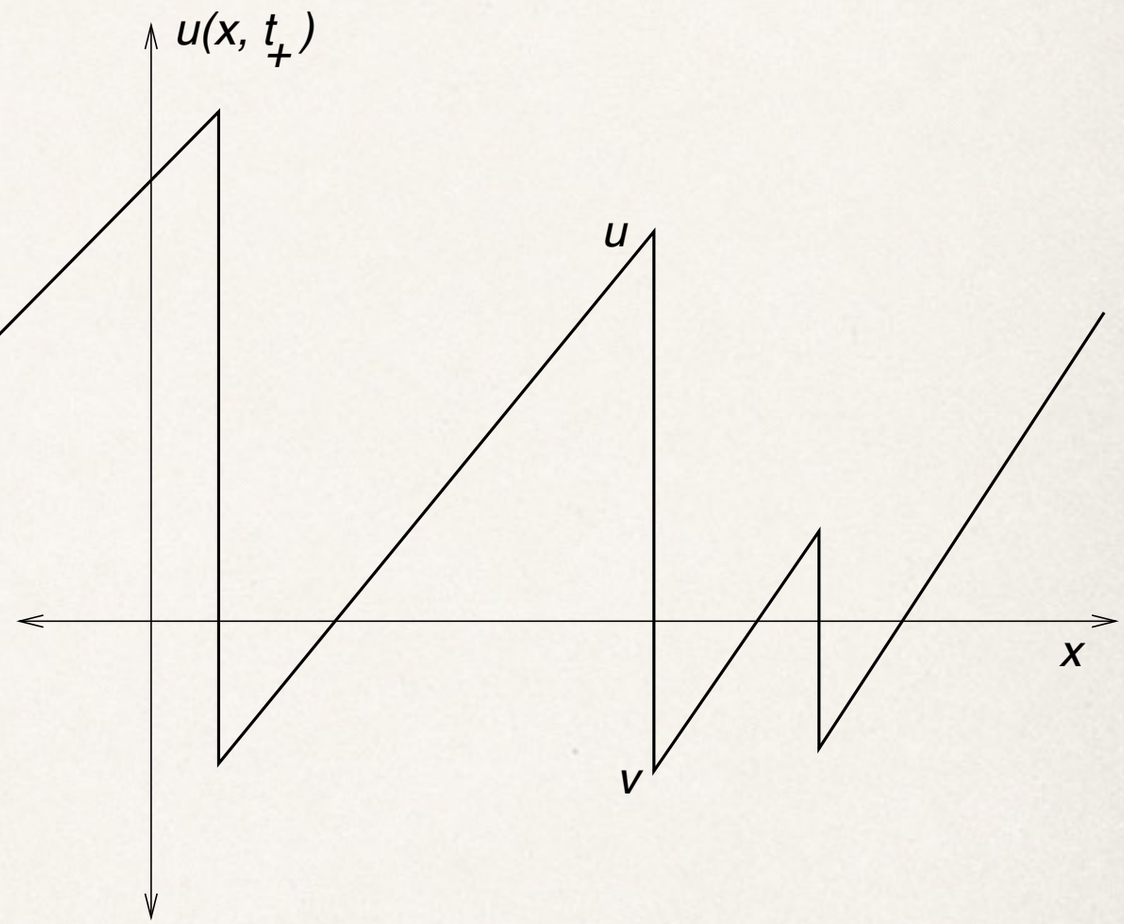


Fig. 1. Binary clustering of shocks

More generally, can include a deterministic drift in the Levy process





"Closure": Bertoin's theorem

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0, \quad x \in \mathbb{R}, t > 0.$$

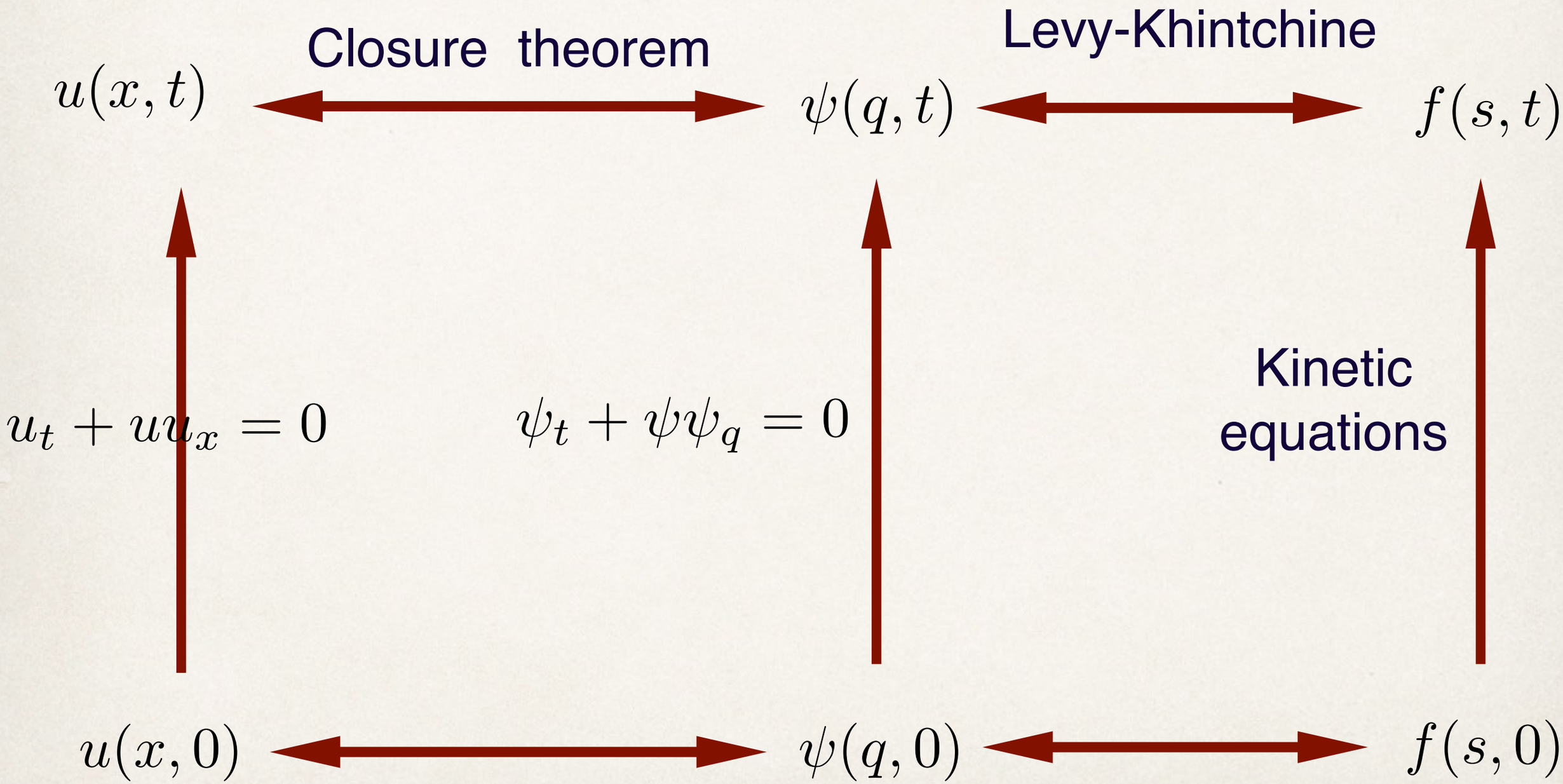
Theorem 1. (Bertoin, CMP, 1996). Assume the initial data is a Levy process (in x) that may include a drift and Brownian motion, but only downward jumps.

Then for each t>0, the entropy solution remains a Levy process with downward jumps.

Remark 1. This theorem should be viewed as an invariant manifold theorem in the space of probability measures on the line.

Remark 2. Levy processes are very "rigid". For example, white noise initial data leads to a Markov process, that is not a Levy process.

Evolution of Levy processes with downward jumps



Smoluchowski's coagulation equations

Unknown : f(s,t) = expected number of shocks of size s per unit length, at time t

K(r,s) =symmetric rate kernel that describes rate of aggregation of shocks of size r and s.

In this case, we have K(r,s) = r + s.

$$\partial_t f(t,s) = Q_+(f,f) - Q_-(f,f)$$

$$Q_+ = \frac{1}{2}\int_0^s K(r,s-r)f(t,r)f(t,s-r)\,dr$$

$$Q_- = \int_0^\infty K(r,s)f(t,r)f(t,s)\,dr$$

Other exact solutions: "monodisperse" data

Assume all shocks are initially of size 1. Then

$$\psi_0(q) = 1 - e^{-q}$$

The method of characteristics gives the solution in implicit form

$$\psi(q,t) + e^{-q} e^{t\psi(q,t)} = 1.$$

Only need to show that this defines a positive jump measure. In this case, one obtains the Borel-Tanner distribution (Stanley, EC 2).

Example: solutions to Burgers equation with Brownian motion initial data

If u(x,0) is Brownian motion, then $\psi(q,0) = q^2$.

$$\psi(q,t) = \tfrac{1}{t^2}\psi_*(qt), \quad \psi_*(q) = q + \tfrac{1}{2} - \sqrt{q + \tfrac{1}{4}}.$$

The Laplace exponent can be inverted to obtain the jump (shock) statistics

$$f_*(s) = \frac{1}{\sqrt{2\pi s^3}}\, e^{-s}.$$

Example: "fat-tailed"- self-similar solutions.

If u(x,0) is an alpha-stable process $\psi(q,0) = q^\alpha, \quad 1 < \alpha \le 2,$

we obtain self-similar solutions in the implicit form

$$\psi(q,t) = \left(q + t\psi(q,t)\right)^\alpha, \quad q,t > 0.$$

$$f_\alpha(s) = \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k-1} s^{k\beta-2}}{k!}\Gamma(1+k-k\beta)\sin \pi k\beta, \quad \beta = 1 - \frac{1}{\alpha}.$$

Precise asymptotics near zero and infinity can be calculated. Solutions satisfy

$$\int_0^\infty s f_\alpha(s)\, ds = 1, \quad \int_0^\infty f_\alpha(s)\, ds = +\infty.$$

$$f_\alpha(s) \sim \frac{1}{s^{1+\alpha}}, \quad s \to \infty.$$

The "equation free" schema (Y. Kevrekidis).

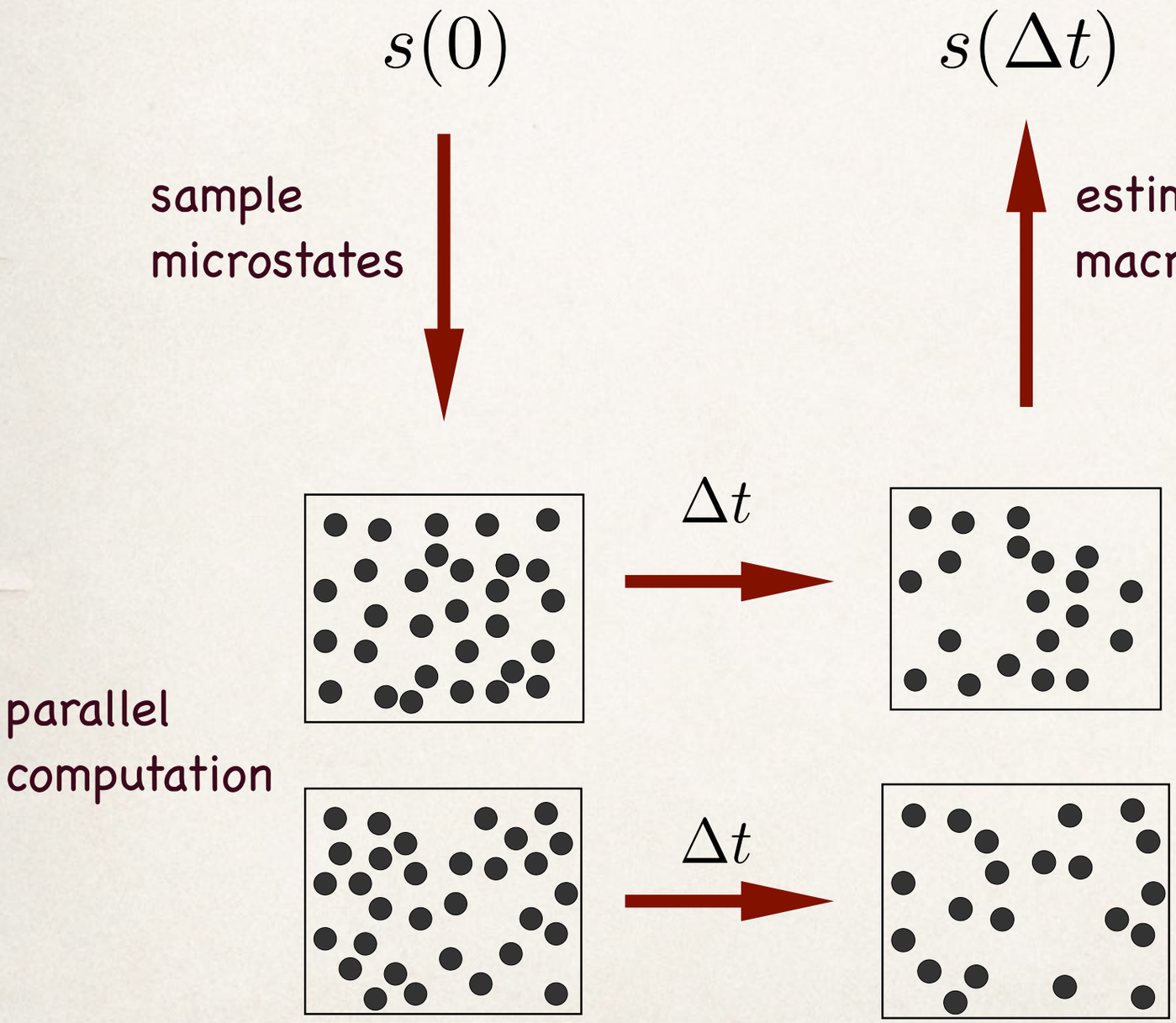
Assumption 1: Fast microscopic evolution. Typical cases of interest are complex microscopic system that model many physical effects.

May be modeled by a legacy code, or a multiphysics code, with which it is expensive or impossible to compute evolution for long time.

Assumption 2: There exists a closed macroscopic evolution for a suitable statistic. However, we do not assume that the equations for this evolution are known! This is why the method is "equation free".

The essence of the method

Run particle code for short bursts in parallel to compute an average flux for the macroscopic statistics.



$$\dot{s}(0) \approx \frac{s(\Delta t) - s(0)}{\Delta t}.$$

$$s(\Delta T) = s(0) + \dot{s}(0)\Delta T.$$

$$O < \Delta t \ll \Delta T$$

Equation free + shock clustering

Microstates: each realization of a random velocity field.

Macroscopic statistics: all contained in the Laplace exponent or jump density. This statistic satisfies a closed equation.

However, unlike most problems, the macroscopic equation is known here.

So we can use the equation-free scheme as a numerical method for shock clustering. Conversely, we can use exact solutions in shock clustering to quantitatively test the equation free method.

History: this test problem was suggested Orszag to Kevrekidis when the method was being developed. But he didn't realize that it was exactly solvable.

Implementation

Step 1. Sampling microstates given jump measure: very easy.

Step 2: Fast computational scheme for sticky particles. Tracking individual shocks, requires O(N^2) steps. Instead, like Brenier-Grenier (1998) we compute a convex hull of N points in O(N log N) steps.

Step 3: Estimation of a Levy measure given many sample paths. This is tricky, because we need to use the empirical measure to estimate the Levy density, and then to resample from the estimated density. Also, have to deal with the fact that all self-similar solutions have infinite number.

Numerical examples: self-similar states, dynamic renormalization.

Test 1. Compute convergence to the self-similar solution with exponential tails beginning with monodisperse data.

In addition, to using the equation free method, we include a rescaling step, so that each iterate satisfies the "pinning conditions"

$$\int_0^\infty s f_k(s)\, ds = 1, \qquad \int_0^\infty s^2 f_k(s)\, ds = 1.$$

Here f_k is the Levy measure at the beginning of the k step.

The reason for adding the second moment condition is that it ensures that the iterates are attracted to the self-similar solution with exponential decay.

[M-Pego (CPAM, 2004; CMP, 2006)]

Numerical examples: self-similar states, dynamic renormalization.

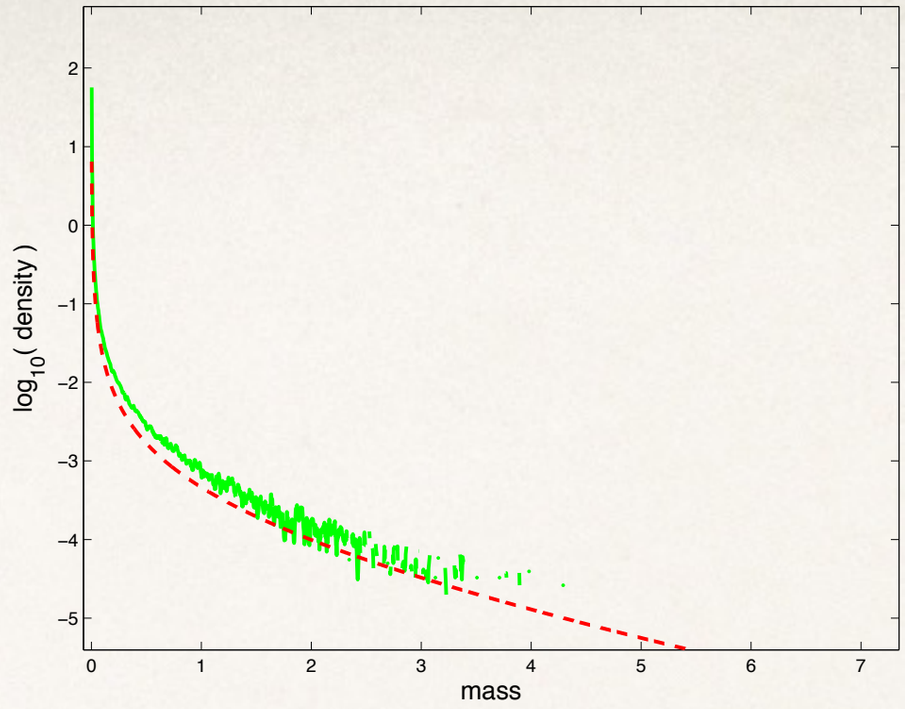
Test 2. Compute convergence to each self-similar solution with fat tails beginning with monodisperse data.

This is not possible without some cheating! We use the "pinning conditions"

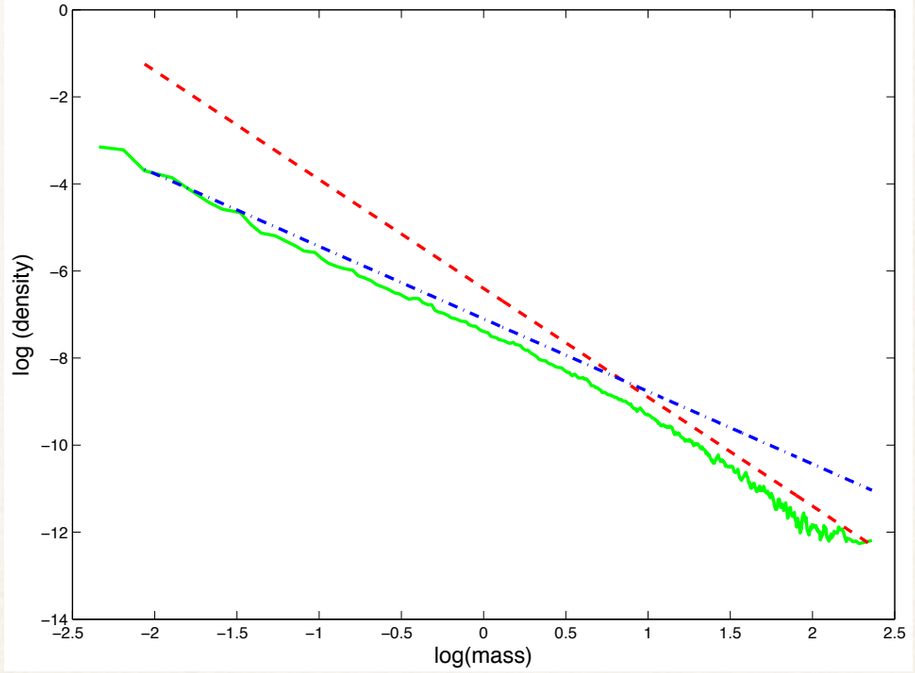
$$\int_0^\infty s f_k(s)\,ds = 1, \qquad \int_0^\infty s^{1+\alpha} f_k(s)\,ds = 1.$$

In fact, all fat-tailed self-similar solutions satisfy

$$\int_0^M s^{1+\alpha} f_\alpha(s)\,ds \sim \log M, \quad M \to \infty.$$



(a) $\rho = 1$



(b) $\rho = 0.5$

Figure 3: Density of exact and computed self-similar solutions for $\rho = 1$ and $\rho = 0.5$. The lines in (b) correspond to rigorous asymptotics of n_ρ as $s \to 0$ and $s \to \infty$.