Stochastic Loewner evolution with branching and the Dyson superprocess

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This talk is based on Vivian's Ph.D thesis (May 2017) as well as more recent joint work.

The help of Steffen Rohde (U. of Washington) is gratefully acknowledged.

Preview

Galton-Watson trees

The Loewner equation

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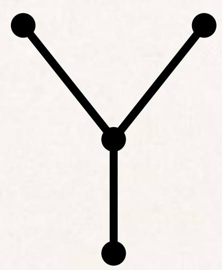

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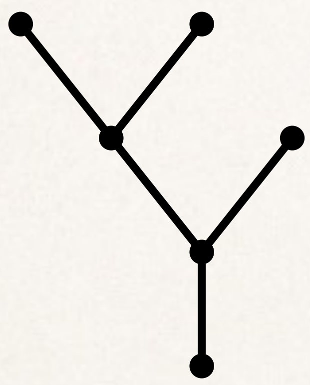

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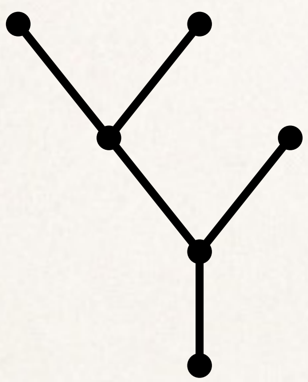

Continuum Random Tree

The CRT is a random metric space that is a universal scaling limit of these trees, conditioned to be large.

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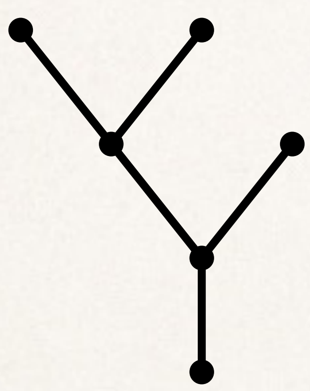

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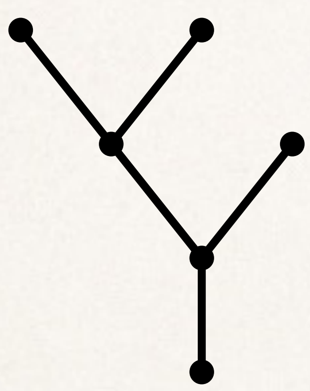

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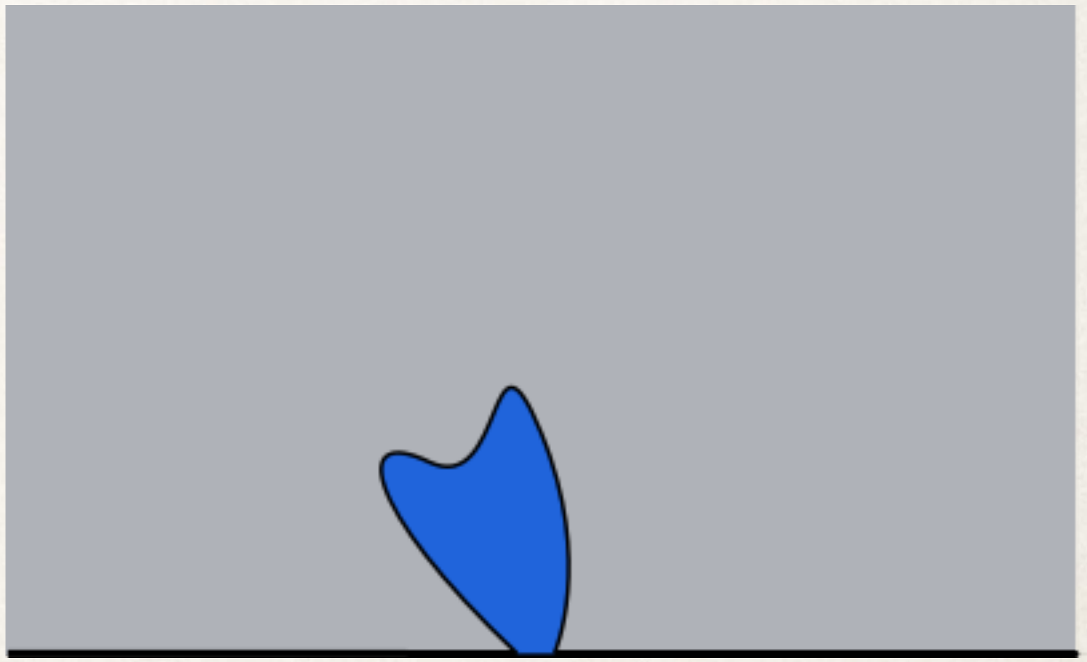

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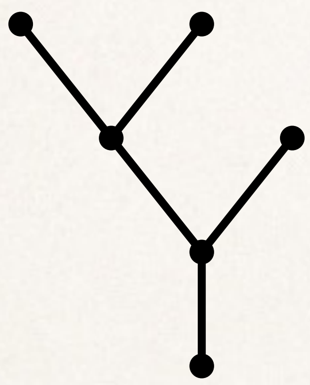

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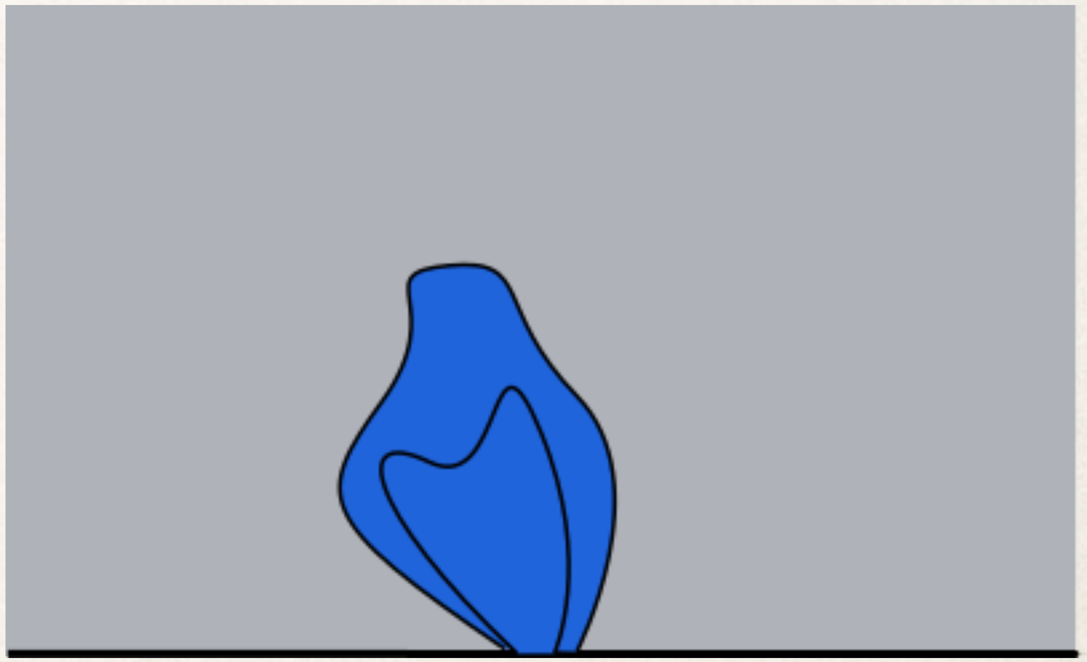

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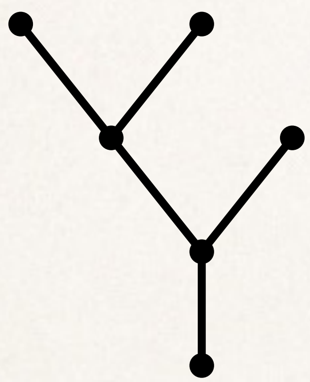

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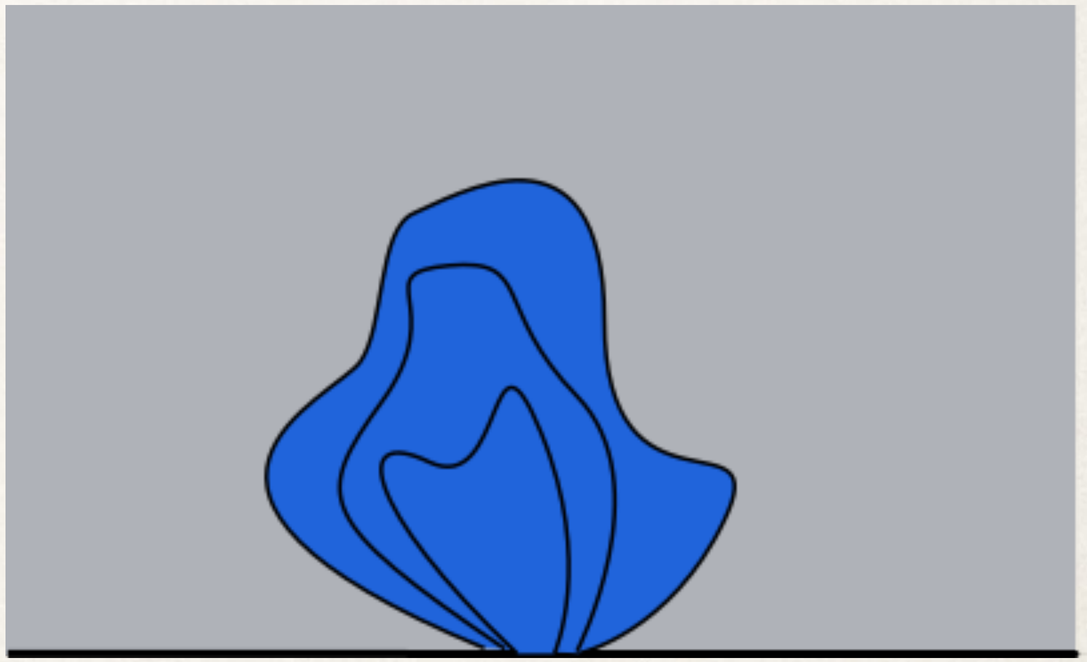


The main questions

Q1. Can we use the Loewner equation to construct natural graph embeddings of Galton-Watson trees in the upper half plane?

Q2. Can we construct a graph embedding of the CRT as a scaling limit of these embeddings of finite Galton-Watson trees?

Q3. What does this construction say about "true trees" (conformally balanced embeddings) and the Brownian map?

At first sight, there is no random matrix theory here. But the above problems are closely related to map enumeration.

Outline

(a) A brief introduction to the CRT.

(b) Loewner evolution with branching (tree embedding).

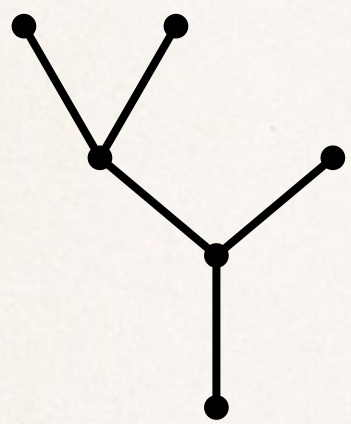
(c) Scaling limits: the SPDE in the case of a Feller diffusion.

(d) Some remark on true trees and the Brownian map.

(a) is (necessary) background. Basic reference: Le Gall (1999).

The continuum random tree

Definition: A plane tree is a rooted combinatorial tree for which the edges are assigned a cyclic order about each vertex.



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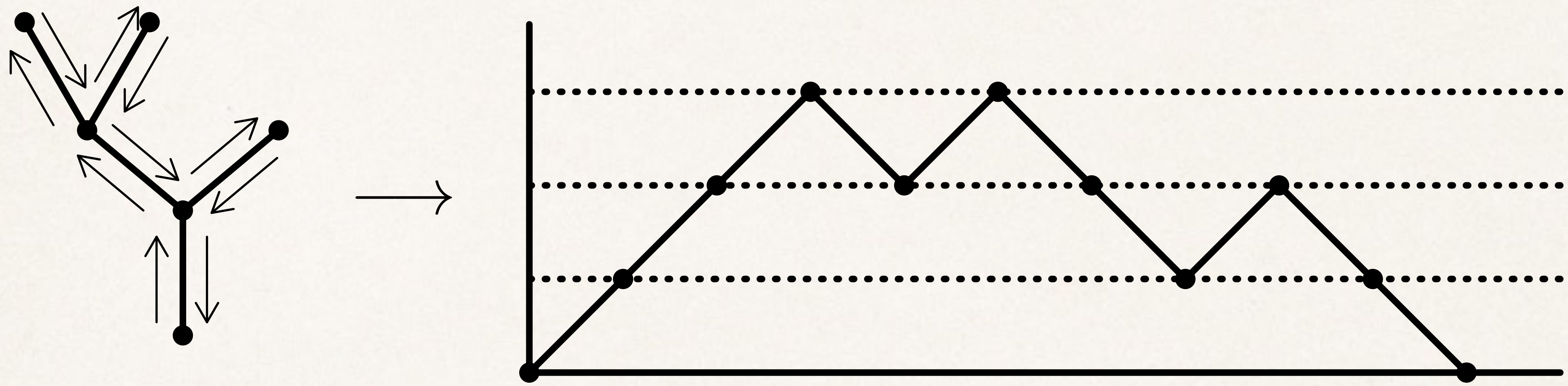

Each plane tree has a corresponding contour function.

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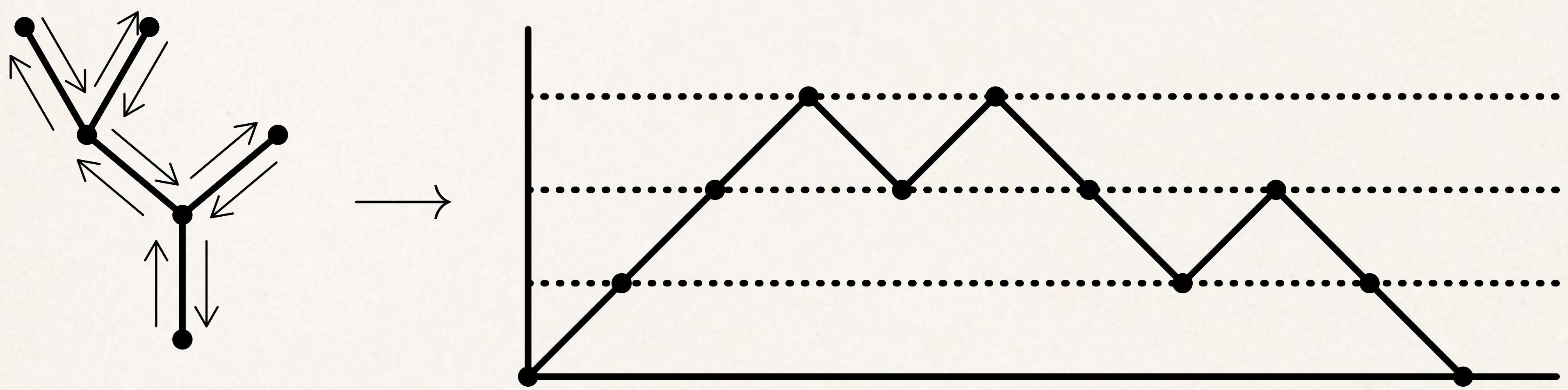

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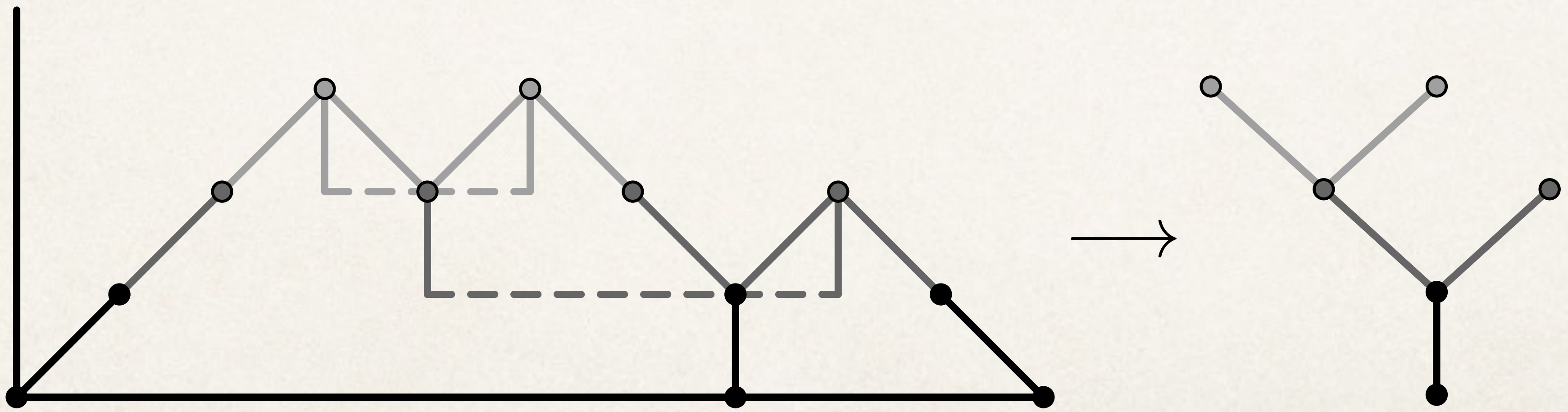

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<u>Def:</u> A *real tree* is a pointed compact metric space with the tree property.

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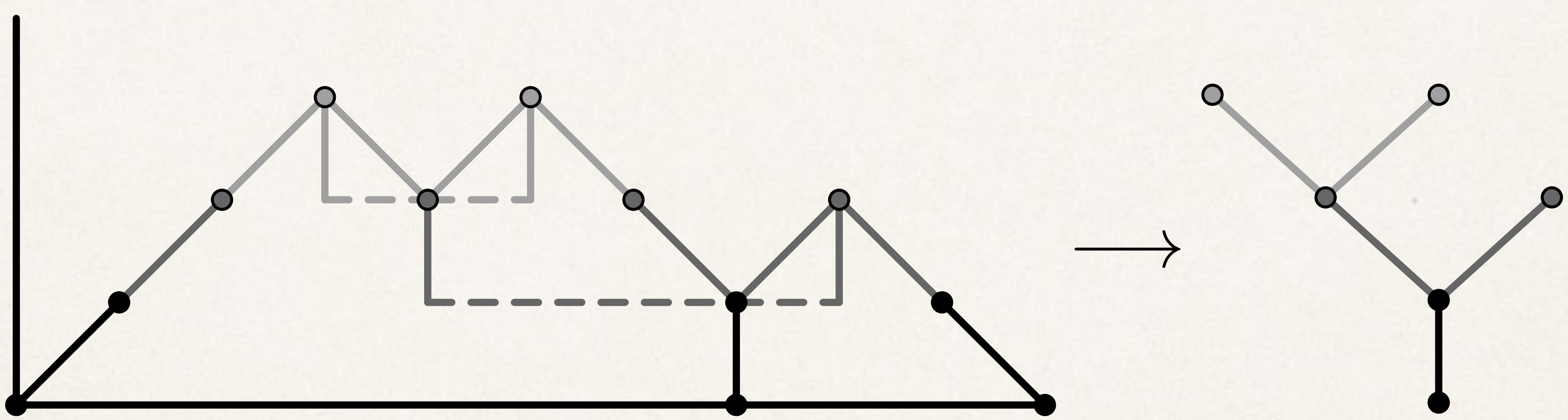
<u>Construction:</u> Let $f : [0,1] \to \mathbb{R}_{\geq 0}$ be continuous, and $f(0) = f(1) = 0$.
Consider

$$d(u,v) = f(u) + f(v) - 2 \min_{s \in [u,v]} f(s)$$

and

$$u \sim_f v \iff d(u,v) = 0.$$



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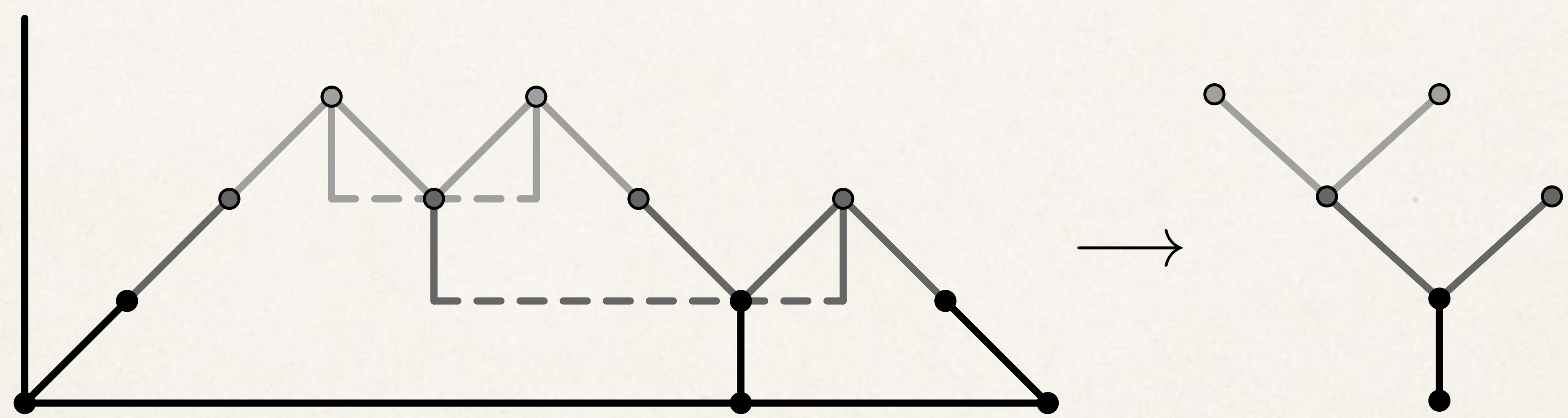

<u>Def:</u> $\mathcal{T}_f = [0,1]/\sim_f$ is the real tree *coded by f*.

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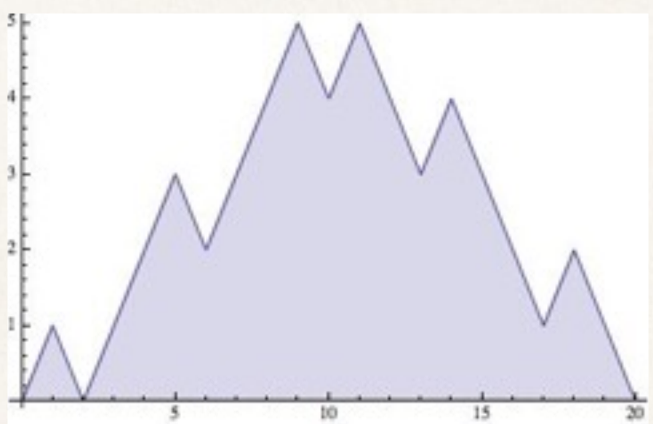
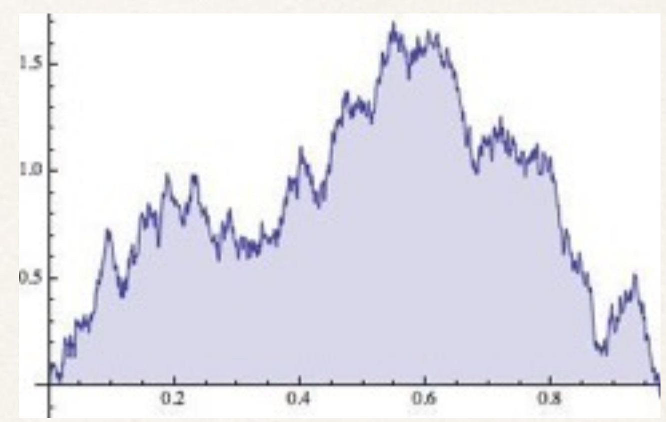
<u>Definition</u>: The continuum random tree (CRT) is the random real tree coded by the normalized Brownian excursion $\mathbb{e}$.

As the uniform distribution on rescaled Dyck paths of length $2n$ converges to $\mathbb{e}$, the uniform distribution on plane trees conditioned to have n edges converges to the CRT.



$\longrightarrow$

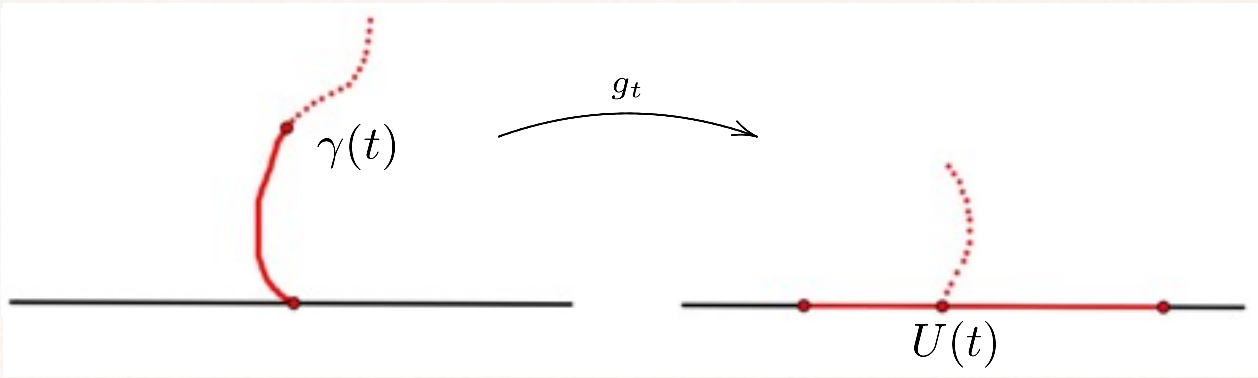


$$\left(\frac{1}{\sqrt{2n}} C_n(2nt)\right)_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(d)} (\mathbb{e}_t)_{0\leq t\leq 1}$$

The CRT was introduced by Aldous (1991-93).

Chordal Loewner evolution

Let $\gamma : (0, T] \to \mathbb{H}$ be a simple curve with $\gamma(0) \in \mathbb{R}$.



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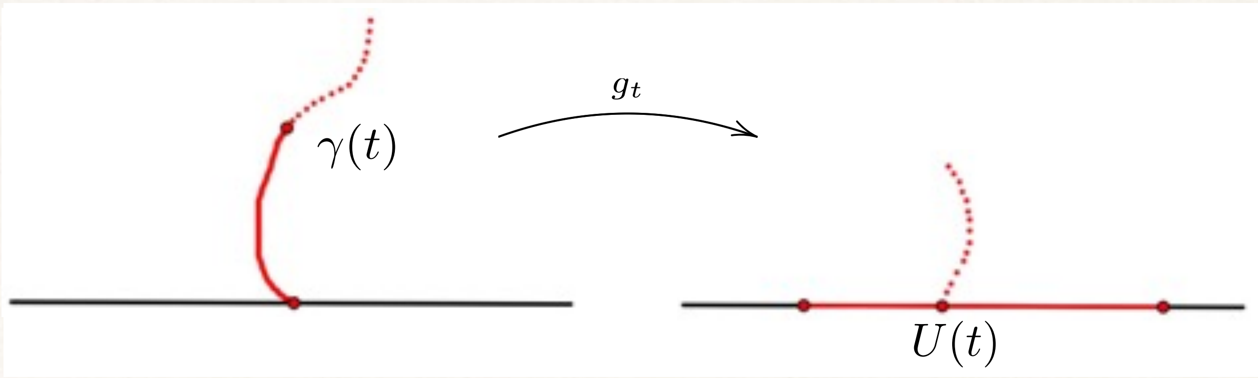

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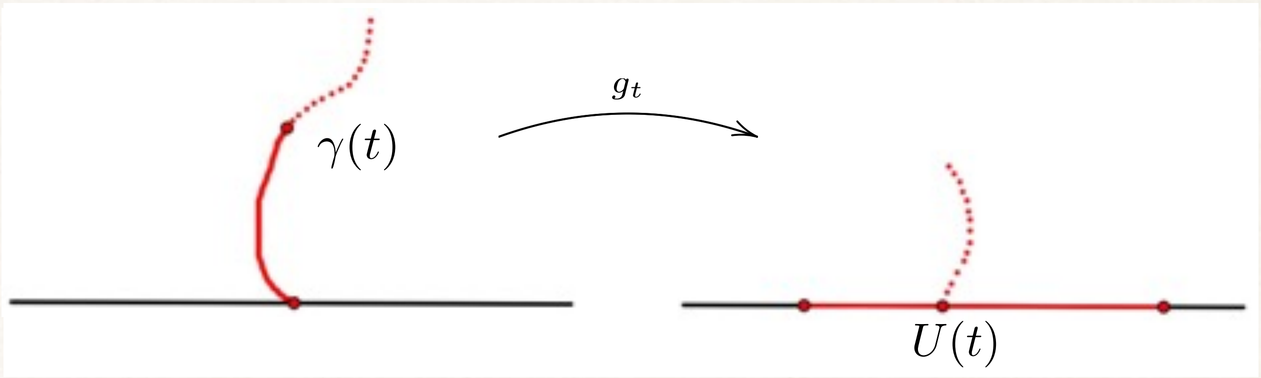

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1) $g_t : \mathbb{H} \setminus \gamma((0, t]) \to \mathbb{H}$

2) $g_t(z) = z + \dfrac{b(t)}{z} + O\left(\dfrac{1}{|z|^2}\right), \quad z \to \infty.$

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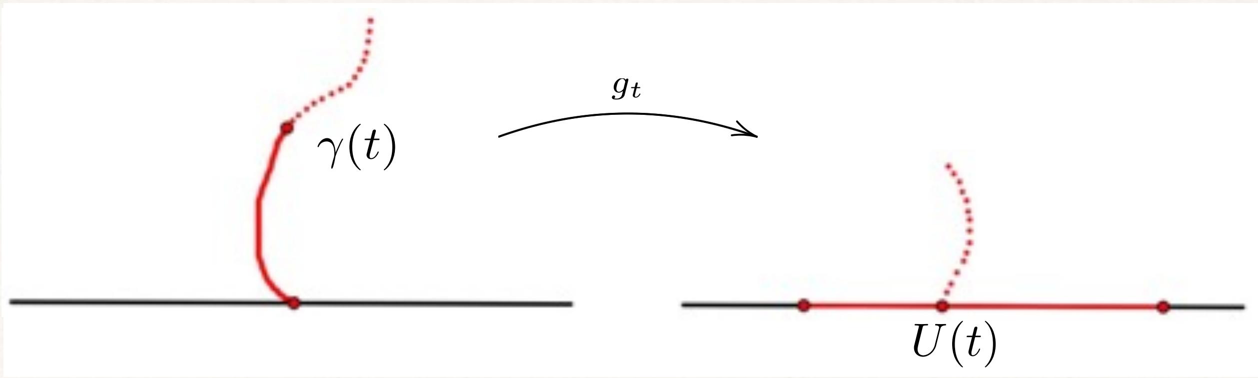

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Loewner (1920s): g_t satisfies the initial value problem

$$\dot{g}_t(z) = \frac{\dot{b}(t)}{g_t(z) - U(t)}, \quad g_0(z) = z.$$

Chordal Loewner evolution

General version: Let $g_t(z)$ denote the solution to the initial value problem

$$\dot{g}_t(z) = \int_{\mathbb{R}} \frac{\mu_t(du)}{g_t(z) - u}, \quad g_0(z) = z.$$

Let $H_t = \{z \in \mathbb{H}\}$ for which $g_t(z) \in \mathbb{H}$ is well defined.

Then g_t is the unique conformal map from H_t onto $\mathbb{H}$ with the hydrodynamic normalization. The hull is the set

$$\mathbb{H} \backslash H_t = K_t.$$

Idea: The measure is supported on points that are escaping $\mathbb{H}$.

Conditions: $\{\mu_t\}_{t \geq 0}$ is a family of nonnegative Borel measures on $\mathbb{R}$ that is right continuous with left limits in the weak topology.
For each t, $\mu_s(\mathbb{R})$ and $\mathrm{supp}\,(\mu_s)$ are each uniformly bounded for $0 \leq s \leq t$.

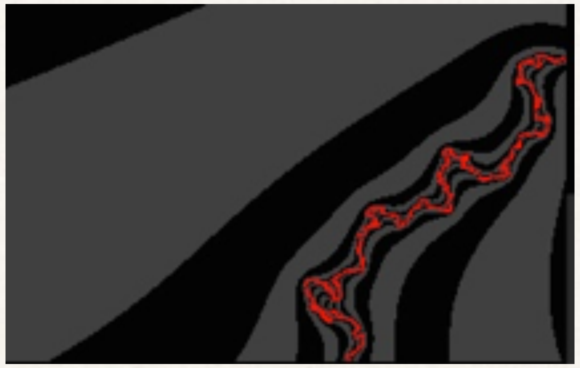
Examples

1) $\mu_t = \delta_{U(t)}$

2) $\mu_t = \sum_{i=1}^{N} \delta_{U_i(t)}$ produces the multislit equation (Schleissinger '13):

$$\dot{g}_t(z) = \sum_{i=1}^{N} \frac{1}{g_t(z) - U_i(t)}.$$

3) $\mu_t = \delta_{\sqrt{\kappa}B_t}$ generates SLE_κ.



Question: Which measures generate embeddings of trees?

What's new

The general form of Loewner evolution is rarely used. However, it is central to our approach. The one line summary of our work is the following conjecture:

Graph embeddings of continuum trees are generated by Loewner evolution when the driving measure is a suitable *superprocess*.

The simplest example in this class is what we call the Dyson superprocess. It is the free probability analogue of the Dawson-Watanabe superprocess.

Loewner evolution driven by Dyson BM with branching

.
(Simulation courtesy of Vivian Healey and Brent Werness.)

SPDE for the Dyson superprocess

$$\mu_t(dx) = \rho(x,t)\,dx, \quad x \in (-\infty,\infty), t > 0,$$

$$\partial_t \rho + \partial_x \left(\rho \mathcal{H} \rho\right) = \sigma \sqrt{\rho}\, \dot{W},$$

where $\dot{W}$ is space-time white noise and $\mathcal{H}$ is the Hilbert transform

$$\left(\mathcal{H}\mu_t\right)(x) = \frac{\text{p.v.}}{\pi} \int_{-\infty}^{\infty} \frac{1}{x-s} \mu_t(ds).$$

The SPDE is formal, but convenient. The measure valued process is actually defined through a martingale problem.

Comparison with the Dawson-Watanabe superprocess

The Dawson-Watanabe superprocess is the scaling limit of branching Brownian motion when the discrete branching processes converges to the Feller diffusion. The spatial motion of each particle is independent.

$$\partial_t \rho = \frac{1}{2}\triangle \rho + \sigma \sqrt{\rho}\,\dot{W}, \quad x \in \mathbb{R}^d, t > 0.$$

The Dyson superprocess is the free probability version of this SPDE:

$$\partial_t \rho + \partial_x \left(\rho \mathcal{H} \rho\right) = \sigma \sqrt{\rho}\,\dot{W}, \quad x \in \mathbb{R}, t > 0.$$

Unlike Dawson-Watanabe, this is a superprocess of interacting particles.

The SPDE and stochastic Loewner evolution

Consider the Cauchy-Stieltjes transform

$$f(z,t) = \int_{-\infty}^{\infty} \frac{1}{z-s}\mu_t(ds), \qquad z \in \mathbb{H}.$$

Define the Gaussian analytic function $h(z,t), z \in \mathbb{H}$ with covariance kernel

$$\mathbb{E}\left(h(z,t)\bar{h}(w,t')\right) = \delta(t-t') \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{z-s} \frac{1}{\bar{w}-s} \mu_t(ds).$$

The Dyson superprocess and Loewner evolution

Let $\dot{B}$ denote white-noise (in time alone). Then (formally)

$$\partial_t f + f \partial_z f = \sigma h \dot{B}, \quad z \in \mathbb{H}, t > 0.$$

This SPDE may be solved by the method of characteristics

$$\frac{dz}{dt} = f(z,t), \quad df = \sigma h(z,t)\, dB, \quad z \in \mathbb{H}.$$

The stochastic Loewner evolution is given by the subordination formula

$$\dot{g}_t(z) = f(g_t(z), t), \quad g_0(z) = z, \quad z \in \mathbb{H}.$$

Absolute continuity w.r.t. Lebesgue measure

$$\mu_t(dx) \stackrel{?}{=} \rho(x,t)\,dx$$

Dawson-Watanabe superprocess:

$d = 1$ μ_t is absolutely continuous (Konno-Shiga, Reimers, 1988).

$d \geq 2$ μ_t is singular (Perkins, 1988).

Dyson superprocess: we don't know yet. The basic regularity estimates for free convolution with a semicircular law were obtained by Biane (1997). Unlike Dawson-Watanabe we would like μ_t to be singular with respect to Lebesgue measure.

If a density exists, then the hull cannot be a tree, so this is a crucial property.

Tree Embedding

Question: In the deterministic setting, what conditions guarantee that the hull is a tree?

Fundamental step: What conditions on the driving measure guarantee that the generated hull is a union of two simple curves in $\mathbb{H}$ that meet at a single point on $\mathbb{R}$ at nontrivial angles (not 0 or π)?

Tree Embedding: (α, β)-approach

<u>Setup:</u> Let $U_1, \ldots, U_n$ be n continuous functions $U_i : [0,T] \to \mathbb{R}$ that are mutually nonintersecting $U_i(t) < U_{i+1}(t)$ for all $i = 1, \ldots, n$ and all $t \in [0,T]$, except for $U_j(0) = U_{j+1}(0)$. Let μ_t be the discrete measure

$$\mu_t = c \sum_{i=1}^{n} \delta_{U_i(t)}$$

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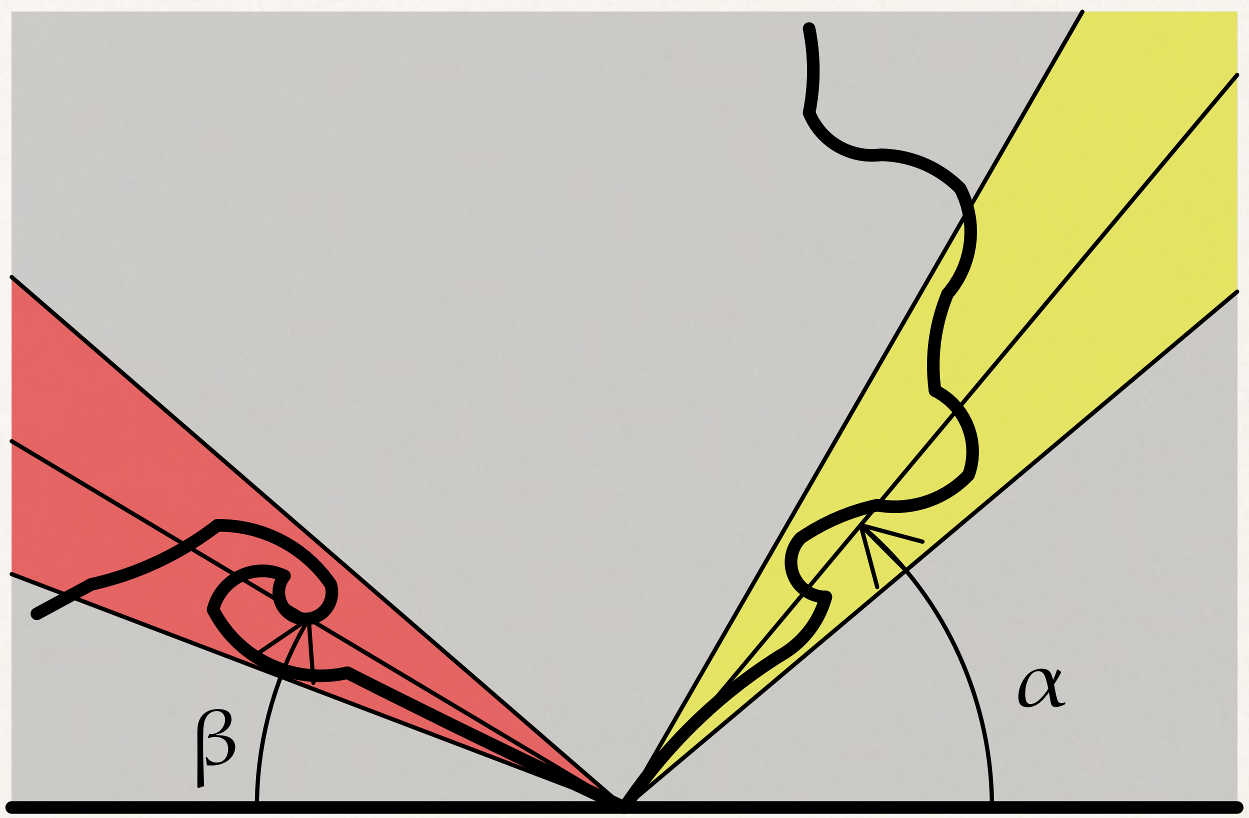
Definition: Let $\alpha, \beta \in (0, \pi)$ such that $\alpha + \beta < \pi$. We say that K_t approaches $\mathbb{R}$ at $U_j(0)$ in **($\boldsymbol{\alpha}$, $\boldsymbol{\beta}$)-direction** if for each $\varepsilon > 0$ there is $s = s_\varepsilon > 0$ such that there are exactly two connected components of K_s that have $U_j(0)$ as a boundary point, and

$$K^{j}_{s_\varepsilon} \subset \{z \in H : \pi - \beta - \varepsilon < \arg(z - U_j(0)) < \pi - \beta + \varepsilon\},$$
$$K^{j+1}_{s_\varepsilon} \subset \{z \in H : \alpha - \varepsilon < \arg(z - U_j(0)) < \alpha + \varepsilon\}.$$

(Motivated by Schleissinger '12.)

Tree Embedding: (α, β)-approach



Tree Embedding: (α, β)-approach

Theorem (Healey): In the setting above, the hulls K_t approach $\mathbb{R}$ in (α, β)-direction at $U_j(0)$ if

$$\lim_{t \searrow 0} \frac{U_j(t) - U_j(0)}{\sqrt{t}} = \phi_1(\alpha, \beta) - \phi_2(\alpha, \beta)$$

$$\lim_{t \searrow 0} \frac{U_{j+1}(t) - U_{j+1}(0)}{\sqrt{t}} = \phi_1(\alpha, \beta) + \phi_2(\alpha, \beta),$$

where $\phi_1(\alpha, \beta)$ and $\phi_2(\alpha, \beta)$ are explicitly computable functions.

Tree Embedding: (α, β)-approach

<u>Balanced case</u>: If $0 < \alpha = \beta < \pi/2$, then ϕ_1 and ϕ_2 simplify to

$$\phi_1(\alpha, \alpha) = 0 \quad \text{and} \quad \phi_2(\alpha, \alpha) = \sqrt{2c}\sqrt{\frac{\pi - 2\alpha}{\alpha}}.$$

<u>Intuitively:</u> Loewner scaling

- If μ_t generates hulls K_t, then $\rho\mu_{t/\rho^2}$ generates the hulls ρK_t.

- So we expect to see $\sqrt{t}$ whenever a hull is preserved under dilation.

A driving measure for any tree

Let $\mathcal{T} = \{v\ , h(v)\}$ be a marked plane tree. (Think of $h(v)$ as the time of death of v.) Let μ_t be indexed by the elements of $\mathcal{T}$ alive at t:

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On time intervals without branching, chose the U_v to evolve according to

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The driving measure

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<u>Theorem (Healey)</u>: If $\mathcal{T}$ is a binary tree such that $h_v \neq h_\eta$, then for each $0 \leq s \leq \max\{h(v)\}$, the hull K_s generated at time s by the Loewner equation driven by μ_t is a graph embedding of the subtree

$$\mathcal{T}_s = \{v \in \mathcal{T} : h(p(v)) < s\}$$

in $\mathbb{H}$, with the image of the root on the real line, and $K_s \subset K_{s'}$ if $s < s'$.

Tree Embedding

Proof (idea):

The proof relies on ODE results about the particle system

$$\dot{U}_v(t) = \sum_{v \neq \eta \in \Delta_t \mathcal{T}} \frac{c_1}{U_v(t) - U_\eta(t)}.$$

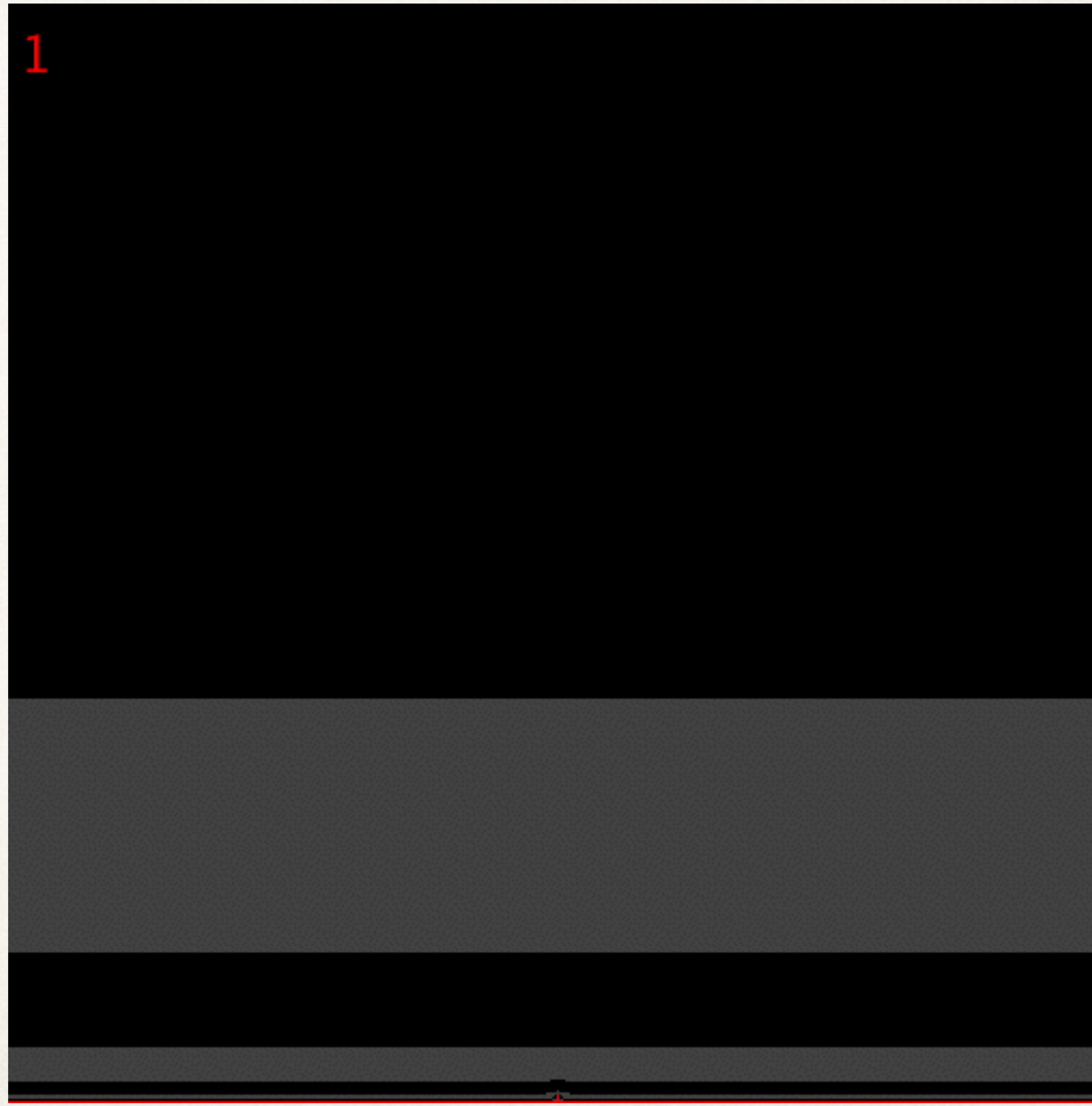
- Extend the solution backward to the initial condition $U_j(0) = U_{j+1}(0)$.
- Show that the solution generates curves away from $t = 0$.
- Show that the generated hull approaches $\mathbb{R}$ in (α, α)-direction for

$$\alpha = \frac{\pi}{2 + \frac{c_1}{2c}}.$$

$\square$

Example: Galton-Watson Trees with deterministic repulsion

If θ is distributed as a binary Galton-Watson tree with exponential lifetimes, then the theorem guarantees that the Loewner equation driven by μ_t generates a graph embedding of θ with probability one.



A sample of a binary Galton-Watson tree with exponential lifetimes.
(Simulation courtesy of Brent Werness.)

Embedding the CRT?

Question 2: Let $\{\theta_k\}$ be a sequence of random trees that (when appropriately rescaled) converges in distribution to the CRT when θ_k is conditioned on having k edges. Does the law of the generated hulls converge to a scaling limit?

Embedding the CRT?

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First step: Find the scaling limit of the corresponding sequence of random driving measures.

Choosing a Sequence of Measures

Let $\{\mathcal{T}_k\}$ be a sequence of random trees, and let $\{c^k\}$ and $\{c_1^k\}$ be two sequences with elements in $\mathbb{R}^+$. For each k, define

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- Same setting as tree embedding theorem.
- How do we choose the trees $\{T_k\}$?

Galton-Watson trees to the CRT

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Question 2a: Can we find a scaling limit of $\{\mu_t^k\}$ defined by

$$\mu_t^k = c^k \sum_{\nu \in \Delta_t \theta_k} \delta_{U_\nu(t)},$$

where the $U_\nu(t)$ evolve according to

$$\dot{U}_\nu(t) = \sum_{\nu \neq \eta \in \Delta_t \theta_k} \frac{c_1^k}{U_\nu(t) - U_\eta(t)},$$

if for each k, θ_k is distributed as a critical binary Galton-Watson tree with exponential lifetimes of mean $\frac{1}{2\sqrt{k}}$, conditioned to have k edges?

The Scaling Limit

For each k, the driving measure μ_t^k is really a measure-valued *process* defined for $t \in [0, \infty)$.
By convergence of driving measures we mean convergence in the Skorokhod space $D_{\mathcal{M}_f(\mathbb{R})}[0, \infty)$ of functions from $[0, \infty)$ to $\mathcal{M}_f(\mathbb{R})$ with càdlàg paths (right continuous with left limits).
Called *superprocesses*.

How to prove convergence of superprocesses?

- Tightness
  Prokhorov: tight $\leftrightarrow$ relatively compact subset of $D_{\mathcal{M}_f(\mathbb{R})}[0, \infty)$.
  In particular, there is at least one limit point.

- Uniqueness of the limit point. (Conv. of finite dimensional marginals.)

Scaling limits: tightness

Theorem (Healey, M.): (a) For each k, let θ_k be distributed as a critical binary Galton-Watson tree with exponential lifetimes of mean $\frac{1}{2\sqrt{k}}$, conditioned to have k edges, and let $\{\mu^k\}$ be the corresponding sequence of measures. If the scaling constants are

$$c^k = c_1^k = \frac{1}{\sqrt{k}}$$

then the sequence $\{\mu^k\}$ is tight in $D_{\mathcal{M}_f(\hat{\mathbb{R}})}[0, \infty)$.

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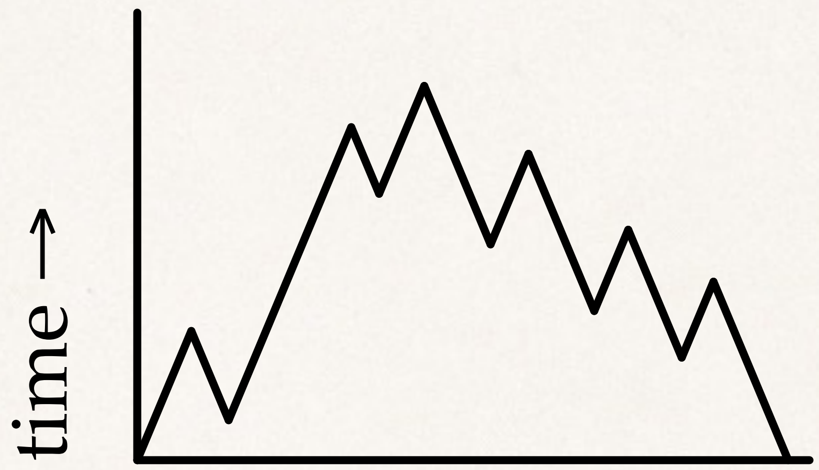
<u>Why these constants?</u>

- Choose $c^k = c_1^k$, since the ratio c_1^k / c^k determines the branching angle.
- $c^k = 1/\sqrt{k}$ is the rescaling for which the total population process of θ_k converges to $L_{\mathbb{e}}^t$, the local time at level t of the normalized Brownian excursion.

The CRT scaling limit

Contour function for critical binary GW trees



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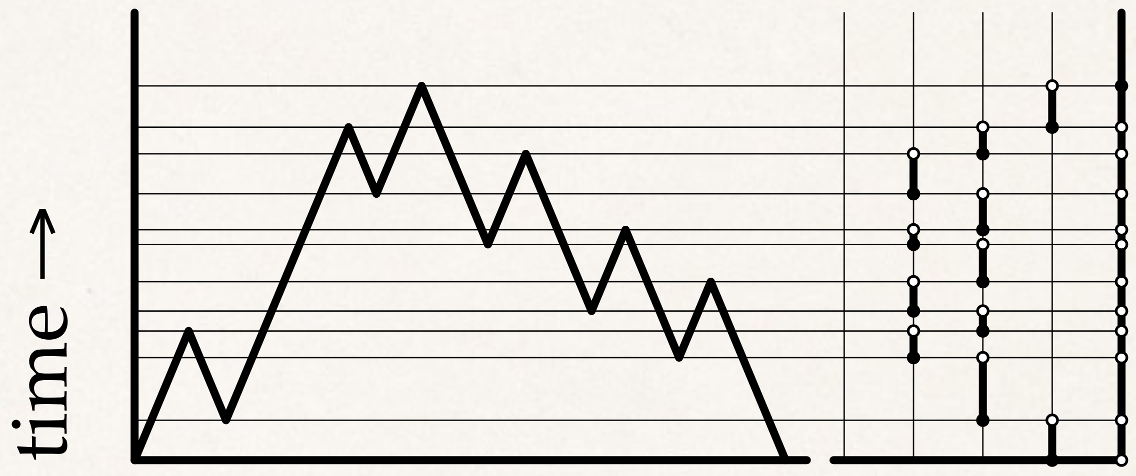

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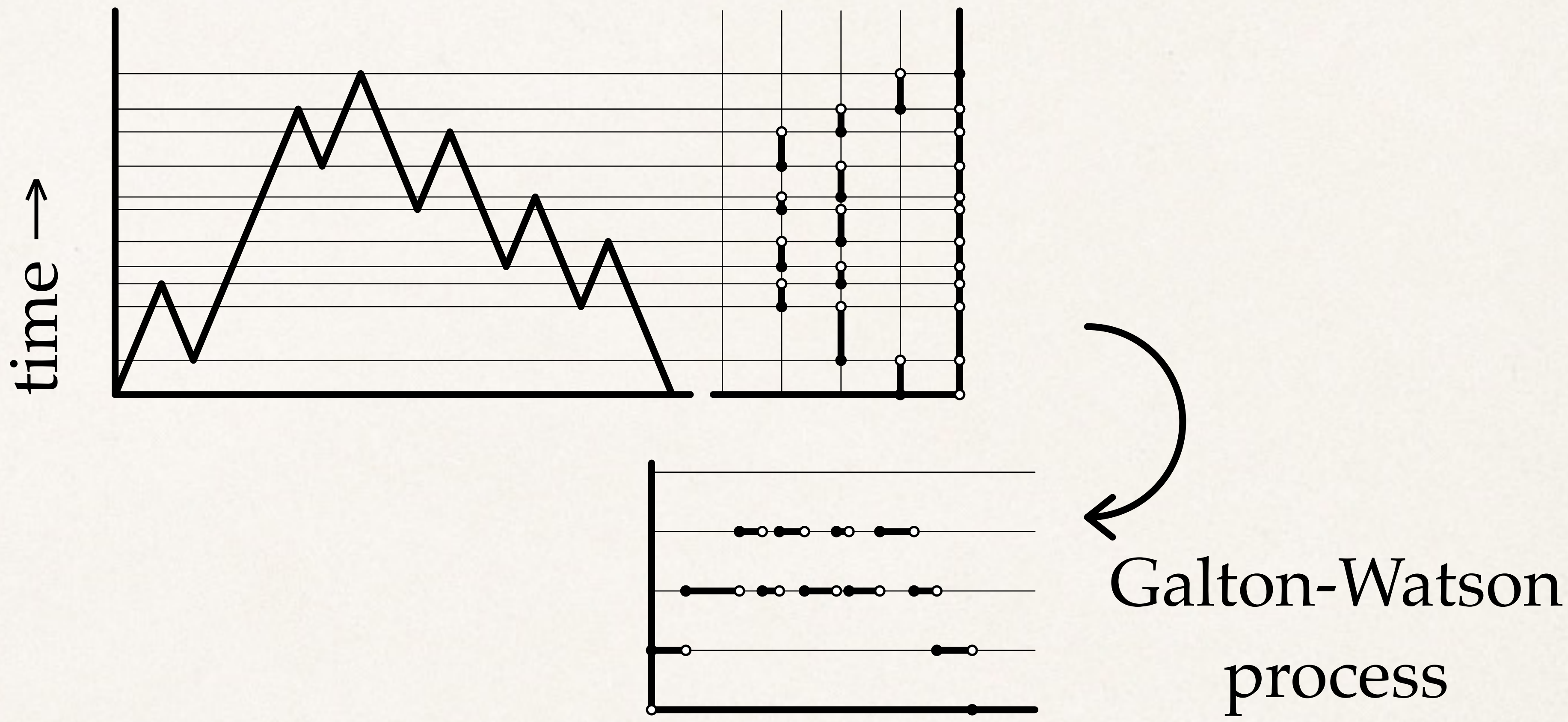

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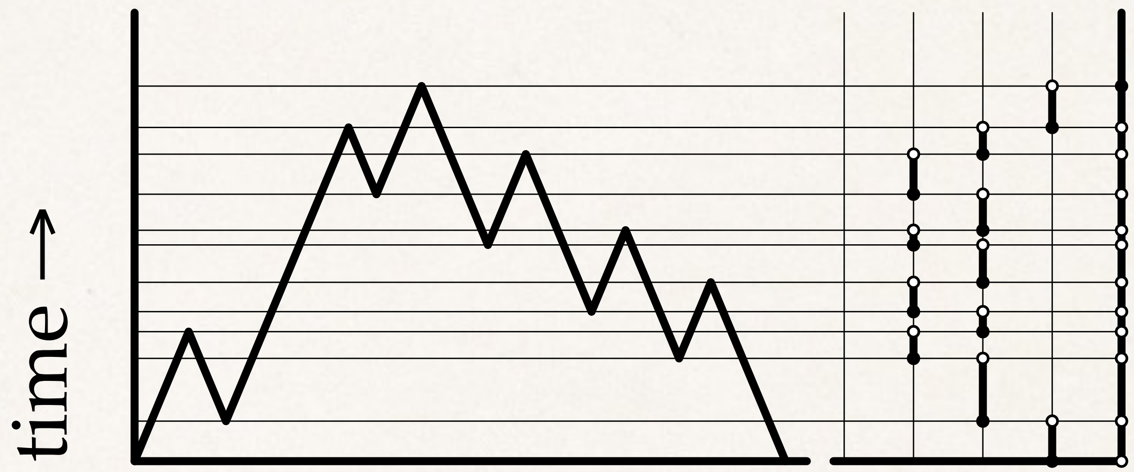

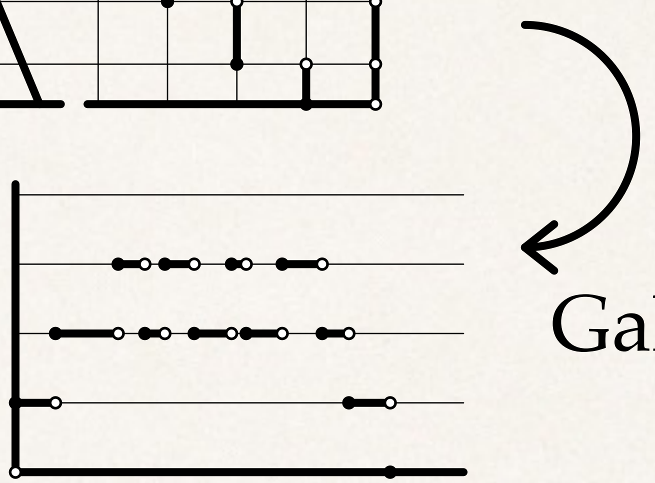

Galton-Watson process

Normalized Brownian excursion $(\mathbb{e})_{0 \leq t \leq 1}$

Local time at level t of normalized Brownian excursion: $L^t_{\mathbb{e}}$

The Scaling limit

Theorem (Pitman): If for each k, N^k is the total population process of θ_k, then

$$\frac{N_t^k}{\sqrt{k}} \to L_{\mathbb{e}}^t,$$

as $k \to \infty$, in the sense of convergence in distribution of random variables in $D_{\mathcal{M}_f(\mathbb{R})}[0, \infty)$.

Conditioned v. unconditioned limits

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as $k \to \infty$, in the sense of convergence in distribution of random variables in $D_{\mathcal{M}_f(\mathbb{R})}[0, \infty)$.

<u>Standard unconditioned result:</u> For each k, let $\widetilde{N}_t^k$ be a discrete critical Galton-Watson process (all lifetimes of length one) whose offspring distribution has finite variance. Then

$$\frac{\widetilde{N}_{kt}^k}{k} \to X_t,$$

as $k \to \infty$, in the sense of convergence in distribution of random variables in $D_{\mathcal{M}_f(\mathbb{R})}[0, \infty)$, where X_t is the Feller diffusion.
(Need $\widetilde{N}_0^k/k \to x_0 > 0$, since the Feller diffusion is absorbing at 0.)

The scaling limit: characterization

Theorem (Healey, M.): In the unconditioned case, each subsequential limit solves the martingale problem for the Dyson superprocess:

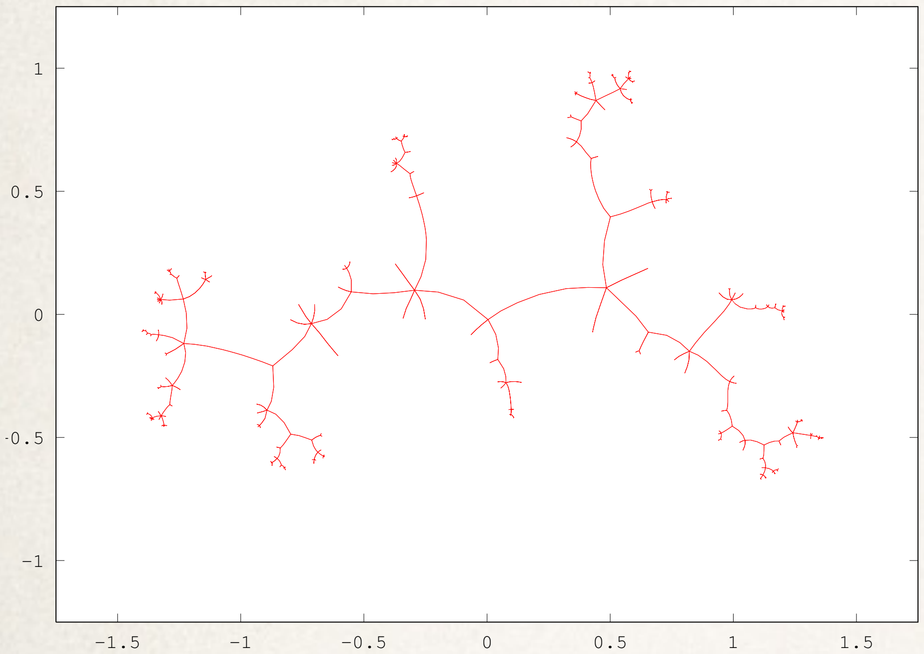
$$\langle \mu_t, \varphi \rangle = \langle \mu_0, \varphi \rangle + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\varphi(x) - \varphi(y)}{x - y} \mu_s(dx) \mu_s(dy) + M_t(\varphi),$$

where the local martingale M has quadratic variation

$$[M(\varphi)]_t = \int_0^t \langle \mu_s, \varphi^2 \rangle \, ds.$$

Remark: (a) Don't know yet if the solution to the martingale problem is unique.
(b) Similar result for the conditioned case (Pitman, Serlet).

Conformally balanced trees in $\mathbb{C}$



Joel Barnes (Ph.D, 2014, U. Washington).

A planar tree is <u>conformally balanced</u> if

(a) each edge has equal harmonic measure from infinity
(b) edge subsets have the same measure from either side.

Balanced trees are in 1-1 correspondence with Shabat polynomials (Bishop, Biane). However, these polynomials are poorly understood.

Conformally balanced trees in $\mathbb{C}$

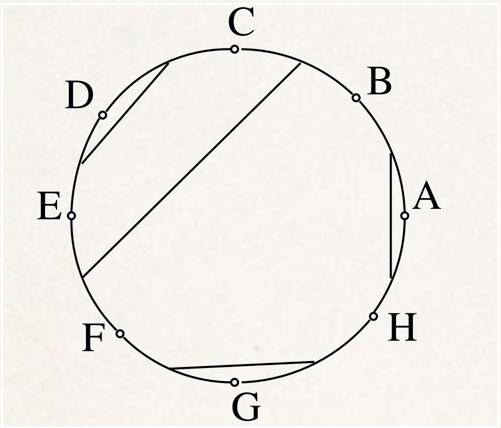


Fig. 1. A noncrossing partition

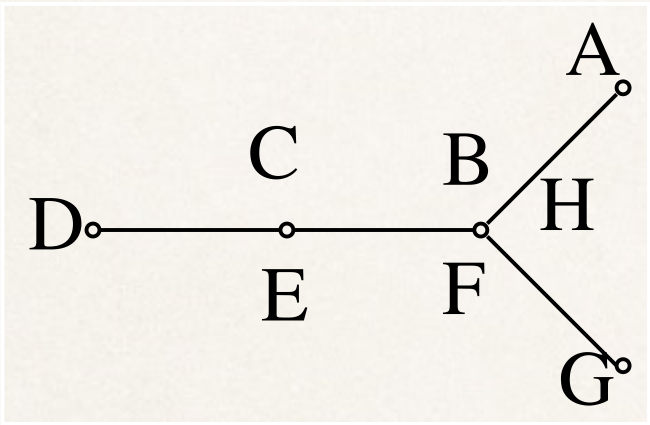


Fig. 2. The planar tree

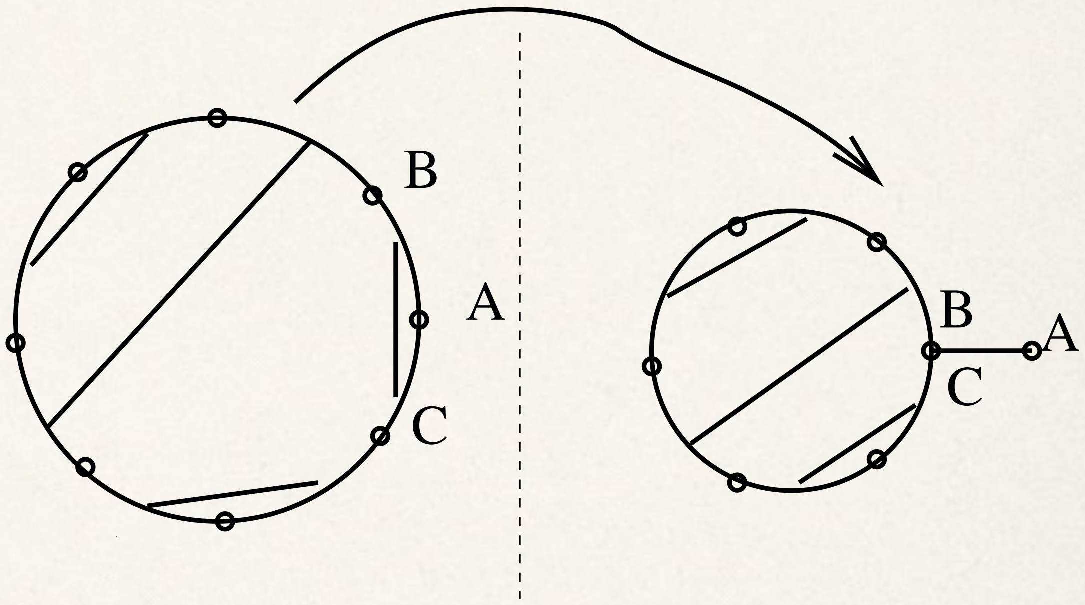


Fig. 3. The domain

Biane (2009): builds a conformal mapping of the exterior domain by welding edges in pairs. End result is a conformal mapping, continuous onto the boundary, that gives exactly the non-crossing partition.

The CVS bijection and the Brownian map

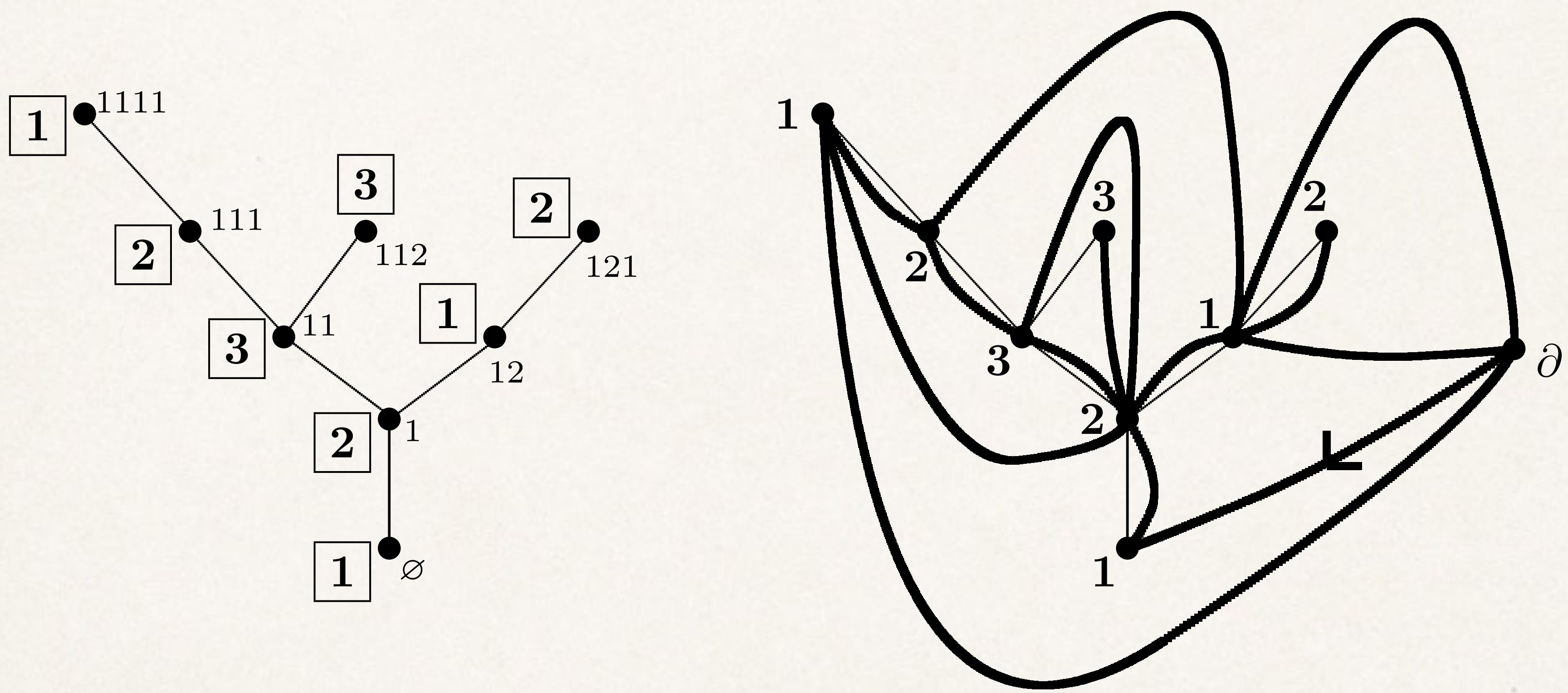
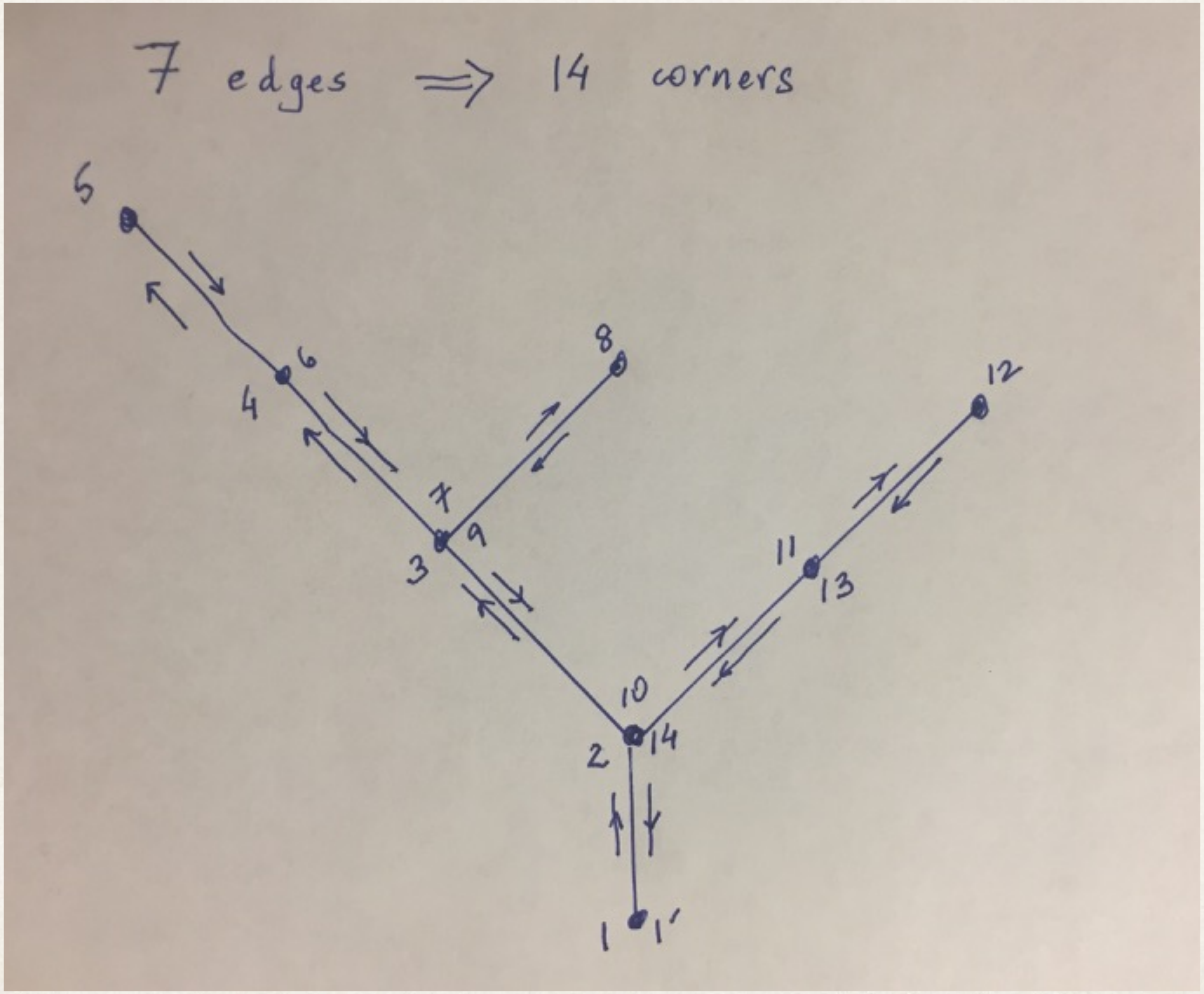


Figure 2. The Cori-Vauquelin-Schaeffer bijection. On the left side, a well-labeled tree (the framed numbers are the labels assigned to the vertices). On the right side, the edges of the associated quadrangulation Q appear in thick curves.

Image from Le Gall (ICM, 2014).

Conformal CVS



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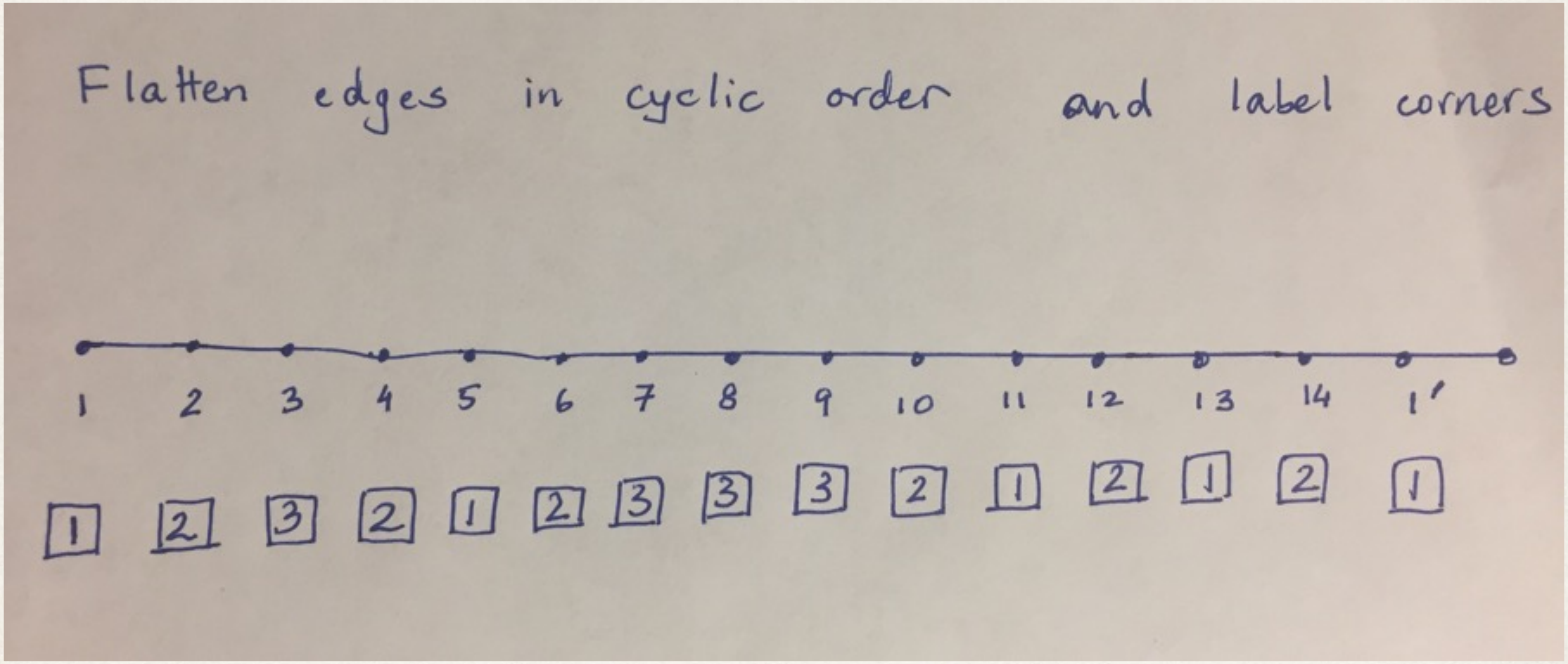

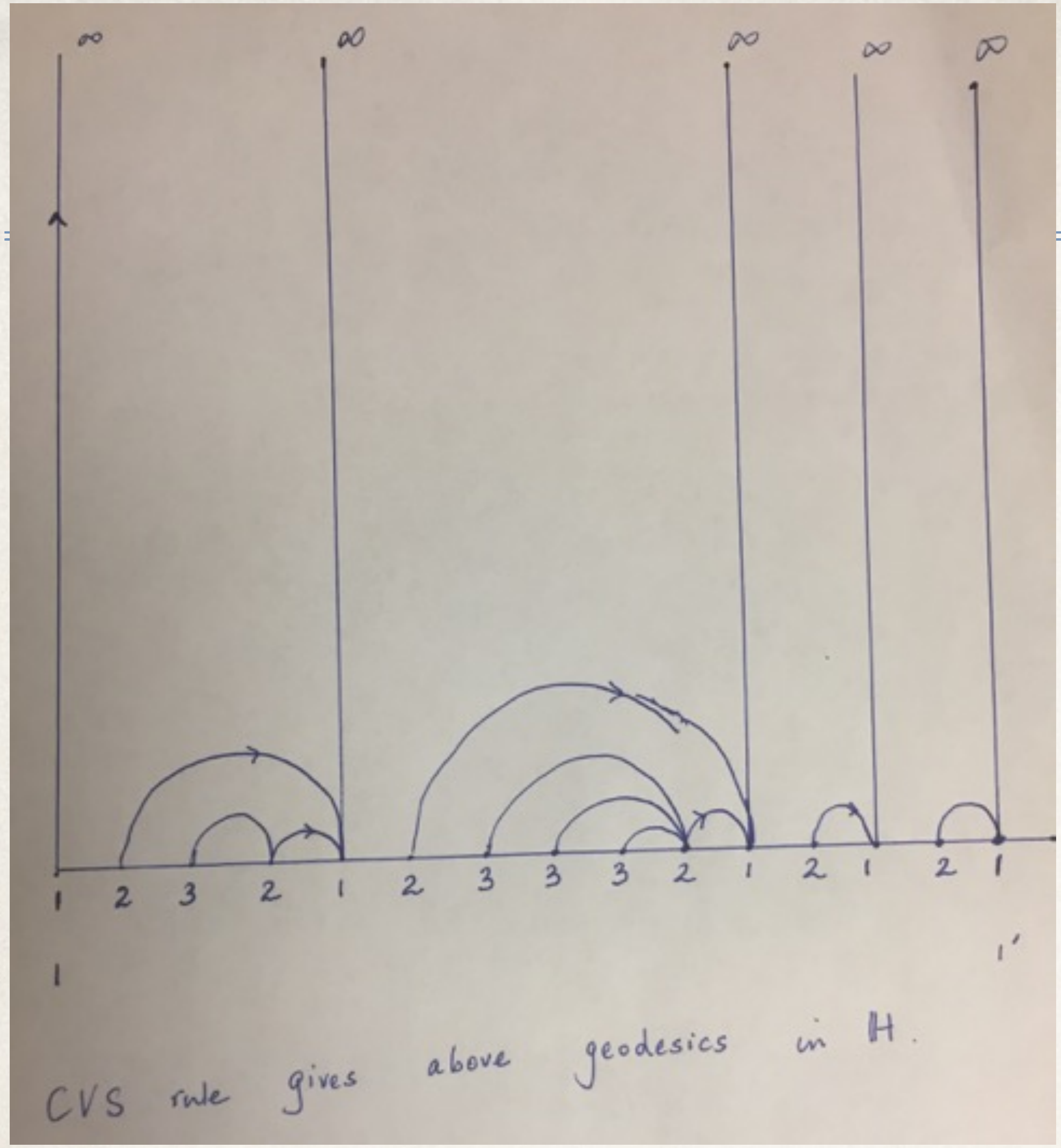

Given the conformal map, each labeling gives a nested family of geodesics in the upper half plane with endpoints on its preimage.

The image of these geodesics under the conformal map is a quadrangulation of the upper half plane.