Stochastic Loewner evolution with branching and the Dyson superprocess

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This talk is based on Vivian's Ph.D thesis (May 2017) as well as more recent joint work.

The help of Steffen Rohde (U. of Washington) is gratefully acknowledged.
Galton-Watson trees  The Loewner equation
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The main questions

**Q1.** Can we use the Loewner equation to construct natural graph embeddings of Galton-Watson trees in the upper half plane?

**Q2.** Can we construct a graph embedding of the CRT as a scaling limit of these embeddings of finite Galton-Watson trees?

**Q3.** What does this construction say about "true trees" (conformally balanced embeddings) and the Brownian map?

At first sight, there is no random matrix theory here. But the above problems are closely related to map enumeration.
(a) A brief introduction to the CRT.

(b) Loewner evolution with branching (tree embedding).

(c) Scaling limits: the SPDE in the case of a Feller diffusion.

(d) Some remark on true trees and the Brownian map.

(a) is (necessary) background. Basic reference: Le Gall (1999).
The continuum random tree

Definition: A plane tree is a rooted combinatorial tree for which the edges are assigned a cyclic order about each vertex.
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Def: A *real tree* is a pointed compact metric space with the tree property.
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Def: A **real tree** is a pointed compact metric space with the tree property.

**Construction:** Let \( f : [0, 1] \to \mathbb{R}_{\geq 0} \) be continuous, and \( f(0) = f(1) = 0 \).

Consider
\[
d(u, v) = f(u) + f(v) - 2 \min_{s \in [u,v]} f(s)
\]
and
\[
u \sim_f v \iff d(u, v) = 0.
\]
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$$d(u, v) = f(u) + f(v) - 2 \min_{s \in [u,v]} f(s)$$

and

$$u \sim_f v \iff d(u, v) = 0.$$ 

**Def:** $\mathcal{T}_f = [0, 1]/ \sim_f$ is the real tree coded by $f$. 
The continuum random tree

**Definition**: The continuum random tree (CRT) is the random real tree coded by the normalized Brownian excursion $\xi$. 
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**Definition:** The continuum random tree (CRT) is the random real tree coded by the normalized Brownian excursion $\Phi$. 

As the uniform distribution on rescaled Dyck paths of length $2n$ converges to $\Phi$, the uniform distribution on plane trees conditioned to have $n$ edges converges to the CRT.

\[
\left( \frac{1}{\sqrt{2n}} C_n(2nt) \right)_{0 \leq t \leq 1} \xrightarrow{n \to \infty} (\Phi_t)_{0 \leq t \leq 1}
\]

The CRT was introduced by Aldous (1991-93).
Let $\gamma : (0, T] \to \mathbb{H}$ be a simple curve with $\gamma(0) \in \mathbb{R}$. 

Chordal Loewner evolution
Let \( \gamma : (0, T] \to \mathbb{H} \) be a simple curve with \( \gamma(0) \in \mathbb{R} \).

The Riemann mapping theorem implies that for each \( t \) there is a unique conformal mapping \( g_t \) such that

1) \( g_t : \mathbb{H} \setminus \gamma((0, t]) \to \mathbb{H} \)
Let $\gamma : (0, T] \rightarrow \mathbb{H}$ be a simple curve with $\gamma(0) \in \mathbb{R}$.

The Riemann mapping theorem implies that for each $t$ there is a unique conformal mapping $g_t$ such that

1) $g_t : \mathbb{H} \setminus \gamma(\mathbb{R}) \rightarrow \mathbb{H}$

2) $g_t(z) = z + \frac{b(t)}{z} + O\left(\frac{1}{|z|^2}\right)$, \quad z \rightarrow \infty.$
Chordal Loewner evolution

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2) $g_t(z) = z + \frac{b(t)}{z} + O \left( \frac{1}{|z|^2} \right), \quad z \to \infty.$

Loewner (1920s): $g_t$ satisfies the initial value problem

$$\dot{g}_t(z) = \frac{\dot{b}(t)}{g_t(z) - U(t)}, \quad g_0(z) = z.$$
Chordal Loewner evolution

General version: Let $g_t(z)$ denote the solution to the initial value problem

$$\dot{g}_t(z) = \int_{\mathbb{R}} \frac{\mu_t(du)}{g_t(z) - u}, \quad g_0(z) = z.$$ 

Let $H_t = \{z \in \mathbb{H}\}$ for which $g_t(z) \in \mathbb{H}$ is well defined.

Then $g_t$ is the unique conformal map from $H_t$ onto $\mathbb{H}$ with the hydrodynamic normalization. The hull is the set

$$\mathbb{H} \setminus H_t = K_t.$$ 

Idea: The measure is supported on points that are escaping $\mathbb{H}$.

Conditions: $\{\mu_t\}_{t \geq 0}$ is a family of nonnegative Borel measures on $\mathbb{R}$ that is right continuous with left limits in the weak topology.
For each $t$, $\mu_s(\mathbb{R})$ and $\text{supp} (\mu_s)$ are each uniformly bounded for $0 \leq s \leq t$. 
Examples

1) \( \mu_t = \delta_{U(t)} \)

2) \( \mu_t = \sum_{i=1}^{N} \delta_{U_i(t)} \) produces the multislit equation (Schleissinger '13):
\[
\dot{g}_t(z) = \sum_{i=1}^{N} \frac{1}{g_t(z) - U_i(t)}.
\]

3) \( \mu_t = \delta_{\sqrt{\kappa}B_t} \) generates SLE_\kappa.

Question: Which measures generate embeddings of trees?
The general form of Loewner evolution is rarely used. However, it is central to our approach. The one line summary of our work is the following conjecture:

Graph embeddings of continuum trees are generated by Loewner evolution when the driving measure is a suitable superprocess.

The simplest example in this class is what we call the Dyson superprocess. It is the free probability analogue of the Dawson-Watanabe superprocess.
Loewner evolution driven by Dyson BM with branching

(Simulation courtesy of Vivian Healey and Brent Werness.)
SPDE for the Dyson superprocess

\[ \mu_t(dx) = \rho(x, t) \, dx, \quad x \in (-\infty, \infty), \, t > 0, \]

\[ \partial_t \rho + \partial_x (\rho \mathcal{H} \rho) = \sigma \sqrt{\rho} \dot{W}, \]

where \( \dot{W} \) is space-time white noise and \( \mathcal{H} \) is the Hilbert transform

\[ (\mathcal{H} \mu_t)(x) = \frac{p.v.}{\pi} \int_{-\infty}^{\infty} \frac{1}{x - s} \mu_t(ds). \]

The SPDE is formal, but convenient. The measure valued process is actually defined through a martingale problem.
Comparison with the Dawson-Watanabe superprocess

The Dawson-Watanabe superprocess is the scaling limit of branching Brownian motion when the discrete branching processes converges to the Feller diffusion. The spatial motion of each particle is independent.

\[
\partial_t \rho = \frac{1}{2} \Delta \rho + \sigma \sqrt{\rho} \dot{W}, \quad x \in \mathbb{R}^d, t > 0.
\]

The Dyson superprocess is the free probability version of this SPDE:

\[
\partial_t \rho + \partial_x (\rho \mathcal{H} \rho) = \sigma \sqrt{\rho} \dot{W}, \quad x \in \mathbb{R}, t > 0.
\]

Unlike Dawson-Watanabe, this is a superprocess of interacting particles.
Consider the Cauchy-Stieltjes transform

$$f(z, t) = \int_{-\infty}^{\infty} \frac{1}{z - s} \mu_t(ds), \quad z \in \mathbb{H}.$$ 

Define the Gaussian analytic function $h(z, t), z \in \mathbb{H}$ with covariance kernel

$$\mathbb{E} \left( h(z, t) \bar{h}(w, t') \right) = \delta(t - t') \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{z - s} \frac{1}{\bar{w} - \bar{s}} \mu_t(ds).$$
The Dyson superprocess and Loewner evolution

Let $\dot{B}$ denote white-noise (in time alone). Then (formally)

$$\partial_t f + f \partial_z f = \sigma h \dot{B}, \quad z \in \mathbb{H}, t > 0.$$ 

This SPDE may be solved by the method of characteristics

$$\frac{dz}{dt} = f(z, t), \quad df = \sigma h(z, t) dB, \quad z \in \mathbb{H}.$$ 

The stochastic Loewner evolution is given by the subordination formula

$$\dot{g}_t(z) = f(g_t(z), t), \quad g_0(z) = z, \quad z \in \mathbb{H}.$$
Absolute continuity w.r.t. Lebesgue measure

\[ \mu_t(dx) \overset{?}{=} \rho(x,t) \, dx \]

Dawson-Watanabe superprocess:

- \( d = 1 \) \( \mu_t \) is absolutely continuous (Konno-Shiga, Reimers, 1988).
- \( d \geq 2 \) \( \mu_t \) is singular (Perkins, 1988).

Dyson superprocess: we don't know yet. The basic regularity estimates for free convolution with a semicircular law were obtained by Biane (1997). Unlike Dawson-Watanabe we would like \( \mu_t \) to be singular with respect to Lebesgue measure.

If a density exists, then the hull cannot be a tree, so this is a crucial property.
Question: In the deterministic setting, what conditions guarantee that the hull is a tree?

Fundamental step: What conditions on the driving measure guarantee that the generated hull is a union of two simple curves in $\mathbb{H}$ that meet at a single point on $\mathbb{R}$ at nontrivial angles (not 0 or $\pi$)?
Tree Embedding: \((\alpha, \beta)\)-approach

**Setup:** Let \(U_1, \ldots, U_n\) be \(n\) continuous functions \(U_i : [0,T] \rightarrow \mathbb{R}\) that are mutually nonintersecting \(U_i(t) < U_{i+1}(t)\) for all \(i = 1, \ldots, n\) and all \(t \in [0,T]\), except for \(U_j(0) = U_{j+1}(0)\). Let \(\mu_t\) be the discrete measure

\[
\mu_t = c \sum_{i=1}^{n} \delta_{U_i(t)}
\]
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**Definition:** Let \(\alpha, \beta \in (0, \pi)\) such that \(\alpha + \beta < \pi\).
Tree Embedding: (α, β)-approach

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$$
\mu_t = c \sum_{i=1}^{n} \delta_{U_i(t)}
$$

Definition: Let $\alpha, \beta \in (0, \pi)$ such that $\alpha + \beta < \pi$. We say that $K_t$ approaches $\mathbb{R}$ at $U_j(0)$ in (α, β)-direction if for each $\varepsilon > 0$ there is $s = s_\varepsilon > 0$ such that there are exactly two connected components of $K_s$ that have $U_j(0)$ as a boundary point, and

$$
K^j_{s_\varepsilon} \subset \{ z \in H : \pi - \beta - \varepsilon < \arg(z - U_j(0)) < \pi - \beta + \varepsilon \},
$$

$$
K^{j+1}_{s_\varepsilon} \subset \{ z \in H : \alpha - \varepsilon < \arg(z - U_j(0)) < \alpha + \varepsilon \}.
$$

(Motivated by Schleissinger ’12.)
Tree Embedding: \((\alpha, \beta)\)-approach
Tree Embedding: $(\alpha, \beta)$-approach

**Theorem (Healey):** In the setting above, the hulls $K_t$ approach $\mathbb{R}$ in $(\alpha, \beta)$-direction at $U_j(0)$ if

$$
\lim_{t \searrow 0} \frac{U_j(t) - U_j(0)}{\sqrt{t}} = \phi_1(\alpha, \beta) - \phi_2(\alpha, \beta)
$$

$$
\lim_{t \searrow 0} \frac{U_{j+1}(t) - U_{j+1}(0)}{\sqrt{t}} = \phi_1(\alpha, \beta) + \phi_2(\alpha, \beta),
$$

where $\phi_1(\alpha, \beta)$ and $\phi_2(\alpha, \beta)$ are explicitly computable functions.
Tree Embedding: ($\alpha$, $\beta$)-approach

**Balanced case:** If $0 < \alpha = \beta < \pi/2$, then $\phi_1$ and $\phi_2$ simplify to

$$
\phi_1(\alpha, \alpha) = 0 \quad \text{and} \quad \phi_2(\alpha, \alpha) = \sqrt{2c} \sqrt{\frac{\pi - 2\alpha}{\alpha}}.
$$

**Intuitively:** Loewner scaling

- If $\mu_t$ generates hulls $K_t$, then $\rho \mu_{t/\rho^2}$ generates the hulls $\rho K_t$.

- So we expect to see $\sqrt{t}$ whenever a hull is preserved under dilation.
A driving measure for any tree

Let $T = \{ \nu, h(\nu) \}$ be a marked plane tree. (Think of $h(\nu)$ as the time of death of $\nu$.) Let $\mu_t$ be indexed by the elements of $T$ alive at $t$:

$$\mu_t = c \sum_{\nu \in \Delta_t T} \delta_{U_{\nu}(t)}.$$
Let $T = \{\nu, h(\nu)\}$ be a marked plane tree. (Think of $h(\nu)$ as the time of death of $\nu$.) Let $\mu_t$ be indexed by the elements of $T$ alive at $t$:

$$
\mu_t = c \sum_{\nu \in \Delta_t T} \delta_{U_\nu(t)}.
$$

On time intervals without branching, chose the $U_\nu$ to evolve according to

$$
\dot{U}_\nu(t) = \sum_{\nu \neq \eta \in \Delta_t T} \frac{c_1}{U_\nu(t) - U_\eta(t)}.
$$
The driving measure

Let $T = \{v, h(v)\}$ be a marked plane tree. (Think of $h(v)$ as the time of death of $v$.) Let $\mu_t$ be indexed by the elements of $T$ alive at $t$:

$$\mu_t = c \sum_{v \in \Delta_t T} \delta_{U_v(t)}.$$ 

On time intervals without branching, chose the $U_v$ to evolve according to

$$\dot{U}_v(t) = \sum_{v \neq \eta \in \Delta_t T} \frac{c_1}{U_v(t) - U_\eta(t)}.$$ 

**Theorem (Healey):** If $T$ is a binary tree such that $h(v) \neq h(\eta)$, then for each $0 \leq s \leq \max\{h(v)\}$, the hull $K_s$ generated at time $s$ by the Loewner equation driven by $\mu_t$ is a graph embedding of the subtree

$$T_s = \{v \in T : h(p(v)) < s\}$$

in $\mathbb{H}$, with the image of the root on the real line, and $K_s \subset K_{s'}$ if $s < s'$. 
Tree Embedding

Proof (idea):

The proof relies on ODE results about the particle system

\[ \dot{U}_v(t) = \sum_{\nu \not= \eta \in \Delta_t \mathcal{T}} \frac{c_1}{U_v(t) - U_\eta(t)}. \]

• Extend the solution backward to the initial condition \( U_j(0) = U_{j+1}(0) \).

• Show that the solution generates curves away from \( t = 0 \).

• Show that the generated hull approaches \( \mathbb{R} \) in \((\alpha, \alpha)\)-direction for

\[ \alpha = \frac{\pi}{2 + \frac{c_1}{2c}}. \]
If $\theta$ is distributed as a binary Galton-Watson tree with exponential lifetimes, then the theorem guarantees that the Loewner equation driven by $\mu_t$ generates a graph embedding of $\theta$ with probability one.

A sample of a binary Galton-Watson tree with exponential lifetimes.
(Simulation courtesy of Brent Werness.)
**Embedding the CRT?**

**Question 2:** Let \( \{\theta_k\} \) be a sequence of random trees that (when appropriately rescaled) converges in distribution to the CRT when \( \theta_k \) is conditioned on having \( k \) edges. Does the law of the generated hulls converge to a scaling limit?
Question 2: Let $\{\theta_k\}$ be a sequence of random trees that (when appropriately rescaled) converges in distribution to the CRT when $\theta_k$ is conditioned on having $k$ edges. Does the law of the generated hulls converge to a scaling limit?

First step: Find the scaling limit of the corresponding sequence of random driving measures.
Choosing a Sequence of Measures

Let \{T_k\} be a sequence of random trees, and let \{c^k\} and \{c_1^k\} be two sequences with elements in \(\mathbb{R}^+\). For each \(k\), define

\[
\mu_t^k = c^k \sum_{\nu \in \Delta_t T_k} \delta_{U_\nu(t)},
\]
Choosing a Sequence of Measures

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\[
\mu^k_t = c^k \sum_{\nu \in \Delta_t T_k} \delta_{U_\nu(t)},
\]

where the \( U_\nu(t) \) evolve according to

\[
\dot{U}_\nu(t) = \sum_{\nu \neq \eta \in \Delta_t T_k} \frac{c_1^k}{U_\nu(t) - U_\eta(t)}.
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Choosing a Sequence of Measures

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\dot{U}_v(t) = \sum_{v \neq \eta \in \Delta_t T_k} \frac{c_1^k}{U_v(t) - U_\eta(t)}.
\]

- Same setting as tree embedding theorem.
- How do we choose the trees \{T_k\}?
**Galton-Watson trees to the CRT**

*Theorem* (Aldous): If $\theta_k$ is distributed as a critical binary Galton-Watson tree with exponential lifetimes of mean $\frac{1}{2\sqrt{k}}$, conditioned to have $k$ edges, then $\theta_k$ converges in distribution to the CRT as $k \to \infty$. 
**Galton-Watson trees to the CRT**

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**Question 2a:** Can we find a scaling limit of $\{\mu^k_t\}$ defined by

$$
\mu^k_t = c^k \sum_{\nu \in \Delta_t \theta_k} \delta_{U_\nu(t)},
$$

where the $U_\nu(t)$ evolve according to

$$
\dot{U}_\nu(t) = \sum_{\nu \neq \eta \in \Delta_t \theta_k} \frac{c^k}{U_\nu(t) - U_\eta(t)},
$$

if for each $k$, $\theta_k$ is distributed as a critical binary Galton-Watson tree with exponential lifetimes of mean $\frac{1}{2\sqrt{k}}$, conditioned to have $k$ edges?
The Scaling Limit

For each $k$, the driving measure $\mu_t^k$ is really a measure-valued process defined for $t \in [0, \infty)$. By convergence of driving measures we mean convergence in the Skorokhod space $D_{\mathcal{M}_f(\mathbb{R})}[0, \infty)$ of functions from $[0, \infty)$ to $\mathcal{M}_f(\mathbb{R})$ with càdlàg paths (right continuous with left limits). Called superprocesses.

How to prove convergence of superprocesses?

• Tightness
  Prokhorov: tight $\iff$ relatively compact subset of $D_{\mathcal{M}_f(\mathbb{R})}[0, \infty)$. In particular, there is at least one limit point.

• Uniqueness of the limit point. (Conv. of finite dimensional marginals.)
Scaling limits: tightness

**Theorem** (Healey, M.): (a) For each \(k\), let \(\theta_k\) be distributed as a critical binary Galton-Watson tree with exponential lifetimes of mean \(\frac{1}{2\sqrt{k}}\), conditioned to have \(k\) edges, and let \(\{\mu^k\}\) be the corresponding sequence of measures. If the scaling constants are

\[
c^k = c^k_1 = \frac{1}{\sqrt{k}}
\]

then the sequence \(\{\mu^k\}\) is tight in \(D_{\mathcal{M}_f(\mathbb{R})}[0, \infty)\).
Scaling limits: tightness

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Theorem (Healey, M.): For each \( k \), let \( \theta_k \) be distributed as a critical binary Galton-Watson tree with exponential lifetimes of mean \( \frac{1}{2\sqrt{k}} \), conditioned to have \( k \) edges, and let \( \{\mu^k\} \) be the corresponding sequence of measures. If the scaling constants are

\[
c^k = c_1^k = \frac{1}{\sqrt{k}}
\]

then the sequence \( \{\mu^k\} \) is tight in \( D_{\mathcal{M}_f(\mathbb{R})}[0, \infty) \).

Why these constants?

• Choose \( c^k = c_1^k \), since the ratio \( c_1^k / c^k \) determines the branching angle.
• \( c^k = 1/\sqrt{k} \) is the rescaling for which the total population process of \( \theta_k \) converges to \( L^t_{\text{e}} \), the local time at level \( t \) of the normalized Brownian excursion.
The CRT scaling limit

Contour function for critical binary GW trees
The CRT scaling limit

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The CRT scaling limit

Contour function for critical binary GW trees
The Scaling limit

Contour function for critical binary GW trees

Normalized Brownian excursion $(\Phi)_{0 \leq t \leq 1}$

Local time at level $t$ of normalized Brownian excursion: $L^t_{\Phi}$

Galton-Watson process
The Scaling limit

**Theorem (Pitman):** If for each $k$, $N^k_t$ is the total population process of $\theta_k$, then

$$\frac{N^k_t}{\sqrt{k}} \rightarrow L^t_e,$$

as $k \to \infty$, in the sense of convergence in distribution in $D_{\mathcal{M}_f(\mathbb{R})}[0, \infty)$.
**Conditioned v. unconditioned limits**

**Theorem (Pitman):** If for each $k$, $N^k_t$ is the total population process of $\theta_k$, then

$$\frac{N^k_t}{\sqrt{k}} \to L^t,$$

as $k \to \infty$, in the sense of convergence in distribution of random variables in $D_{\mathcal{M}_f(\mathbb{R})}[0, \infty)$.

**Standard unconditioned result:** For each $k$, let $\tilde{N}^k_t$ be a discrete critical Galton-Watson process (all lifetimes of length one) whose offspring distribution has finite variance. Then

$$\frac{\tilde{N}^k_{kt}}{k} \to X_t,$$

as $k \to \infty$, in the sense of convergence in distribution of random variables in $D_{\mathcal{M}_f(\mathbb{R})}[0, \infty)$, where $X_t$ is the Feller diffusion.

(Need $\tilde{N}^k_0/k \to x_0 > 0$, since the Feller diffusion is absorbing at 0.)
The scaling limit: characterization

**Theorem** (Healey, M.): In the unconditioned case, each subsequential limit solves the martingale problem for the Dyson superprocess:

\[
\langle \mu_t, \varphi \rangle = \langle \mu_0, \varphi \rangle + \int_0^t \int \int \frac{\varphi(x) - \varphi(y)}{x - y} \mu_s(dx) \mu_s(dy) + M_t(\varphi),
\]

where the local martingale \( M \) has quadratic variation

\[
[M(\varphi)]_t = \int_0^t \langle \mu_s, \varphi^2 \rangle ds.
\]

**Remark:** (a) Don't know yet if the solution to the martingale problem is unique.
(b) Similar result for the conditioned case (Pitman, Serlet).
A planar tree is **conformally balanced** if

(a) each edge has equal harmonic measure from infinity
(b) edge subsets have the same measure from either side.

Balanced trees are in 1-1 correspondence with Shabat polynomials (Bishop, Biane). However, these polynomials are poorly understood.
Conformally balanced trees in $\mathbb{C}$

Let us consider, in the complex plane, the $2^n$th roots of unity and the arcs of the unit circle joining them. Let $\pi$ be a partition of these arcs into $n$ pairs, which is noncrossing. This means that if one draws the $n$ segments joining the middles of the arcs which are in the same pair of $\pi$, these segments do not cross. The quotient of the unit circle by the equivalence relation identifying the two arcs of each pair is a planar tree. Each planar tree can be so obtained, and the corresponding partition is unique, up to some rotation of the circle.

Here is an example, for $n=4$, of a noncrossing partition and its associated planar tree.

![Fig. 1. A noncrossing partition](image1)

![Fig. 2. The planar tree](image2)

Let us now assume that we identify each pair of arcs according to the natural length. This means that if we identify $[2k\pi/2n, (2k+1)\pi/2n]$ with $[(2l-1)\pi/2n, 2l\pi/2n]$ (for parity reasons, these are the only possible identifications) we have to match the points $(2k+\theta)\pi/2n$ and $(2l-\theta)\pi/2n$, for $\theta \in [0, 1]$. Shabat Polynomials and Harmonic Measure

Proposition 1. Let $\pi$ be a noncrossing pair partition of the unit circle. Then there exists a unique conformal mapping from the outside of the unit disk to $\mathbb{C}$, with a Laurent expansion which extends continuously to the boundary of the circle and such that the equivalence relation on the unit circle induced by this map is the noncrossing pair partition $\pi$. The image of the unit circle by this map is a embedding of the tree associated with $\pi$.

The equivalence relation on the circle in the proposition is the one for which two points are in the same class if they have the same image by the continuous extension.

Sketch of proof. The conformal mapping of the proposition can be constructed in the following way. Choose a leaf of the tree (a vertex with only one adjacent edge), it corresponds to some $2^n$th root of unity whose adjacent arcs are in the same part of the partition $\pi$. We can assume, without loss of generality, that this root of unity is 1, then the maps $\phi_\theta(z) = \frac{z^2 + 1 + 2\sin(\theta/2)z + (z+1)\sqrt{z^2+1}-2z\cos(\theta)}{(2z)}$ glue the two intervals according to their natural length. These maps define a conformal mapping from the exterior of the disk to a domain which is the complement of the disk centered at 0, with radius $\cos(2(\theta/2))$, and of the segment $[\cos(2(\theta/2)), (1+\sin(\theta/2))^2]$. For $\theta = \pi/n$ we have glued the two intervals.

Biane (2009): builds a conformal mapping of the exterior domain by welding edges in pairs. End result is a conformal mapping, continuous onto the boundary, that gives exactly the non-crossing partition.

![Fig. 3. The domain](image3)
The CVS bijection and the Brownian map

Figure 2. The Cori-Vauquelin-Schaeffer bijection. On the left side, a well-labeled tree (the framed numbers are the labels assigned to the vertices). On the right side, the edges of the associated quadrangulation $Q$ appear in thick curves.

Image from Le Gall (ICM, 2014).
7 edges \implies 14 corners
Conformal CVS

Flatten edges in cyclic order and label corners
Given the conformal map, each labeling gives a nested family of geodesics in the upper half plane with endpoints on its preimage.

The image of these geodesics under the conformal map is a quadrangulation of the upper half plane.