# Scalar conservation laws with random data

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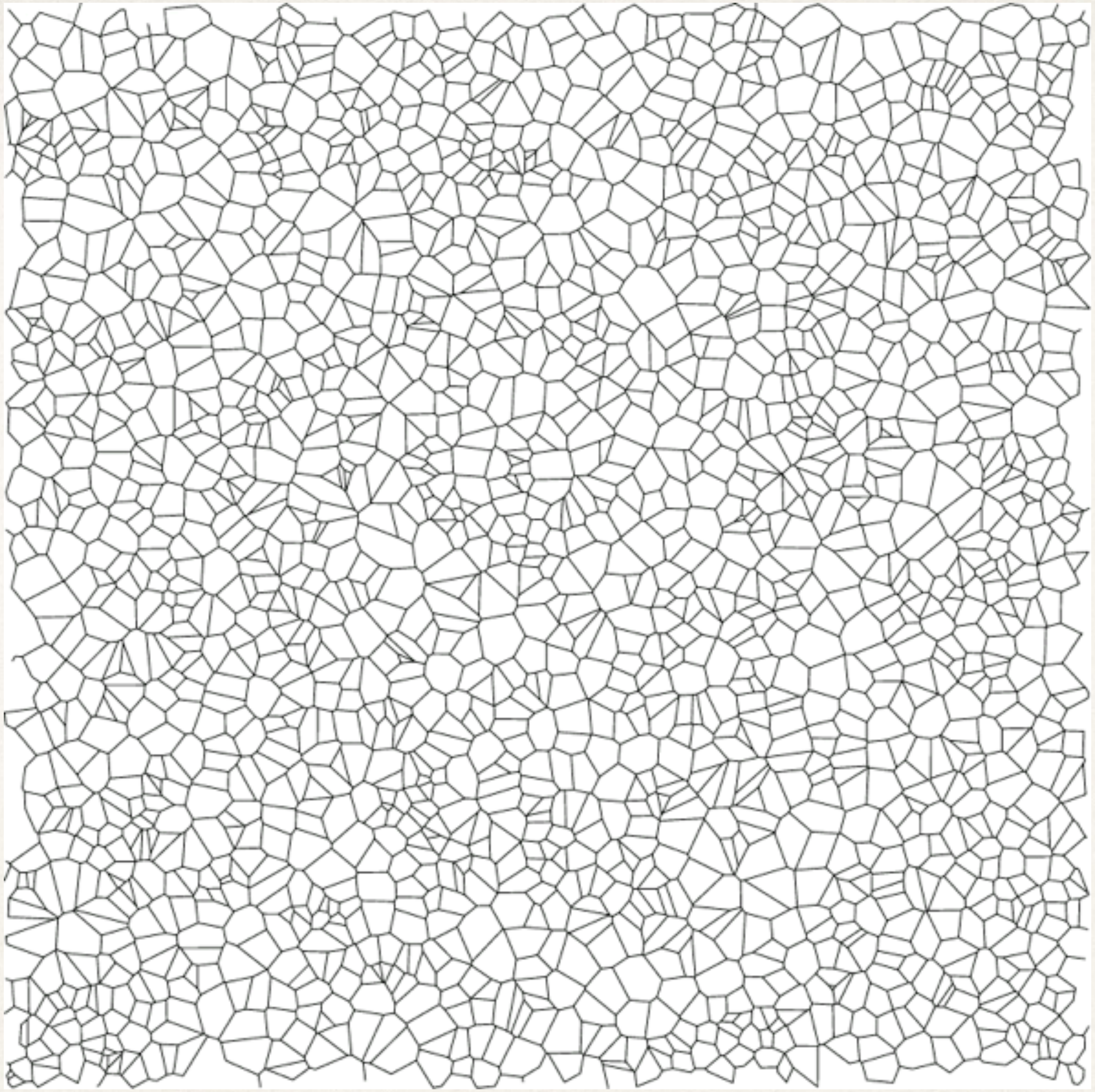
# Our initial motivation: Domain coarsening in materials science



Evolution of a random network by motion by mean-curvature + Herring boundary condition. (Courtesy: Emanuel Lazar).

# Big picture

The first clear mathematical formulation of the evolution of random data by a PDE is due to E. Hopf in his 1952 paper "Statistical hydromechanics and a functional calculus".

Rigorous understanding of this approach is very limited. In most examples, we don't know how to construct "natural" ensembles, nor do we understand how the PDE "propagates" ensembles.

This talk is a description of what happens in the simplest possible setting: a nonlinear scalar PDE in 1-D with random initial data.

# Two stories

(1) Burgers equation with random data.

(a) Groeneboom's solution: Burgers with white noise.

(b) Bertoin's theorem: Burgers with Levy process data.

Roughly, the classical problem of Burgers is completely solved (though this doesn't seem to be as well known as it should be).

(2) Scalar conservation laws with convex flux

(a) Lax equations (M, Srinivasan, 2010; Kaspar, Rezakhanlou 2015).

(b) Complete integrability (M, 2012; Luen-Chau Li 2015).

(2) explains why (1) is solvable. First step beyond Burgers equation.

(3) Unknown: Systems of conservation laws on the line; Hamilton-Jacobi or scalar conservation laws in higher dimensions.

# Burgers model (1930s)

Consider the scalar conservation law

$$\partial_t u + \partial_x \left( \frac{u^2}{2} \right) = 0, \quad x \in \mathbb{R}, t > 0.$$

with white noise as initial data.

Main question: How does the equation evolve white noise or other random data?

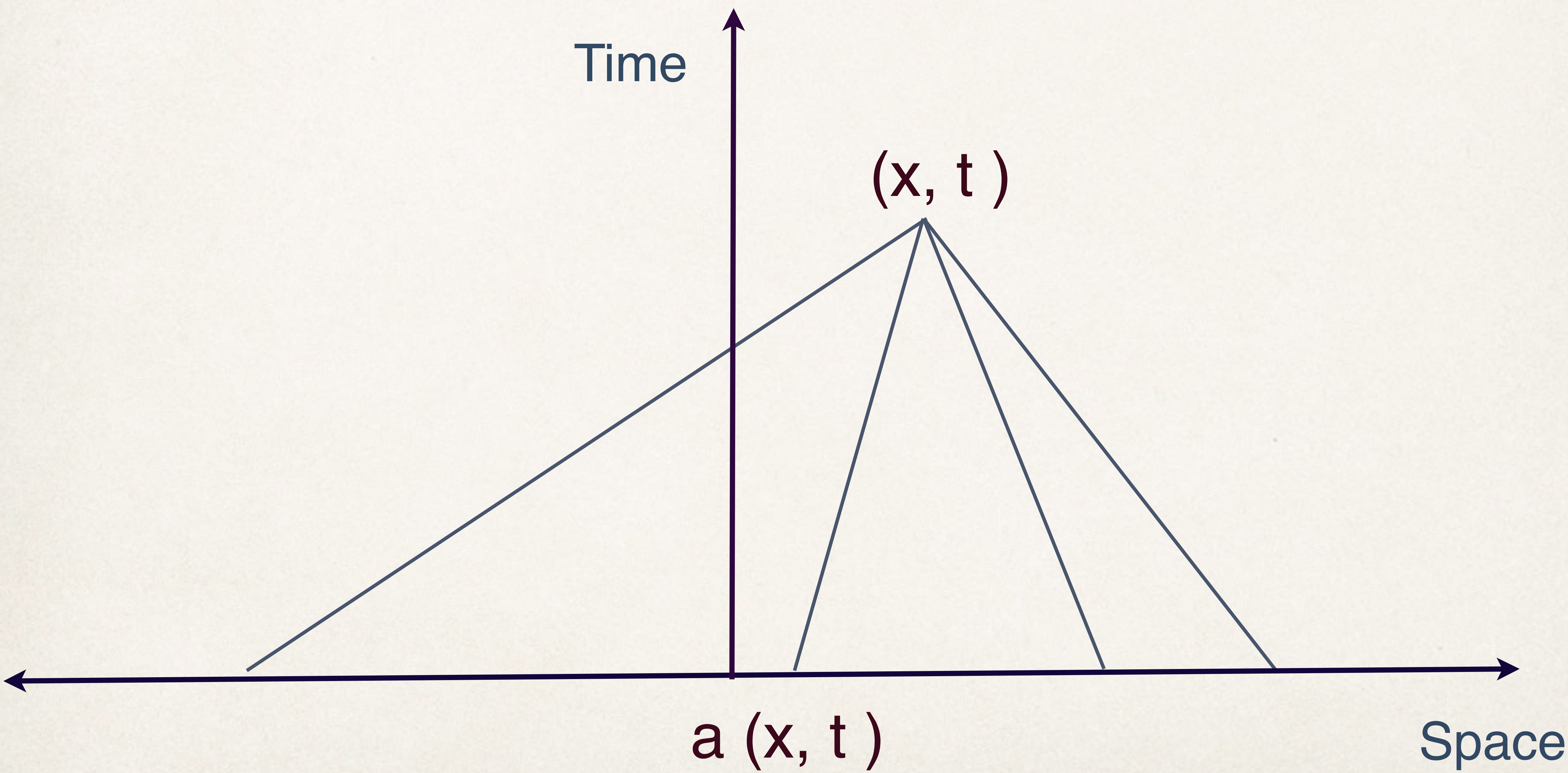
The unique entropy solution is given by a variational principle.

$$u(x,t) = \frac{x - a(x,t)}{t}$$

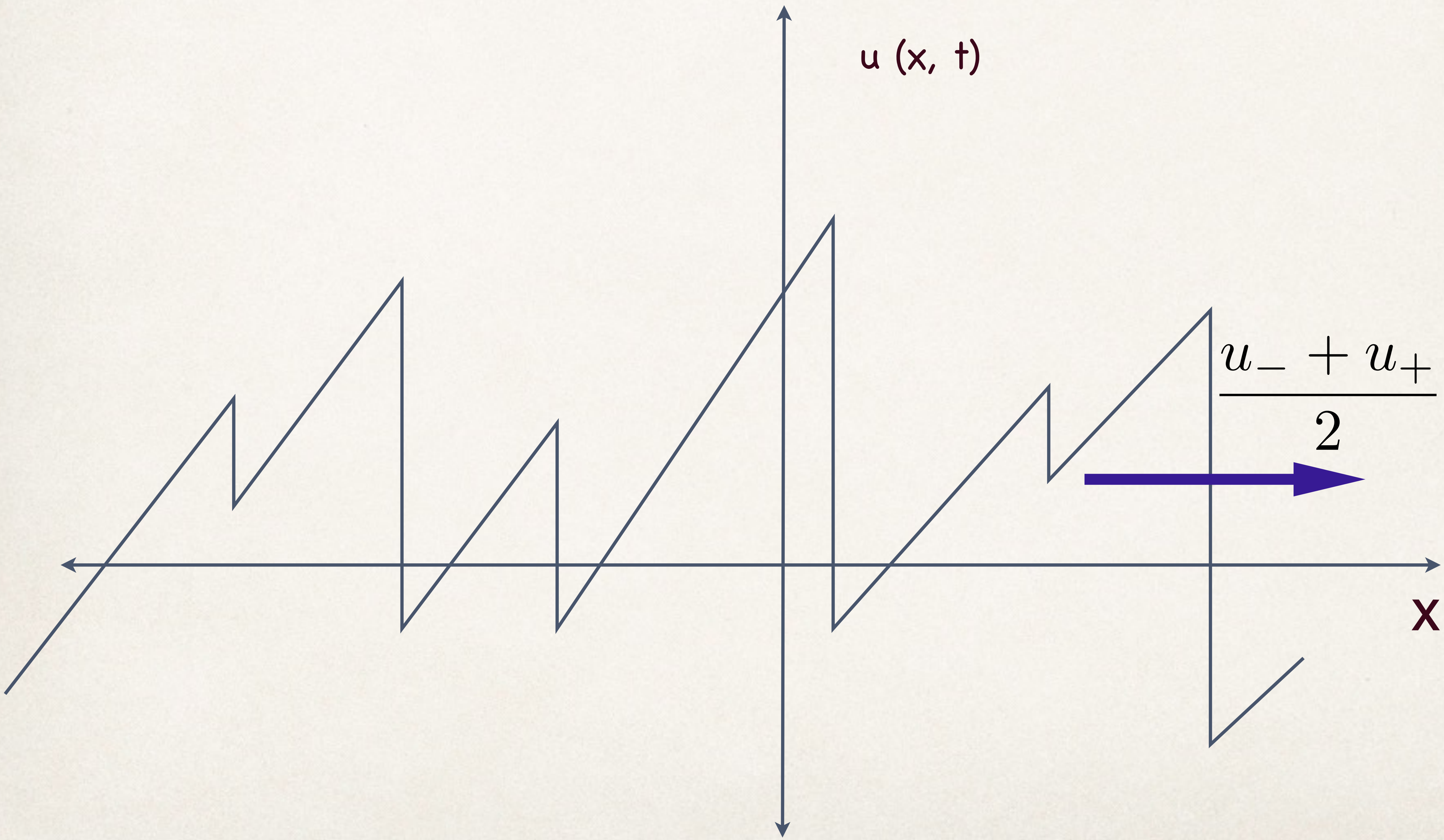
$$a(x,t) = \mathrm{argmin}_y^+ \left\{ U_0(y) + \frac{(x-y)^2}{2t} \right\}$$

$$U_0(y) = \int_0^y u_0(s)\, ds$$

Roughly, a(x,t) gives the `correct' characteristic through the point (x,t) in space-time.



u(x,t) is of bounded variation. Jumps in inverse Lagrangian, a, give rise to shocks in u. These correspond to `double-touches' in the geometric principle.



# The entropy solution emerging from white noise.

Shocks form instantaneously. They then collide and cluster.

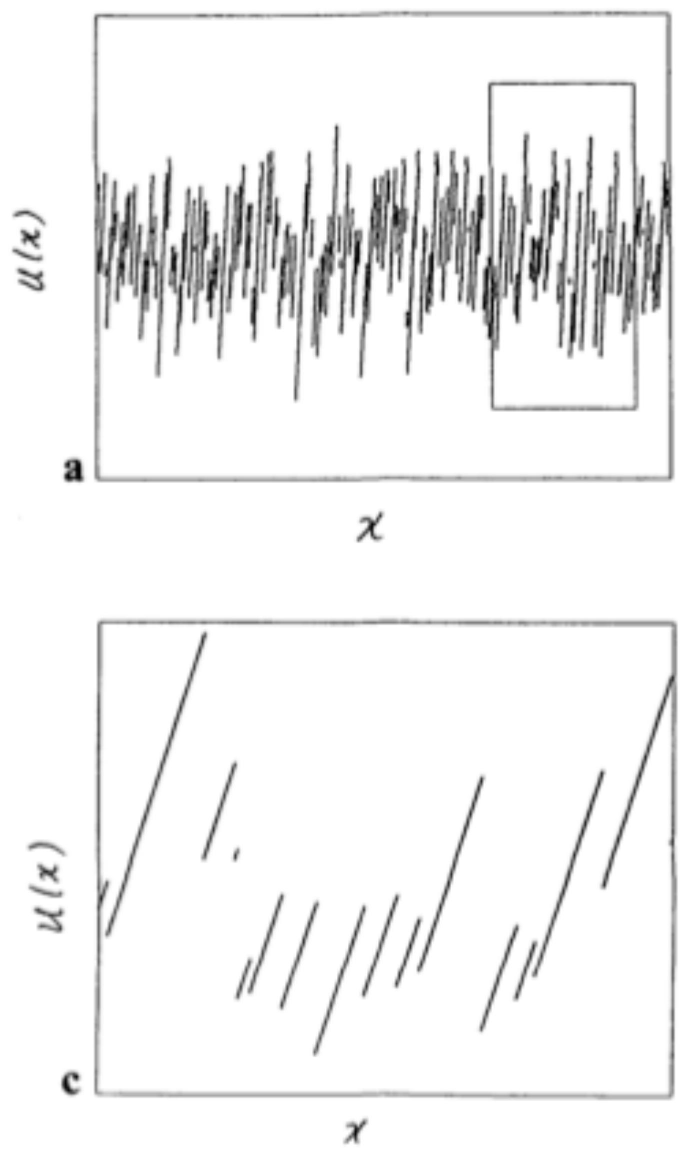



**Fig. 4.** **a** Solution $u(x)$, solution to the inviscid Burgers equation at $t = 1$ with white noise initial velocity (type $B$). **b, c** Show successive small-scale zooming. Note the hierarchy of ramp-like structures with slope 1 and the isolated character of shock points

She, Aurell, Frisch, Comm. Math. Phys. 148,(1992),
Sinai, Comm. Math. Phys. 148, (1992).

## Groeneboom (1985): The exact solution with white noise.

It is enough to determine the solution at time 1. Then:

(1) The solution u(x,1) is a Markov process in x.

(2) The generator of u(x,1) is the integro-differential operator:

$$\mathcal{A}\varphi(u) = \varphi'(u) + \int_{-\infty}^{u} n_*(u,v)\,(\varphi(v) - \varphi(u))\,dv$$

with a jump density $n_*$ given explicitly on the next slide.

Both (1) and (2) are surprising :

(1) because it is "structural": white noise --> Markov ?!

(2) because it is so explicit.

# The exact solution with white noise (contd.)

The jump density $n_*$ depends on two positive function J and K

$$n_*(u,v) = \frac{J(v)}{J(u)} K(u-v), \quad u > v.$$

The Laplace transforms of J and K are given in terms of Airy functions.

$$j(q) = \frac{1}{\mathrm{Ai}(q)}, \quad k(q) = -2\frac{d^2}{dq^2}\log(\mathrm{Ai}(q)).$$

Observe that the expression for k is reminiscent of determinantal soliton formulas (e.g. Dyson's formula for the Gelfand-Levitan-Marcenko equation).

# "Closure": Bertoin's theorem

$$\partial_t u + \partial_x \left( \frac{u^2}{2} \right) = 0, \quad x \in \mathbb{R}, t > 0.$$

Theorem 1. (Bertoin, 1996). Assume the initial data is a Levy process (in x) that may include a drift and Brownian motion, but only downward jumps.

Then for each t>0, the entropy solution remains a Levy process with downward jumps.

Remark 1. This theorem should be viewed as an invariant manifold theorem in the space of probability measures on the line. This idea was first proposed by Carraro and Duchon (1993), and later Chabanol and Duchon.

Remark 2. Levy processes are very "rigid". For example, white noise initial data leads to a Markov process, that is not a Levy process.

# Our approach

Rather than solve for a fixed initial condition such as white noise, and a fixed PDE such as Burgers equation, we study the consequences of the "structural" aspects of the Hopf-Lax formula when applied to random initial data.

That is, we look for a class of stochastic process that is invariant under the entropy solution for an arbitary scalar conservation law. We then find equations that describe the flow on this "invariant manifold of stochastic processes".

# The global picture

Consider:

(1) a scalar conservation law with smooth, convex flux f

$$\partial_t u + \partial_x f(u) = 0, \quad x \in \mathbb{R}, t > 0.$$

(2) initial data that is Markov in x with only downward jumps (or limits of such data).

Then the evolution of the law of u(x,t), t >0, is completely integrable.

There are many significant gaps in a fully rigorous understanding. But the overall structure of the problem is now quite clear.

# A brief explanation of our perspective

The set of spectrally negative Markov processes (in x) is preserved by the entropy solution for every flux f. So these laws form an "invariant manifold" within the space of stochastic processes.

The "coordinates" on the "invariant manifold" are given by the generators of the Markov process. These are linear operators that characterize the stochastic process.

The "flow" on the "invariant manifold" is given by a Lax pair for the generators.

The Lax pair is an infinite-dimensional analog of Euler's equations for rigid body dynamics (geodesic flow on a Lie group with a Manakov metric). In particular, it is Hamiltonian and algebraically completely integrable.

# Deconstructing the results

There are three apparently unrelated aspects:

I. (A) Preservation of the Markov property by entropy solutions.
(B) The Lax equation for the evolution of generators.

II. (A) Kinetic equations for clustering.
(B) Exact solutions.

III. (A) Euler's equations for a rigid body (a free top).
(B) Geodesic flow on Lie groups.

The connection between (I), (II) and (III) is the Lax equation which is a Hamiltonian system as well as a kinetic equation of Vlasov-Boltzmann type.

# The Markov property, entropy solutions, and the Lax equation

## The general question

Let f be convex . Consider the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \quad x \in \mathbb{R}, t > 0,$$

when the initial data is a Markov process with only downward jumps.

The entropy solution to the pde is given by the Hopf-Lax formula.

This induces a nonlinear transformation of the law of the initial data (sample path by sample path). We wish to compute this evolution.

# Spectrally negative Markov processes are preserved by the Hopf-Lax formula

Thm. 1. [M., Srinivasan, JSP 2010]

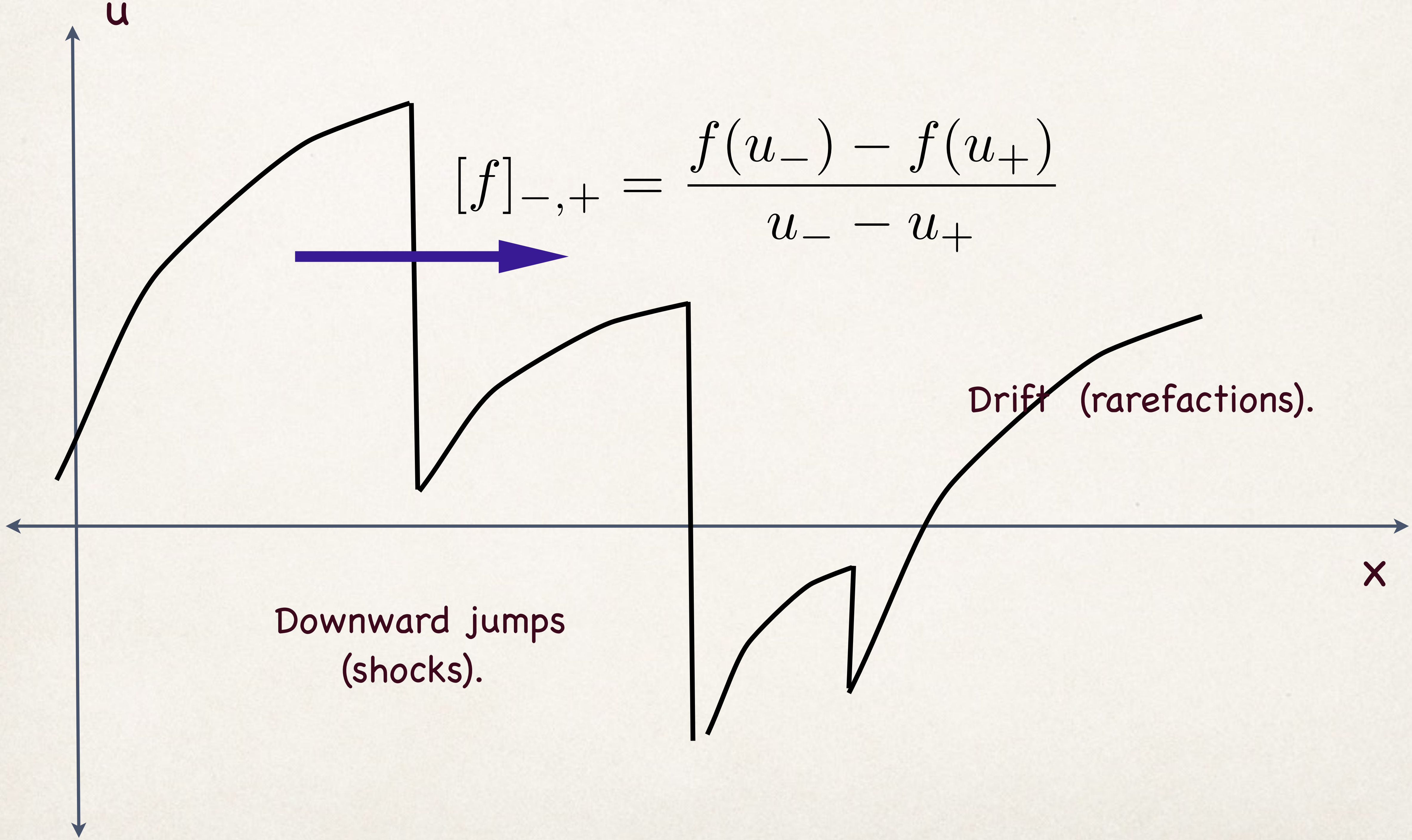
Assume the initial velocity $u(x,0)$ is a strong Markov process (in $x$) with only downward jumps.

Then so is the solution $u(x,t)$ for every $t>0$.

Proof follows Bertoin (1998). Main technical tool is a splitting time argument at the global minimum of a Markov process (Getoor, Millar).

In what follows, we assume that Feller processes are preserved-- this is not proven in generality, but let's see where it leads.

Typical profile of entropy solutions:
Bounded variation + downward jumps




$$[f]_{-,+} = \frac{f(u_-) - f(u_+)}{u_- - u_+}$$

# Markov processes and their generators

A Feller process is characterized by its transition semigroup Q and generator A. For suitable test functions, we have

$$\mathcal{A}\varphi = \lim_{h\downarrow 0} \frac{Q_h\varphi - \varphi}{h}.$$

## Generators of spectrally negative Feller processes

A Feller process with BV sample paths and only downward jumps has an infinitesimal generator

$$\mathcal{A}\varphi(u) = \underbrace{b(u)}\,\varphi'(u) + \int_{-\infty}^{u} \underbrace{n(u,v)}\,(\varphi(v) - \varphi(u))\,dv.$$

Drift at level u.

(rarefactions)

Jumps from u to v.

(shocks)

Since the process is Markov, it has an infinitesimal generator that depends on (x,t). Conceptually, we have the following picture.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
u(x,t) \arrow[r, leftrightarrow, "\text{Invariance theorem}"'] & \mathcal{A}_{(x,t)} \\
u(x,0) \arrow[u, "\partial_t u + \partial_x f(u) = 0."] \arrow[r, leftrightarrow, "\text{Markov property}"] & \mathcal{A}_{(x,0)} \arrow[u, "??"']
\end{tikzcd}
$$

## The evolution in time

Given the generator (in x):

$$\mathcal{A}\varphi(u) = b(u)\varphi'(u) + \int_{-\infty}^{u} n(u,v)\left(\varphi(v) - \varphi(u)\right)\, dv.$$

Define an associated operator that measures the infinitesimal change in the process in t. Here f is the flux function in the scalar conservation law:

$$\begin{aligned}\mathcal{B}\varphi(u) &= -f'(u)b(u)\varphi'(u)\\ &\quad - \int_{-\infty}^{u} \frac{f(v)-f(u)}{v-u} n(u,v)\left(\varphi(v) - \varphi(u)\right)\, dv.\end{aligned}$$

B is also a generator of a Markov process when f is monotone.

# The Lax equation

$$\begin{tikzcd}[column sep=huge, row sep=huge]
u(x,t) \arrow[r, leftrightarrow, "\text{Invariance theorem}"] & \mathcal{A}(t) \arrow[r, leftrightarrow, "\text{Characterization of generator}"] & (b_t, n_t) \\
u(x,0) \arrow[u, "\partial_t u + \partial_x f(u) = 0."] \arrow[r, leftrightarrow] & \mathcal{A}(0) \arrow[u, "\dot{\mathcal{A}} = {[\mathcal{A}, \mathcal{B}]}"'] \arrow[r, leftrightarrow] & (b_0, n_0) \arrow[u, "\text{Kinetic equations}"']
\end{tikzcd}$$

This Lax pair was derived by several methods in [M, Srinivasan, J. Stat. Phys. (2010), however it was not rigorously established.

Theorem: [Kaspar, Rezakhanlou, 2015].

Assume the flux f is smooth and convex.

Assume the initial data is a bounded, pure jump, monotonically decreasing Feller process.

Then for each t>0, the process u(., t) is a Feller process, and its generator evolves in time according to the Lax equations.

The case of Burgers equation is special, since one can always "remove" the drift, as observed in [M, Pego '2008]. For general f, one cannot removed the rarefactions, so the Lax equation with and without rarefactions are genuinely different.

# The kinetic equations

# The Lax equation and kinetic theory

The operators A and B in the Lax equation are integro-differential operators that are completely described by the drift and jump measure (b,n).

$$\mathcal{A}\varphi(u) = b(u)\,\varphi'(u) + \int_{-\infty}^{u} n(u,v)\,\left(\varphi(v) - \varphi(u)\right)\,dv.$$

$$\begin{aligned}\mathcal{B}\varphi(u) &= -f'(u)b(u)\varphi'(u)\\ &\quad - \int_{-\infty}^{u} \frac{f(v)-f(u)}{v-u} n(u,v)\left(\varphi(v)-\varphi(u)\right)\,dv.\end{aligned}$$

We can compute the commutator [A,B] to obtain kinetic equations.

## Kinetic equations of shock clustering for arbitrary f

$$\partial_t b = -f''(u) b^2.$$

Rarefactions

Shocks

$$\partial_t n + \partial_u (n V_u) + \partial_v (n V_v) \\ = Q(n,n) + n\left(\left([f]_{u,v} - f'(u)\right)\partial_u b - b f''(u)\right).$$

$$Q(n,n)(u,v) = \int_v^u \left([f]_{u,w} - [f]_{w,v}\right) n(u,w) n(w,v)\, dw \\ - \int_{-\infty}^u \left([f]_{u,v} - [f]_{v,w}\right) n(u,v) n(v,w)\, dw \\ - \int_{-\infty}^u \left([f]_{u,w} - [f]_{u,v}\right) n(u,v) n(u,w)\, dw.$$

Collision kernel

# The kinetic equation for Burgers turbulence

$$\dot{b} = -b^2, \qquad \partial_t n(u,v,t) = \underbrace{D(b,n)}_{\text{Drift}} + \underbrace{Q(n,n)}_{\text{Collisions}}.$$

$$D(b,n) = \left(\frac{u-v}{2}\right)\left(b(u)\partial_u n - \partial_v(b(v)n)\right)$$

$$\begin{aligned}
Q(n,n) = \frac{u-v}{2}\int_v^u n(u,w)n(w,v)\,dv & \qquad \text{Birth} \\
-n(u,v)\int_{-\infty}^{v} n(v,w)\left(\frac{u-w}{2}\right)dw & \\
& \qquad \text{Death} \\
-n(u,v)\int_{-\infty}^{u} n(u,w)\left(\frac{w-v}{2}\right)dw &
\end{aligned}$$

## Groeneboom's solution (Burgers with white noise)

$$b(u,t) = \frac{1}{t}, \quad n(u,v,t) = \frac{1}{t^{1/3}} n_* \left( \frac{u}{t^{2/3}}, \frac{v}{t^{2/3}} \right).$$

$$n_*(u,v) = \frac{J(v)}{J(u)} K(u-v),$$

where J and K have Laplace transforms:

$$j(q) = \frac{1}{Ai(q)}, \quad k(q) = -2 \frac{d^2}{dq^2} \log Ai(q).$$

# Exact solutions to the kinetic equations

It is an interesting calculation to check that Groeneboom's solution satisfies our kinetic equation. The calculation involves some "miraculous" cancellations and an identity involving Painleve 2.

Another class of exact solutions arises from Levy processes (Bertoin's theorem). These can be dealt with more elegantly using the Laplace transform. The kinetic equation then simplifies to Smoluchowski's coagulation equation (Bertoin (ICM 2002), M-Pego, CMP, 2007).

So far, the only exact solutions are in the Burgers case.

Euler's equation for a rigid body,
and geodesic flow with Manakov metrics

## Euler's equation for a free top

$$\dot{A} = [A, B] = AB - BA.$$

Here A is the angular momentum and B the angular velocity in the body frame (i.e. as viewed by someone tumbling with the top).

A and B are linearly related by the moment of inertia.

In 3d these are vectors, so we can write them as antisymmetric matrices and the cross-product of the vectors becomes the Lie bracket.

The equations generalize in this form to n-dimensions and yield a flow on so(n), the Lie algebra of nxn rotations.

# Geodesic flow on SO(n)

$$\dot{A} = [A, B].$$

A positive definite linear map A -> B defines a norm on so(n) and a left-invariant metric on the associated group of rotations SO(n). An important example is the diagonal metric:

$$B_{ij} = F_{ij} A_{ij}, \quad F_{ij} > 0.$$

Fundamental facts:

(a) Geodesic flow with respect to this metric is given by the Lax equation.

(b) The Lax equation is Hamiltonian. The natural symplectic structure is induced by the group action on coadjoint orbits (Kostant-Kirillov).

# Hamiltonian systems and integrability

The integration of Hamiltonian systems relies on an explicit understanding of its symmetries (obvious and hidden).

(1) Noether's theorem: group invariance => integrals.

(2) Liouville's theorem: m integrals in involution for a 2m dimensional system => the flow may be linearized in principle ("soft").

(3) Separation of variable (Jacobi) : construct the linearizing transformation and action-angle variables explicitly. This typically involves special functions, complex analysis and algebraic geometry and each integrable system has its own individuality ("hard").

Particularly important for us is the reformulation of integrable systems of the 19th century as Lax pairs by Jurgen Moser (1970s).

# Not all geodesic flows are integrable

$$\dot{A} = [A, B], \quad B_{ij} = F_{ij} A_{ij}$$

(1) Lax equations is isospectral, so we have n conserved eigenvalues.

(2) But n is the size of the matrix, so size of the system $O(n^2)$. How do we find additional integrals?

(3) This can't be done in general, but for special "Manakov metrics"

$$F_{ij} = \frac{f(u_i) - f(u_j)}{u_i - u_j}.$$

we find additional integrals by introducing a spectral parameter.

# The spectral parameter

Main observation: for these metrics we have an additional equation

$$[A, \mathcal{M}] - [\mathcal{N}, B] = 0.$$

$$\mathcal{M} = \mathrm{diag}(u_1, \ldots, u_n), \quad \mathcal{N} = \mathrm{diag}(f(u_1), \ldots, f(u_n))$$

This allows us to introduce a spectral parameter z, and obtain a family of Lax equations.

$$\frac{d}{dt}(A - z\mathcal{M}) = [A - z\mathcal{M}, B + z\mathcal{N}]$$

This provides additional commuting integrals and an algebraic curve that remains fixed under the evolution. The explicit linearization is based on the structure of this algebraic curve (e.g. hyperelliptic).

Hamiltonian structure, geodesic flows of Markov operators, and complete integrability.

We now have discussed two Lax equations.

(a) Shock clustering:

$$\dot{\mathcal{A}} = [\mathcal{A}, \mathcal{B}]$$

$$\mathcal{A}\varphi(u) = b(u)\,\varphi'(u) + \int_{-\infty}^{u} n(u,v)\,\left(\varphi(v) - \varphi(u)\right)\,dv.$$

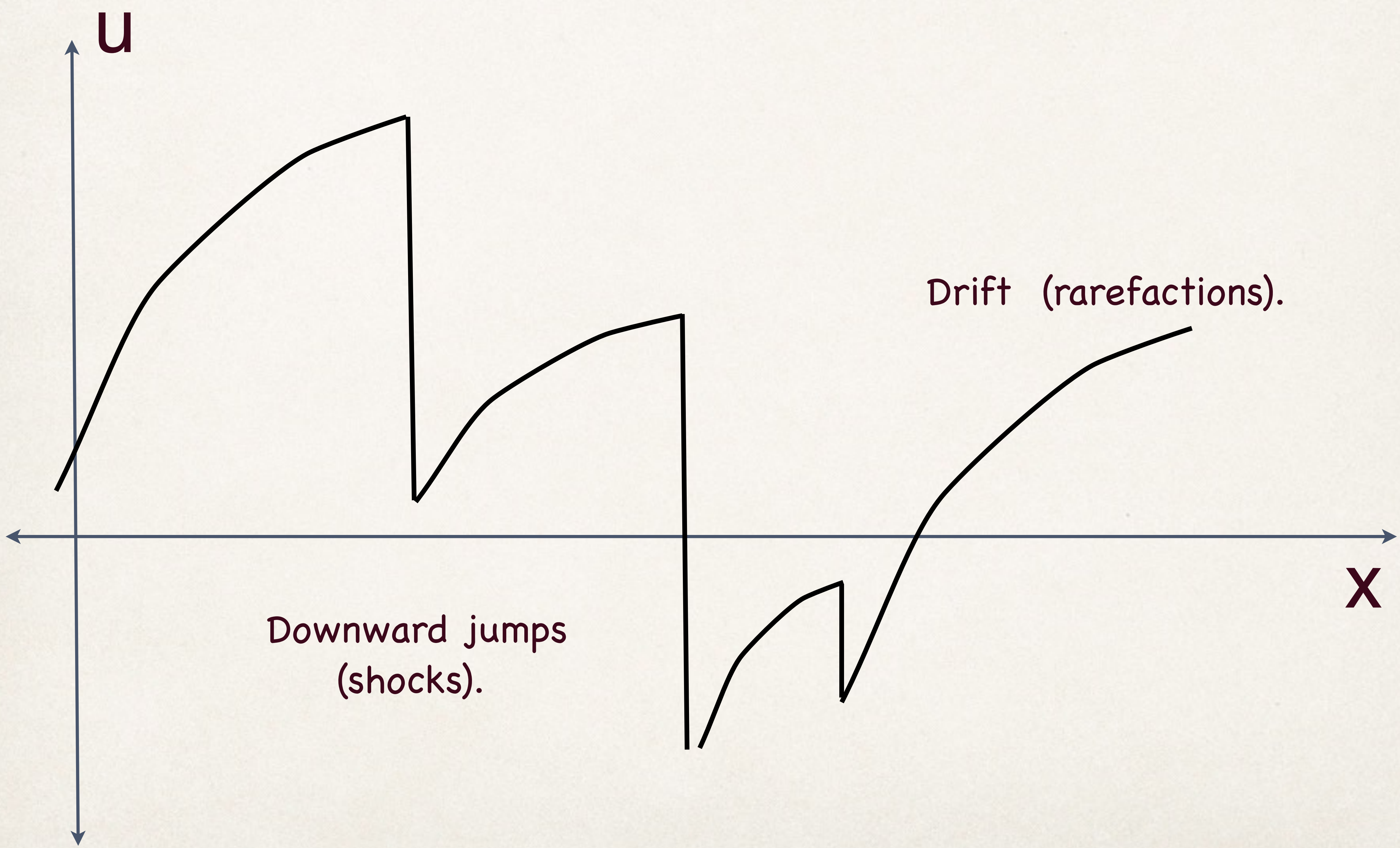
$$\mathcal{B}\varphi(u) = -f'(u)b(u)\varphi'(u)$$
$$-\int_{-\infty}^{u} \frac{f(v)-f(u)}{v-u} n(u,v)\left(\varphi(v)-\varphi(u)\right)\,dv.$$

(b) Geodesic flow on SO(n) with the Manakov metric.

$$\dot{A} = [A, B], \quad B_{ij} = F_{ij}A_{ij}$$

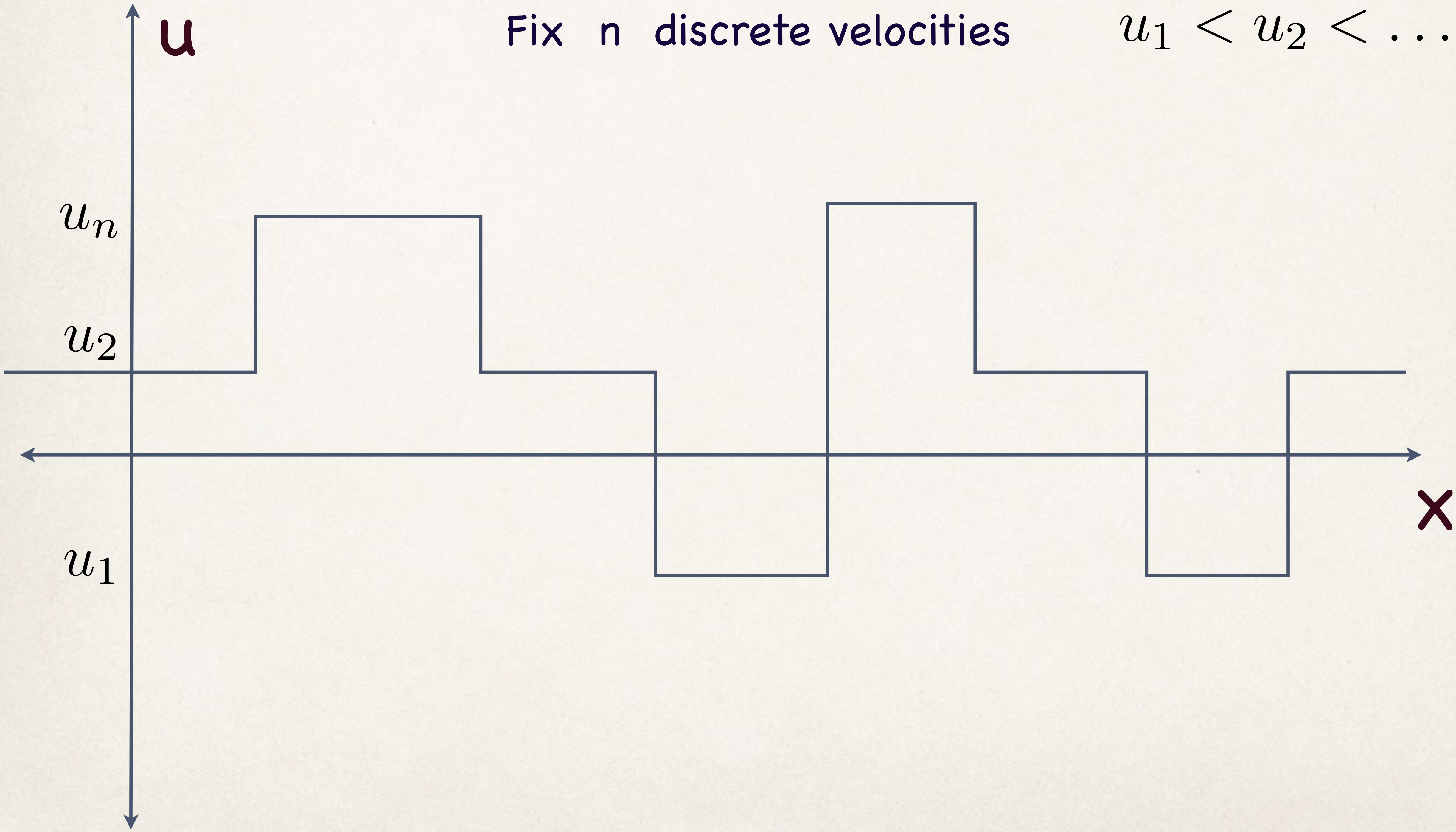
$$F_{ij} = \frac{f(u_i) - f(u_j)}{u_i - u_j}.$$

# Markov processes with downward jumps



Continuous time Markov chain with discrete states.

Fix n discrete velocities $u_1 < u_2 < \ldots u_n.$



# Generators of the discretized process

The generator A is an n x n matrix with positive entries on the off-diagonal. The diagonal entries are determined by setting the sum of each row to zero.

If we don't require positivity, the set of such matrices is a Lie algebra. Let's call the associated group the Markov group.

Of course, positivity is needed for a probabilistic interpretation, and the analysis of the infinite-dimensional Lax equation must really deal with semigroups, not groups, but this helps us get started.

# Discrete Lax equations for shock clustering

$$\dot{A} = [A, B]$$

$$B_{ij} = F_{ij} A_{ij}, \quad i \neq j. \qquad B_{ii} = -\sum_{j \neq i} B_{ij}.$$

$$F_{ij} = \frac{f(u_i) - f(u_j)}{u_i - u_j}, \quad i \neq j.$$

We have now come full circle. The Lax equation was derived from probabilistic considerations, but it is nothing but a Manakov top on the Markov group. We can now introduce several methods from integrable systems.

Caveat: except for the triangular case, this does not describe a true stochastic evolution. The discretization is introduced only to understand the underlying integrable structure.

# Hamiltonian structure

Thm. 2 [M, 2012, Luen-Chau Li, 2015] The discrete Lax equations are Hamiltonian.

Remark: It is not obvious that a Lax equation is Hamiltonian. The ingredients of a Hamiltonian flow are a symplectic manifold (manifold+symplectic structure) and a Hamiltonian defined on this manifold.

Here we need to introduce a symplectic structure on the cone of generators. This is done as follows.

Coadjoint orbits of a Lie group carry a natural symplectic structure (Kirillov-Kostant). This is why we introduced the "Markov group". By "twisting" the usual Lie bracket of matrices using a splitting of gl(n,R) we obtain a symplectic structure for which the Lax equation is Hamiltonian.

# Geodesic flow

Thm. 3 [M, 2012] The discrete Lax equations are obtained from a least action principle on the Markov group with a quadratic action defined by the flux f.

When $f' > 0$, the action is equivalent to a left-invariant metric, and minimizers of the action are geodesics on the Markov group with respect to this metric.

Remark: This is in direct analogy with geodesic flow on a Lie group and the proof is routine. However, it is still surprising that one gets a "geometric view" of the evolution of the law with a metric determined by the flux f in the conservation law.

# Complete integrability

Thm. 4 [M, 2012]. The discrete Lax equations are a completely integrable Hamiltonian system. More precisely:

(a) the Lax equation admits $n(n-1)/2$ linearly independent, commuting integrals.

(b) the Lax equation may be formulated as a matrix Riemann-Hilbert problem and linearized by a loop-group factorization.

Thm 5 [Luen-Chau Li]. The flow can be explicitly linearized on the Jacobi variety of the spectral curve. The solution is expressed using Riemann theta functions.

Remark: Thm 4 follows the Adler-Kostant-Symes framework for algebraic complete integrability. Thm 5 is harder since it requires a detailed understanding of the spectral curve. The general approach follows classic work in integrable systems but there is a lot of work to be done.

Main problem: so far, this picture correspond to a true probabilistic evolution only for monotone date.

## Integrable systems and stochastic analysis

Summary: Shock clustering defines a nonlinear transformation of spectrally negative Markov processes on path space. A Lax equation describes the kinetics of shock clustering. This equation is also completely integrable.

The moral of the story is:

shock clustering = infinite dimensional integrable top.

## Several questions are open

(1) Exact solutions from the new point of view -- exact solutions to Burgers with Ornstein-Uhlenbeck data are still not known. The forms of stochastic forcing that respect the Lax structure are not known.

(2) Are the equations correct for arbitrary f?

(3) Rigorous formulation of weak solutions to the kinetic equations.

(4) Piecewise linear flux functions (Dafermos' polygonal approximation). This is important to understand kinetics of shock clustering for systems of conservation laws.

(5) Reconciling excursion theory with kinetic theory. The approach of Kaspar-Rezakhanlou is fundamentally different from all past work in this problem, including Bertoin's. It is very interesting to understand how unexpected symmetries under path transformations can be understood from the underlying particle system.