PDE, Part II

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1 Scalar Conservation Laws

$$u_t + (f(u))_x = 0$$

$x \in \mathbb{R}$, $t > 0$, typically f convex. $u(x,0) = u_0(x)$ (given). Prototypical example: *Inviscid Burgers Equation*

$$f(u) = \frac{u^2}{2}.$$

Motivation for Burgers Equation. Fluids in 3 dimensions are described by *Navier-Stokes* equations.

$$\begin{aligned} u_t + u \cdot Du &= -Dp + \nu \Delta u \\ \operatorname{div} u &= 0. \end{aligned}$$

Unknown: $u\colon \mathbb{R}^3 \to \mathbb{R}^3$ velocity, $p\colon \mathbb{R}^3 \to \mathbb{R}$ pressure. ν is a parameter called *viscosity*. Get rid of incompressibility and assume $u\colon \mathbb{R} \to \mathbb{R}$.

$$u_t + u\, u_x = \nu u_{xx}.$$

Burgers equation (1940s): small correction matters only when u_x is large (Prantl). Method of characteristics:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0.$$

Same as $u_t + u\,u_x = 0$ if u is smooth. We know how to solve $u_t + c\,u_x = 0$. ($c \in \mathbb{R}$ constant) (*1D transport equation*). Assume

$$u = u(x(t), t)$$

By the chain rule

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u_x \frac{\mathrm{d}x}{\mathrm{d}t} + u_t.$$

If $\mathrm{d}x/\mathrm{d}t = u$, we have $\mathrm{d}u/\mathrm{d}t = u\,u_x + u_t = 0$. More precisely,

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= 0 \quad \text{along paths} \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= u(x(t), t) = u_0(x(0)). \end{aligned}$$

Suppose $u_0(x)$ is something like this:

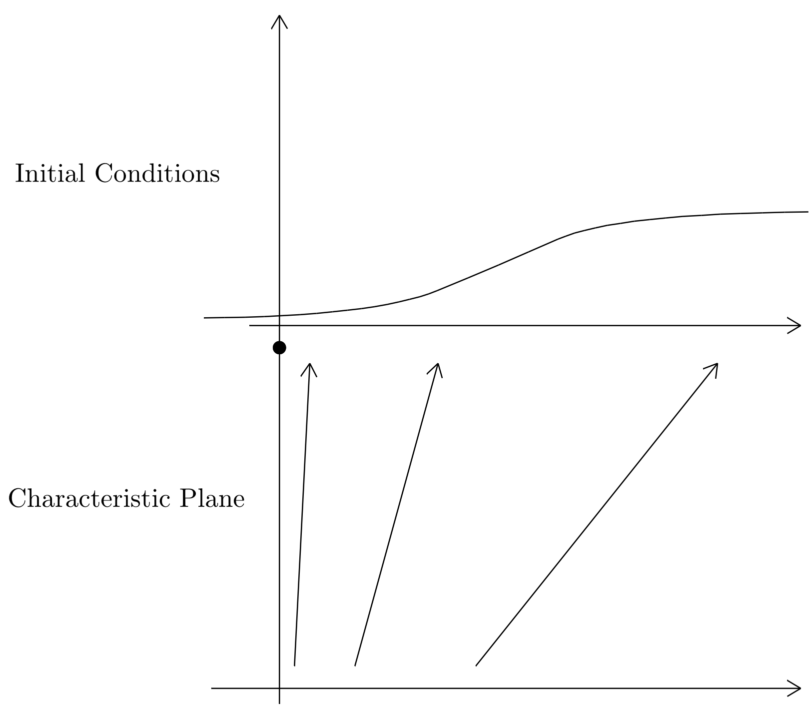

Figure 1.1.

Analytically, $u(x,t) = u_0(x_0)$, $\mathrm{d}x/\mathrm{d}t = u_0(x_0) \Rightarrow x(t) = x(0) + t\,u_0(x_0)$. Strictly speaking, (x,t) is fixed, need to determine x_0. Need to invert $x = x_0 + t\,u_0(x_0)$ to find x_0 and thus $u(x,t) = u_0(x_0)$.

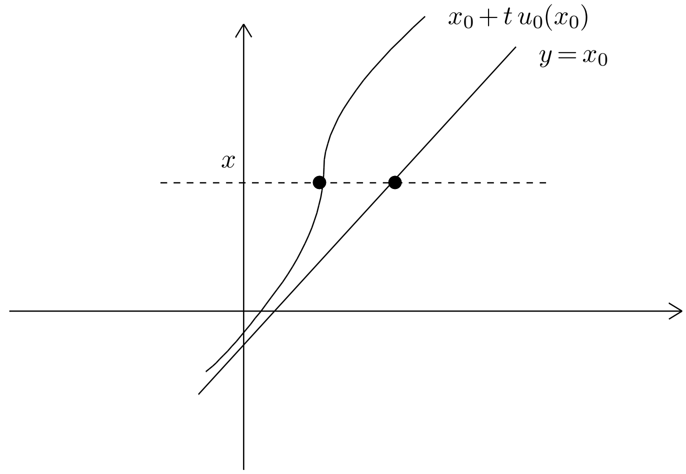

Figure 1.2.

As long as $x_0 + t\,u_0(x_0)$ is increasing, this method works. Example 2:

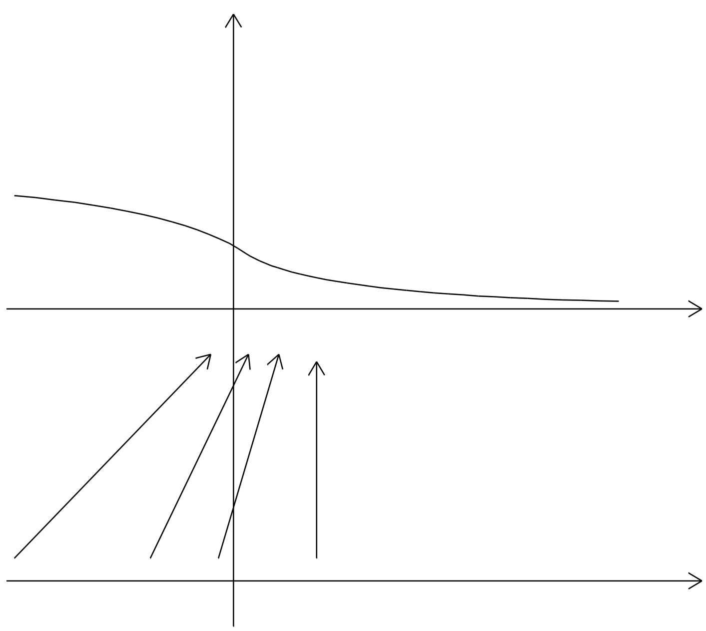

Figure 1.3.

This results in a sort-of breaking wave phenomenon. Analytically, the solution method breaks down when

$$0 = \frac{\mathrm{d}x}{\mathrm{d}x_0} = 1 + t\,u_0'(x_0).$$

No classical (smooth) solutions for all $t > 0$. Let's try weak solutions then. Look for solutions in $\mathcal{D}'$. Pick any test function $f \in C_c^\infty(\mathbb{R} \times [0, \infty))$:

$$\int_0^\infty \int_{\mathbb{R}} \varphi \left[u_t + \left(\frac{u^2}{2} \right)_x \right] = 0, \quad u(x, 0) = u_0(x).$$

Integrate by parts:

$$\int_0^\infty \int_{\mathbb{R}} \left[\varphi_t u + \varphi_x \frac{u^2}{2} \right] \mathrm{d}x\, \mathrm{d}t + \int_{\mathbb{R}} \varphi(x, 0) u_0(x) \mathrm{d}x = 0. \tag{1.1}$$

Definition 1.1. *$u \in L^1_{\mathrm{loc}}([0, \infty] \times \mathbb{R})$ is a weak solution if (1.1) holds for all $\varphi \in C_c^1([0, \infty) \times \mathbb{R})$.*

1.1 Shocks and the Rankine-Hugoniot condition

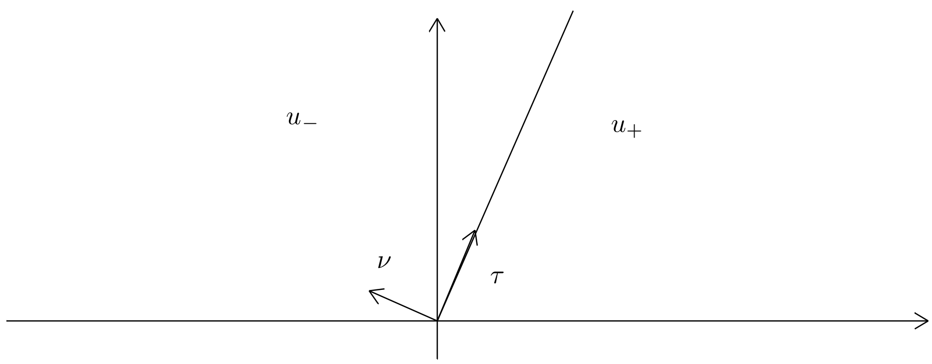

Figure 1.4. Solution for a simple discontinuity (ν and τ are unit vectors.)

Let φ have compact support in $\mathbb{R} \times (0, \infty)$ which crosses the the line of discontinuity. Apply (1.1). Ω_- is the part of the support of φ to the left of the line of discontinuity, Ω_+ the one to the right.

$$\begin{aligned} 0 &= \int_{\Omega_-} \varphi_t u_- + \varphi_x \left(\frac{u_-^2}{2} \right) \mathrm{d}x\, \mathrm{d}t + \int_{\Omega_+} \varphi_t u_+ + \varphi_x \left(\frac{u_+^2}{2} \right) \mathrm{d}x\, \mathrm{d}t \\ &= \int_{\Omega_-} (\varphi u_-)_t + \left(\varphi \frac{u_-^2}{2} \right)_t \mathrm{d}x\, \mathrm{d}t + \cdots \\ &= -\int_\Gamma \varphi \left[u_- \nu_t + \left(\frac{u_-^2}{2} \right) \nu_x \right] \mathrm{d}s + \int_\Gamma \varphi \left[u_+ \nu_t + \left(\frac{u_+^2}{2} \right) \nu_x \right] \mathrm{d}s \end{aligned}$$

Notation $[\![g]\!] = g_+ - g_-$ for any function that jumps across discontinuity. Thus, we have the integrated jump condition

$$\int_\Gamma \varphi \left[[\![u]\!] \nu_t + \left[\!\!\left[\frac{u^2}{2} \right]\!\!\right] \nu_x \right] \mathrm{d}s.$$

Since φ is arbitrary,

$$[u] \nu_t + \left[\!\!\left[\frac{u^2}{2} \right]\!\!\right] \nu_x = 0.$$

For this path,

$$\tau = (\dot{x}, 1) \frac{1}{\sqrt{\dot{x}^2 + 1}}, \quad \nu = (-1, \dot{x}) \frac{1}{\sqrt{\dot{x}^2 + 1}}.$$

($\dot{x}$ is the speed of the shock.)

$$\Rightarrow \dot{x} = \frac{\left[\!\!\left[\frac{u^2}{2} \right]\!\!\right]}{[\![u]\!]} = \frac{u_- + u_+}{2}.$$

Rankine-Hugoniot condition:

$$\text{shock speed} = \frac{[\![f(u)]\!]}{[\![u]\!]}$$

for a scalar conservation law $u_t + (f(u))_x = 0$.

Definition 1.2. *The* Riemann problem *for a scalar conservation law is given by*

$$u_t + (f(u))_x = 0,$$

$$u_0(x) = \begin{cases} u_- & x < 0, \\ u_+ & x \geqslant 0. \end{cases}$$

Example 1.3. Let's consider the Riemann problem for the Burgers equation: $f(u) = u^2/2$.

$$u_0(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geqslant 0. \end{cases}$$

By the derivation for "increasing" initial data above, we obtain

$$u(x,t) = \mathbf{1}_{\{x \geqslant y(t)\}}, \quad y(t) = \frac{[\![u^2/2]\!]}{[\![u]\!]} = \frac{t}{2}.$$

The same initial data admits another (weak) solution. Use characteristics:

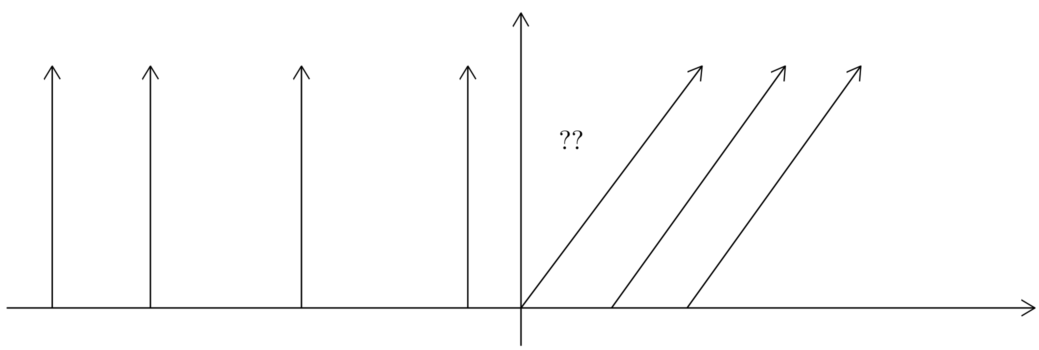

Figure 1.5.

Rarefaction wave: Assume $u(x,t) = v(x/t) =: v(\xi)$. Then

$$\begin{aligned} u_t &= v'\left(-\frac{x}{t^2}\right) = \frac{-\xi v'}{t}, \\ u_x &= v'\left(\frac{1}{t}\right) = \frac{1}{t} v'. \end{aligned}$$

So, $u_t + u\, u_x = 0 \Rightarrow -\xi/t\, v' + v/t\, v' = 0 \Rightarrow v'(-\xi + v) = 0$. Choose $v(\xi) - \xi$. Then

$$u(x,t) = \frac{x}{t}.$$

Thus we have a second weak solution

$$u(x,t) = \begin{cases} 0 & x < 0, \\ x/t & 0 \leqslant \frac{x}{t} \leqslant 1, \\ 1 & \frac{x}{t} > 1. \end{cases}$$

So, which if any is the *correct* solution? Resolution:

- $f(u) = u^2/2$: E. Hopf, 1950
- General convex f: Lax, Oleinik, 1955.
- Scalar equation in $\mathbb{R}^n$: Kružkov.

1.2 Hopf's treatment of Burgers equation

Basic idea: The "correct" solution to

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

must be determined through a limit as $\varepsilon \downarrow 0$.

$$u_t^\varepsilon + u^\varepsilon u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon.$$

This is also called to the *vanishing viscosity method*. Then, apply a clever change of variables. Assume u has compact support. Let

$$U(x,t) = \int_{-\infty}^{x} u(y,t)\mathrm{d}y.$$

(Hold $\varepsilon > 0$ fixed, drop superscript.)

$$U_t = \int_{-\infty}^{x} u_t(y,t)\mathrm{d}y = -\int_{-\infty}^{x}\left(\frac{u^2}{2}\right)_y \mathrm{d}y + \varepsilon \int_{-\infty}^{x} u_{yy}(y,t)\mathrm{d}y.$$

Then

$$U_t = -\frac{u^2}{2} + \varepsilon u_x$$

or

$$U_t + \frac{U_x^2}{2} = \varepsilon U_{xx}. \tag{1.2}$$

Equations of the form $U_t + H(Du) = 0$ are called *Hamilton-Jacobi equations*. Let

$$\psi(x,t) = \exp\left(-\frac{U(x,t)}{2\varepsilon}\right)$$

(Cole-Hopf)

$$\begin{aligned}
\psi_t &= \psi\left(-\frac{1}{2\varepsilon}U_t\right)\\
\psi_x &= \psi\left(-\frac{1}{2\varepsilon}U_x\right)\\
\psi_{xx} &= \psi\left(-\frac{1}{2\varepsilon}U_x\right)^2 + \psi\left(-\frac{1}{2\varepsilon}U_{xx}\right).
\end{aligned}$$

Use (1.2) to see that

$$\psi_t = \varepsilon \psi_{xx},$$

which is the heat equation for $x \in \mathbb{R}$, and

$$\psi_0(x) = \exp\left(-\frac{U_0(x)}{2\varepsilon}\right).$$

Since $\psi > 0$, uniqueness by Widder.

$$\psi(x,t) = \frac{1}{\sqrt{4\pi t\varepsilon}} \int_{\mathbb{R}} \exp\left(-\frac{1}{2\varepsilon}\left[\frac{(x-y)^2}{2t} + U_0(y)\right]\right)\mathrm{d}y.$$

Define

$$G(t,x,y) = \frac{(x-y)^2}{2t} + U_0(y),$$

which is called the *Cole-Hopf* function. Finally, recover $u(x,t)$ via

$$\begin{aligned}
u(x,t) = -2\varepsilon\psi_x/\psi &= -2\varepsilon \frac{\int_{\mathbb{R}} \frac{-2(x-y)}{2\varepsilon\, 2t}\exp\left(-\frac{G}{2\varepsilon}\right)\mathrm{d}y}{\int_{\mathbb{R}}\exp\left(-\frac{G}{2\varepsilon}\right)\mathrm{d}y} = \frac{\int_{\mathbb{R}} \frac{x-y}{t}\exp\left(-\frac{G}{2\varepsilon}\right)\mathrm{d}y}{\int_{\mathbb{R}}\exp\left(-\frac{G}{2\varepsilon}\right)\mathrm{d}y}\\
&= \frac{x}{t} - \frac{1}{t}\cdot\frac{\int_{\mathbb{R}} y\exp\left(-\frac{G}{2\varepsilon}\right)\mathrm{d}y}{\int_{\mathbb{R}}\exp\left(-\frac{G}{2\varepsilon}\right)\mathrm{d}y}.
\end{aligned}$$

Heuristics: We want $\lim_{\varepsilon \to 0} u^{\varepsilon}(x,t)$.

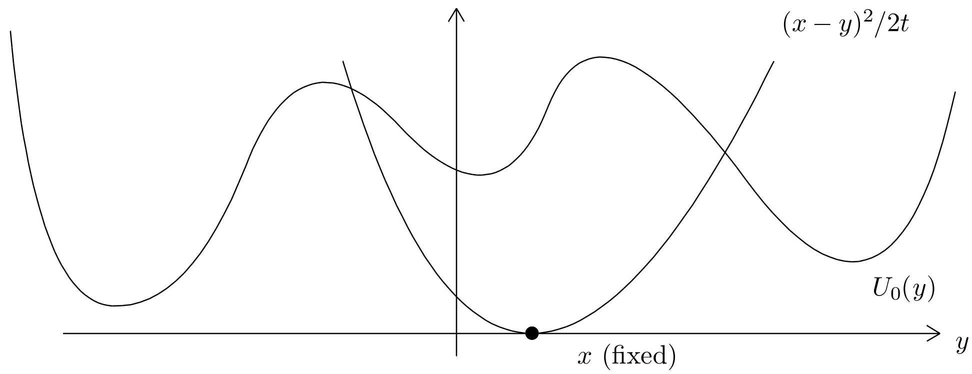

Figure 1.6.

Add to get $G(x,y,t)$. We hold x,t fixed and consider $\varepsilon \downarrow 0$. Let $a(x,t)$ be the point where $G=0$. We'd expect

$$\lim_{\varepsilon \to 0} u^{\varepsilon}(x,t) = \frac{x-a(x,t)}{t}.$$

Problems:

- G may not have a unique minimum.
- G need not be C^2 near minimum.

Assumptions:

- U_0 is continuous (could be weakened)
- $U_0(y) = o(|y|^2)$ as $|x| \to \infty$.

Definition 1.4. [The inverse Lagrangian function]

$$\begin{aligned} a_-(x,t) &= \inf \left\{ z \in \mathbb{R} \colon G(x,z,t) = \min_y G \right\} = \inf \operatorname{argmin} G, \\ a_+(x,t) &= \sup \left\{ z \in \mathbb{R} \colon G(x,z,t) = \min_y G \right\} = \sup \operatorname{argmin} G, \end{aligned}$$

Lemma 1.5. *Use our two basic assumptions from above. Then*

- *These functions are well-defined.*
- $a_+(x_1,t) \leqslant a_-(x_2,t)$ *for* $x_1 < x_2$. *In particular,* a_-, a_+ *are increasing (non-decreasing).*
- a_- *is left-continuous,* a_+ *is right-continuous:* $a_+(x,t) = a_+(x_+,t)$.
- $\lim_{x \to \infty} a_-(x,t) = +\infty$, $\lim_{x \to -\infty} a_+(x,t) = -\infty$.

In particular, $a_+ = a_-$ *except for a countable set of points* $x \in \mathbb{R}$ *(These are called* shocks*).*

Theorem 1.6. (Hopf) *Use our two basic assumptions from above. Then for every* $x \in \mathbb{R}$, $t > 0$

$$\frac{x-a_+(x,t)}{t} \leqslant \limsup_{\varepsilon \to 0} u^{\varepsilon}(x,t) \leqslant \liminf_{\varepsilon \to 0} u^{\varepsilon}(x,t) \leqslant \frac{x-a_-(x,t)}{t}.$$

In particular, for every $t > 0$ *except for* x *in a countable set, we have*

$$\lim_{\varepsilon \to 0} u^{\varepsilon}(x,t) = \frac{x-a_+(x,t)}{t} = \frac{x-a_-(x,t)}{t}.$$

Graphical solution I (Burgers): Treat $U_0(y)$ as given.

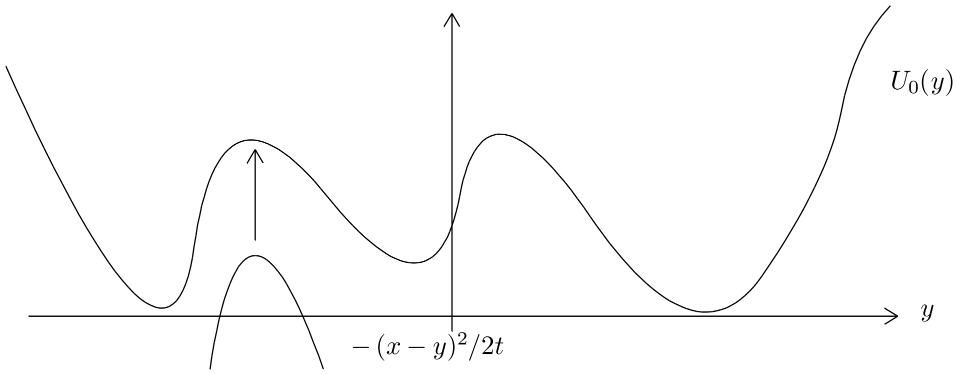

Figure 1.7.

$U_0(y) > C - (x-y)^2/2t$ is parabola is below $U_0(y)$. Then

$$U_0(y) + \frac{(x-y)^2}{2t} - C > 0,$$

where C is chosen so that the two terms "touch".

Graphical solution II: Let

$$H(x,y,t) = G(x,y,t) - \frac{x^2}{2t} = U_0(y) + \frac{(x-y)^2}{2t} - \frac{x^2}{2t} = U_0(y) + \frac{y^2}{2t} - \frac{x\,y}{t}.$$

Observe H, G have minima at same points for fixed x, t.

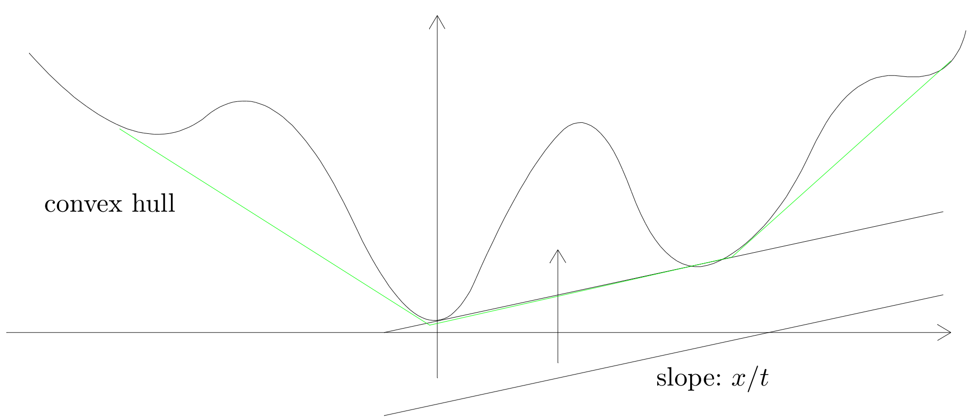

Figure 1.8.

Definition 1.7. *If* $f\colon \mathbb{R}^n \to \mathbb{R}$ *continuous, then the* convex hull *of* f *is*

$$\sup_g \{f \geqslant g\colon g \text{ convex}\}.$$

a_+, a_- defined by $U_0(y) + y^2/2t$ same as that obtained from the convex hull of $U_0(y) + y^2/2t \Rightarrow$ Irreversibility.

Remark 1.8. Suppose $U_0 \in C^2$. Observe that at a critical point of G, we have

$$\partial_y G(x, y, t) = 0,$$

which means

$$\partial_y \left[U_0(y) + \frac{(x-y)^2}{2t} \right] = 0,$$

so

$$u_0(y) + \frac{(y-x)}{t} = 0 \Rightarrow x = y + t\, u_0(y).$$

Every y such that $y + t\, u_0(y) = x$ gives a Lagrangian point that arrives at x at the time t.

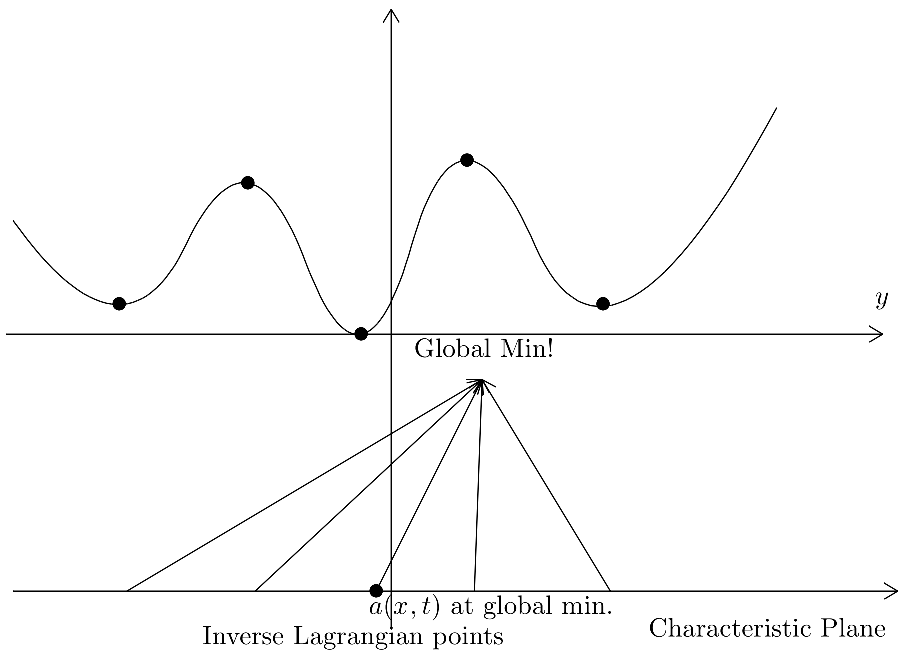

Figure 1.9.

Remark 1.9. Main point of Cole-Hopf method is that we have a solution formula independent of ε, and thus provides a uniqueness criteria for suitable solutions. (Kružkov)

- Eberhard Hopf, CPAM 1950 "The PDE $u_t + u\, u_x = \mu u_{xx}$"
- S.N. Kružkov, Math USSR Sbornik, Vol. 10, 1970 #2.

$$S_{(x,t)} = \left\{ z \in \mathbb{R}: G(x, z, t) = \min_y G \right\}$$

Proof. [Lemma 1.5] Observe that $G(x, y, t)$ is continuous in y, and

$$\lim_{|y| \to \infty} \frac{G(x, y, t)}{|y|^2} = \lim_{|y| \to \infty} \frac{(x-y)^2}{2t|y|^2} + \frac{U_0(y)}{|y|^2} = \frac{1}{2t} > 0.$$

Therefore, minima of G exist and $S_{(x,t)}$ is a bounded set for $t > 0$.

$$\begin{aligned} \Rightarrow a_-(x, t) = \inf S_{(x,t)} &> -\infty, \\ a_+(x, t) = \sup S_{(x,t)} &< \infty. \end{aligned}$$

Proof of monotinicity: Fix $x_2 > x_1$. For brevity, let $z = a_+(x_1, t)$. We'll show $G(x_2, y, t) > G(x_2, z, t)$ for any $y < z$. This shows that $\min_y G(x_2, y, t)$ can only be achieved in $[z, \infty)$, which implies $a_-(x_2, t) \geqslant z = a_+(x_1, t)$. Use definition of G:

$$\begin{aligned} G(x_2, y, t) - G(x_2, z, t) &= \frac{(x-y)^2}{2t} + U_0(y) - \frac{(x_2-z)^2}{2t} - U_0(z) \\ &= \left[\frac{(x_1-y)^2}{2t} + U_0(y)\right] - \left[\frac{(x_1-z)^2}{2t} + U_0(z)\right] + \frac{1}{2t}\big[(x_2-y)^2 - (x_1-y)^2 + (x_1 - z)^2 - (x_2-z)^2\big] \\ &= \underbrace{G(x, y, t) - G(x, z, t)}_{a)} + \frac{1}{t}\Big[\underbrace{(x_2-x_1)(z-y)}_{b)}\Big] \end{aligned}$$

$a) \geqslant 0$ because $G(x, z, t) = \min G(x, \cdot, t)$, $b) > 0$ because $x_2 > x_1$, by assumption $z > y$. By definition, $a_-(x_2, t) \leqslant a_+(x_2, t)$. So in particular,

$$a_+(x_1, t) \leqslant a_+(x_2, t),$$

so a_+ is increasing. Proof of other properties is similar. □

Corollary 1.10. *$a_-(x, t) = a_+(x, t)$ at all but a countable set of points.*

Proof. We know a_-, a_+ are increasing functions and bounded on finite sets. Therefore,

$$\lim_{y \to x_-} a_\pm(y, t), \qquad \lim_{y \to x_+} a_\pm(y, t)$$

exist at all $x \in \mathbb{R}$. Let $F = \{x: a_+(x_-, t) < a_-(x_+, t)\}$. Then F is countable.

Claim: $a_-(x, t) = a_+(x, t)$ for $x \notin F$.

$$a_+(y_1, t) \leqslant a_-(y_2, t) \leqslant a_+(y_3, t).$$

Therefore,

$$\lim_{y \to x} a_-(y, t) = a_+(x, t).$$

□

Remark 1.11. Hopf proves a stronger version of Theorem 1.6:

$$\frac{x - a_+(x, t)}{t} \leqslant \liminf_{\varepsilon \to 0, \xi \to x, \tau \to t} u^\varepsilon(\xi, \tau) \leqslant \limsup_{\varepsilon \to 0, \xi \to x, \tau \to t} u^\varepsilon(\xi, \tau) \leqslant \frac{x - a_-(x, t)}{t}.$$

Proof. (of Theorem 1.6) Use the explicit solution to write

$$u^\varepsilon(x, t) = \frac{\int_{\mathbb{R}} \frac{x-y}{t} \cdot \exp\left(\frac{-P}{2t}\right) \mathrm{d}y}{\int_{\mathbb{R}} \exp\left(\frac{-P}{2t}\right) \mathrm{d}y},$$

where $P(x, y, t) = G(x, y, t) - m(x, t)$ with $m(x, t) = \min_y G$.

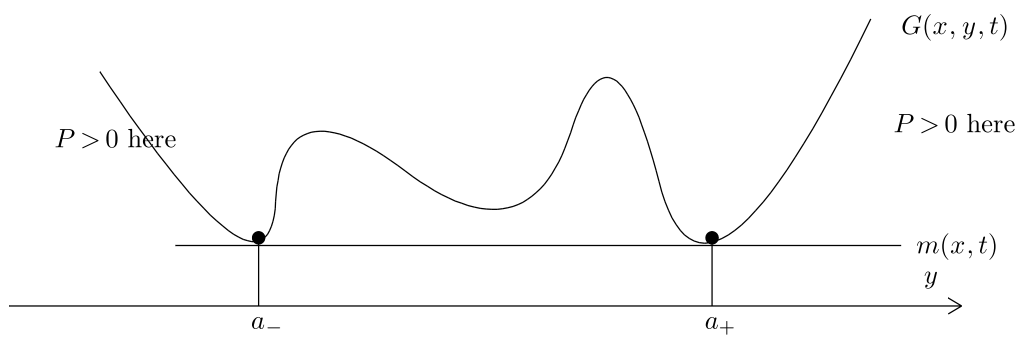

Figure 1.10.

Fix x, t. Fix $\eta > 0$, let a_+ and a_- denote $a_+(x,t)$ and $a_-(x,t)$. Let

$$\begin{aligned} l &:= \frac{x - a_+ - \eta}{t} \\ &\leqslant \frac{x - a_- - \eta}{t} =: L. \end{aligned}$$

Lower estimate

$$\liminf_{e \to 0} u^\varepsilon(x,t) \geqslant \frac{x - a_+}{t} - \eta.$$

Consider

$$u^\varepsilon(x,t) - l = \frac{\int_{\mathbb{R}} \left(\frac{x-y}{t} - l\right) \cdot \exp\left(\frac{-P}{2\varepsilon}\right) \mathrm{d}y}{\int_{\mathbb{R}} \exp\left(\frac{-P}{2\varepsilon}\right) \mathrm{d}y} = \frac{\int_{\mathbb{R}} \left(\frac{a_+ + \eta - y}{t} - l\right) \cdot \exp\left(\frac{-P}{2\varepsilon}\right) \mathrm{d}y}{\int_{\mathbb{R}} \exp\left(\frac{-P}{2\varepsilon}\right) \mathrm{d}y}.$$

Estimate the numerator as follows:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{a_+ + \eta - y}{t} \cdot \exp\left(\frac{-P}{2\varepsilon}\right) \mathrm{d}y &= \underbrace{\int_{-\infty}^{a_+}}_{\geqslant 0} + \int_{a_+}^{\infty} \\ \int_{\mathbb{R}} &\geqslant \int_{a_+ + \eta}^{\infty} \frac{a_+ + \eta - y}{t} \exp\left(\frac{-P}{2\varepsilon}\right) \mathrm{d}y \end{aligned}$$

On the interval $y \in [a_+ + \eta, \infty]$, we have the uniform lower bound

$$\frac{P(x,y,t)}{(y - a_+)^2} \geqslant \frac{A}{2} > 0$$

for some constant A depending only on η. Here we use

$$\frac{P(x,y,t)}{|y|^2} = \frac{U_0(y)}{|y|^2} + \frac{(x-y)^2}{2t|y|^2} - \frac{m(x,t)}{|y|^2} \to \frac{1}{2t} > 0$$

as $|y| \to \infty$. We estimate

$$\begin{aligned} \int_{a_+ + \eta}^{\infty} \frac{|a_+ + \eta - y|}{t} e^{-P/2\varepsilon} \mathrm{d}y &\leqslant \int_{a_+ + \eta}^{\infty} \frac{|a_+ + \eta - y|}{t} \exp\left(-\frac{A}{4\varepsilon}(y - a_+)^2\right) \mathrm{d}y \\ &= \int_{\eta}^{\infty} \frac{(y - \eta)}{t} \exp\left(-\frac{A y^2}{4\varepsilon}\right) \mathrm{d}y \\ &< \int_{\eta}^{\infty} \frac{y}{t} \exp\left(-\frac{A y^2}{4\varepsilon}\right) \mathrm{d}y \\ &= \frac{1}{t} \frac{\varepsilon}{A} \int_{\sqrt{\frac{A}{\varepsilon}}\eta}^{\infty} y e^{-y^2/2} \mathrm{d}y = \frac{1}{t} \cdot \frac{\varepsilon}{A} e^{-\frac{A\eta^2}{2\varepsilon}}. \end{aligned}$$

For the denominator,

$$\int_{\mathbb{R}} \exp\left(\frac{-P}{2\varepsilon}\right) \mathrm{d}y:$$

Since P is continuous, and $P(x, a_+, t) = 0$, there exists δ depending only on η such that

$$P(x,y,t) \leqslant \frac{A}{2} \eta$$

for $y \in [a_+, a_+ + \delta]$. Thus,

$$\int_{\mathbb{R}} e^{-P/2\varepsilon} \mathrm{d}y \geqslant \int_{a_+}^{a_+ + \delta} e^{-P/2\varepsilon} \mathrm{d}y \geqslant \int_{a_+}^{a_+ + \delta} e^{-(A/2\varepsilon)\eta^2} \mathrm{d}y = \delta e^{-(A/2\varepsilon)\eta^2}.$$

Combine our two estimates to obtain

$$u^\varepsilon(x,t) - l \geqslant \frac{-\varepsilon e^{-(A/2\varepsilon)\eta^2}}{A\, t\, \delta e^{-(A/2\varepsilon)\eta^2}} = -\varepsilon \cdot \frac{1}{A\, t \delta}.$$

Since A, δ depend only on η,

$$\liminf_{\varepsilon \to 0} u^{\varepsilon}(x, t) \geqslant l = \frac{x - a_+ - \eta}{t}.$$

Since $\eta > 0$ arbitrary,

$$\liminf_{\varepsilon \to 0} u^{\varepsilon}(x, t) = \frac{x - a_+}{t}.$$

□

Corollary 1.12. *$\lim_{\varepsilon \to 0} u^{\varepsilon}(x, t)$ exists at all but a countable set of points and defines $u \in \mathrm{BV}_{\mathrm{loc}}$ with left and right limits at all $x \in \mathbb{R}^n$.*

Proof. We know

$$a_+(x, t) = a_-(x, t)$$

at all but a countable set of shocks. So,

$$\lim_{\varepsilon \to 0} u^{\varepsilon}(x, t) = \frac{x - a_+(x, t)}{t} = \frac{x - a_-(x, t)}{t}$$

at these points. $\mathrm{BV}_{\mathrm{loc}}$ because we have the difference of increasing functions. □

Corollary 1.13. *Suppose $u_0 \in \mathrm{BC}(\mathbb{R})$ (bounded, continuous). Then*

$$u(\cdot, t) = \lim_{\varepsilon \to 0} u^{\varepsilon}(\cdot, t) \in \mathrm{BC}(\mathbb{R}).$$

and u is a weak solution to

$$u_t + \left(\frac{u^2}{2}\right)_x = 0.$$

Proof. Suppose $\varphi \in C_c^{\infty}(\mathbb{R} \times (0, \infty))$. Then we have

$$\varphi\left(u_t^{\varepsilon} + \left(\frac{u^{\varepsilon}}{2}\right)_x\right) = (e\, u_{xx}^{\varepsilon})\varphi$$

$$\int_0^{\infty} \int_{\mathbb{R}} \left[\varphi_t u^{\varepsilon} + \varphi_x \frac{(u^{\varepsilon})^2}{2}\right] \mathrm{d}x\, \mathrm{d}t = \varepsilon \int_0^{\infty} \int_{\mathbb{R}} \varphi_{xx} u^{\varepsilon} \mathrm{d}x\, \mathrm{d}t.$$

We want

$$-\int_0^{\infty} \left[\varphi_t u + \varphi_x \frac{u^2}{2}\right] \mathrm{d}x\, \mathrm{d}t = 0.$$

Suppose

$$u_t^{\varepsilon} + u^{\varepsilon} u_x^{\varepsilon} = \varepsilon u_{xx}^{\varepsilon}, \quad u^{\varepsilon}(x, 0) \in \mathrm{BC}(\mathbb{R}).$$

Maximum principle yields

$$\|u^{\varepsilon}(\cdot, t)\|_{L^{\infty}} \leqslant \|u_0\|_{L^{\infty}}.$$

Use DCT+$\lim_{\varepsilon \to 0} u^{\varepsilon}(x, t) = u$ a.e. to pass to limit. □

1.3 Two basic examples of Solutions

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

$u(x, 0) = u_0(x)$, $U_0(x) = \int_0^x u_0(y) \mathrm{d}y$. Always consider the Cole-Hopf solution.

$$u(x, t) = \frac{x - a(x, t)}{t},$$

$$a(x, t) = \operatorname{argmin} \underbrace{\frac{(x - y)^2}{2t} + U_0(y)}_{G(x, y, t)}.$$

Example 1.14. $u_0(x) = \mathbf{1}_{\{x>0\}}$. Here,

$$U_0(y) = \int_0^y \mathbf{1}_{\{y'>0\}} \mathrm{d}y' = y\mathbf{1}_{\{y>0\}}$$

Then

$$G(x,y,t) = \frac{(x-y)^2}{2t} + y\mathbf{1}_{\{y>0\}} \geqslant 0,$$

and

$$G(x,y,t) = 0 = x\mathbf{1}_{\{x>0\}} = 0$$

if $x \leqslant 0$. So, $a = x$ for $x \leqslant 0$. Differentiate G and set $= 0$

$$0 = \frac{y-x}{t} + 1 \quad (\text{assuming } y > 0)$$

So, $y = x - t$. Consistency: need $y > 0 \Rightarrow x > t$. Gives $u(x,t) = 1$ for $x > t$.

$$\begin{aligned} G(x,y,t) &= \frac{x^2}{2t} + \frac{y^2}{2t} - \frac{x\,y}{t} + y\mathbf{1}_{\{y>0\}} \\ &= \frac{x^2}{2t} + \frac{y^2}{2t} + y\Big(\mathbf{1}_{\{y>0\}} - \frac{x}{t}\Big). \end{aligned}$$

Consider $0 < x/t < 1$, $t > 0$. *Claim:* $G(x,y,t) \geqslant x^2/2t$ and $a = 0$.

- Case I: $y < 0$, then $G(x,y,t) - x^2/2t = y^2/2t - x\,y/t > 0$.
- Case II: $y > 0$, then $G(x,y,t) - x^2/2t = y^2/2t + (1 - x/t)y > 0$.

$$a(x,t) = \begin{cases} x & x \leqslant 0, \\ 0 & 0 < x \leqslant t, \\ x - t & x \geqslant t. \end{cases}$$

Then

$$u(x,t) = \frac{x - a(x,t)}{t} = \begin{cases} 0 & x \leqslant 0, \\ x/t & 0 < x \leqslant t, \\ 1 & t \leqslant x. \end{cases}$$

Example 1.15. $u_0(x) = -\mathbf{1}_{\{x>0\}}$. Then

$$u(x,t) = -\mathbf{1}_{\{x > -t/2\}}.$$

Shock path: $x = -t/2$.

Here are some properties of the Cole-Hopf solution:

- $u(\cdot, t) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}) \ \rightarrow$ difference of two increasing functions
- $u(x_-, t)$ and $u(x_+, t)$ exist at all $x \in \mathbb{R}$. And $u(x_-, t) \geqslant u(x_+, t)$. In particular,

  $$u(x_-, t) > u(x_+, t)$$

  at jumps. This is the *Lax-Oleinik entropy condition*. It says that chracteristics always enter a shock, but never leave it.
- Suppose $u(x_-, t) > u(x_+, t)$. We have the Rankine-Hugoniot condidtion:

  $$\text{Velocity of shock} = \frac{\left[\!\left[\frac{u^2}{2}\right]\!\right]}{[\![u]\!]} = \frac{1}{2}(u(x_+, t) + u(x_-, t)).$$

  Claim: If x is a shock location

  $$\frac{1}{2}(u(x_-, t) + u(x_+, t)) = \frac{1}{a(x_+, t) - a(x_-, t)} \int_{a_-}^{a_+} u_0(y)\mathrm{d}y.$$

  $$\underbrace{(a_+ - a_-)(\text{velocity of shock})}_{\text{final momentum}} = \underbrace{\int_{a_-}^{a_+} u_0(y)\mathrm{d}y}_{\text{initial momentum}}$$

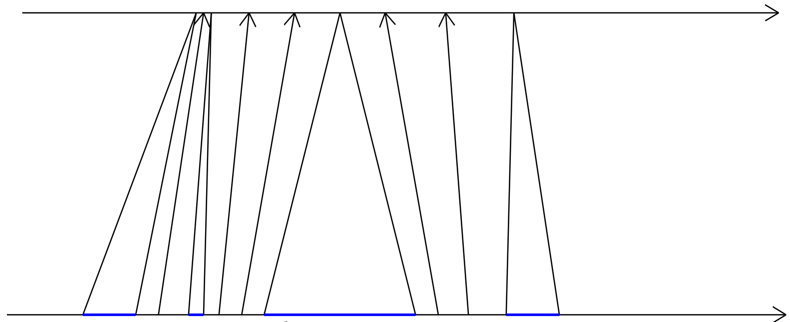

Figure 1.11. The “clustering picture”.

1.4 Entropies and Admissibility Criteria

$$\begin{aligned} u_t + D\cdot(f(u)) &= 0 \\ u(x,0) &= u_0(x) \end{aligned}$$

for $x\in\mathbb{R}^n$, $t>0$. Many space dimensions, but u is a scalar $u\colon \mathbb{R}^n\times(0,\infty)\to\mathbb{R}$, $f\colon\mathbb{R}\to\mathbb{R}^n$ (which we assume to be C^1, but which usually is C^∞). Basic calculation: Suppose $u\in C_c^\infty(\mathbb{R}^n\times[0,\infty))$, and also suppose we have a convex function $\eta\colon\mathbb{R}\to\mathbb{R}$ (example: $\eta(u)=u^2/2$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\eta(u)\mathrm{d}x = \int_{\mathbb{R}^n}\eta'(u)\,u_t\mathrm{d}x = -\int_{\mathbb{R}^n}\eta'(u)D_x(f(u))\mathrm{d}x.$$

Suppose we have a function $q\colon\mathbb{R}\to\mathbb{R}^n$ such that

$$D_x q(u) \;=\; \eta'(u)D_x(f(u)),$$

i.e.

$$\begin{aligned} \partial_{x_1}q_1(u)+\partial_{x_2}q_2(u)+\cdots+\partial_{x_n}q_n(u) &= q_1'u_{x_1}+q_2'u_{x_2}+\cdots+q_n'u_{x_n} \\ &\overset{\text{RHS}}{=} \eta'(u)f_1'u_{x_1}+\eta'(u)f_2'u_{x_2}+\cdots+\eta'(u)f_n'u_{x_n}. \end{aligned}$$

Always holds: Simply define $q_i'=\eta'(u)f_i'$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\eta(u)\mathrm{d}x = -\int_{\mathbb{R}^n}D_x\cdot(q(u))\mathrm{d}x = 0$$

provided $q(u)=0$.

Example 1.16. Suppose $u_t+u\,u_x=0$. Here $f'(u)=u$. If $\eta(u)=u^2/2$, $q'(u)=\eta'(u)f'(u)=u^2$. So, $q(u)=u^3/3$. Smooth solution to Burgers Equation:

$$\partial_t\left(\frac{u^2}{2}\right)+\partial_x\left(\frac{u^3}{3}\right)=0.$$

(called the *companion balance law*) And

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{u^2}{2}\mathrm{d}x=0,$$

which is *conservation of energy*.

Consider what happens if we add viscosity

$$\begin{aligned} u_t^\varepsilon + D_x\cdot(f(u^\varepsilon)) &= \varepsilon\Delta u^\varepsilon, \\ u^\varepsilon(x,0) &= u_0(x). \end{aligned}$$

In this case, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\eta(u^\varepsilon)\mathrm{d}x &= \int_{\mathbb{R}^n}\eta'(u^\varepsilon)u_t^\varepsilon\mathrm{d}x = \underbrace{-\int_{\mathbb{R}^n}D_x\cdot(q(u^\varepsilon))\mathrm{d}x}_{=0} + \varepsilon\int_{\mathbb{R}^n}\eta'(u^\varepsilon)\mathrm{d}x\\ &= -\varepsilon\int_{\mathbb{R}^n}\underbrace{\eta'(u^\varepsilon)}_{\geqslant 0}|Du^\varepsilon|^2\mathrm{d}x < 0\end{aligned}$$

because η is *convex*. If a solution to our original system is $\lim_{\varepsilon\to 0} u^\varepsilon$ of solutions of the viscosity system, we must have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\eta(u)\mathrm{d}x \leqslant 0.$$

Fundamental convex functions (*Kružkov entropies*): $(u-k)_+$, $(k-u)_+$, $|u-k|$.

Definition 1.17. (Kružkov) *A function $u\in L^\infty(\mathbb{R}^n\times(0,\infty))$ is an* entropy *(or* admissible*)* solution *to the original system, provided*

1. *For every $\varphi\in C_c^\infty(\mathbb{R}^n\times(0,\infty))$ with $\varphi\geqslant 0$ and every $k\in\mathbb{R}$ we have*

$$\int_0^\infty [|u-k|\varphi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\cdot D_x\varphi]\mathrm{d}x\,\mathrm{d}t \geqslant 0. \tag{1.3}$$

2. *There exists a set of measure zero such that for $t\notin F$, $u(\,\cdot\,,t)\in L^\infty(\mathbb{R}^n)$ and for any ball $B(x,r)$*

$$\lim_{t\to 0, t\in F}\int_{B(x,r)}|u(y,t)-u_0(y)|\mathrm{d}y = 0.$$

An alternative way to state Condition 1 above is as follows: For every (entropy, entropy-flux) pair (η, q), we have

$$\partial_t\eta(u) + \partial_x(q(u)) \leqslant 0 \tag{1.4}$$

in $\mathcal{D}'$. Recover (1.3) by choosing $\eta(u)=|u-k|$. (1.3)$\Rightarrow$(1.4) because all convex η can be generated from the fundamental entropies.

(1.3) means that if we multiply by $\varphi\geqslant 0$ and integrate by parts we have

$$-\int_0^\infty\int_{\mathbb{R}^n}[\varphi_t\eta(u) + D_x\varphi\cdot q(u)]\mathrm{d}x\,\mathrm{d}t \leqslant 0.$$

Positive distributions are measures, so

$$\partial_t\eta(u) + \partial_x(q(u)) = -m_\eta,$$

where m_η is some measure that depends on η. To be concrete, consider Burgers equation and $\eta(u)=u^2/2$ (energy). Dissipation in Burgers equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\frac{(u^\varepsilon)^2}{2}\mathrm{d}x &= -\int_{\mathbb{R}}(u^\varepsilon)^2u_x^\varepsilon + \varepsilon\int_{\mathbb{R}}u^\varepsilon u_{xx}^\varepsilon\mathrm{d}x\\ &= -\varepsilon\int_{\mathbb{R}}(u_x^\varepsilon)^2\mathrm{d}x.\end{aligned}$$

But what is the limit of the integral term as $\varepsilon\to 0$? Suppose we have a situation like in the following figure:

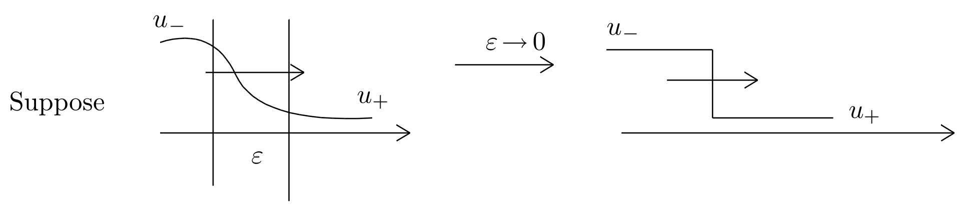

Figure 1.12.

Traveling wave solution is of the form

$$u^\varepsilon(x,t) = v\left(\frac{x - c\,t}{\varepsilon}\right),$$

where $c = [\![f(u)]\!]/[\![u]\!] = (u_- + u_+)/2$. And

$$-c\,v' + \left(\frac{v^2}{2}\right)' = v''.$$

Integrate and obtain

$$-c(v - u_-) + \frac{v^2}{2} - \frac{u_-^2}{2} = v'.$$

For a traveling wave

$$\begin{aligned} \varepsilon \int_{\mathbb{R}} (u_x^\varepsilon)^2 \mathrm{d}x &= \frac{\varepsilon}{\varepsilon} \int_{\mathbb{R}} \left(v'\left(\frac{x - c\,t}{\varepsilon}\right)\right)^2 \frac{\mathrm{d}x}{\varepsilon} \\ &= \int_{\mathbb{R}} (v')^2 \mathrm{d}x \end{aligned}$$

independent of ε! In fact,

$$\begin{aligned} \int_{\mathbb{R}} (v')^2 \mathrm{d}x &= \int_{\mathbb{R}} v' \cdot \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x \\ &= \int_{u_-}^{u_+} \left[-c(v - u_-) + \left(\frac{v^2}{2} - \frac{u_-^2}{2}\right)\right] \mathrm{d}v \\ &= (u_- - u_+)^3 \int_0^1 s(1-s)\mathrm{d}s = \frac{(u_- - u_+)^3}{6}. \end{aligned}$$

Always have $u_- > u_+$. Heuristic picture:

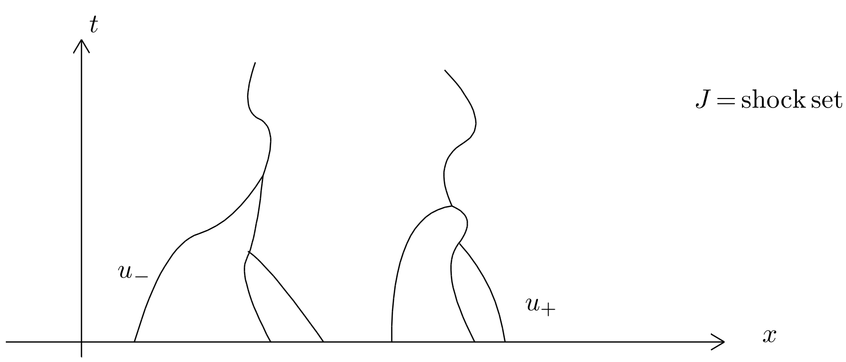

Figure 1.13.

The dissipation measure is concentrated on J and has density

$$\frac{(u_+ - u_-)^2}{6}.$$

1.5 Kružkov's uniqueness theorem

In what follows, $Q = \mathbb{R}^n \times (0, \infty)$. Consider entropy solutions to

$$\begin{aligned} u_t + D_x \cdot (f(u)) &= 0 \quad (x,t) \in Q \\ u(x,0) &= u_0(x) \end{aligned}$$

Here, $u: Q \to \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$, $M := \|u\|_{L^\infty(Q)}$. Characteristics:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f'(u) \quad \text{or} \quad \frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(u), \quad i = 1, \ldots, n.$$

Let $c_* = \sup_{u \in [-M, M]} |f'(u)|$ be the maximum speed of characteristics. Consider the area given by

$$K_R = \left\{ (x, t) \colon |x| \leqslant R - c_* t, 0 \leqslant t \leqslant \frac{R}{c_*} \right\}$$

Define $r := R / c_*$.

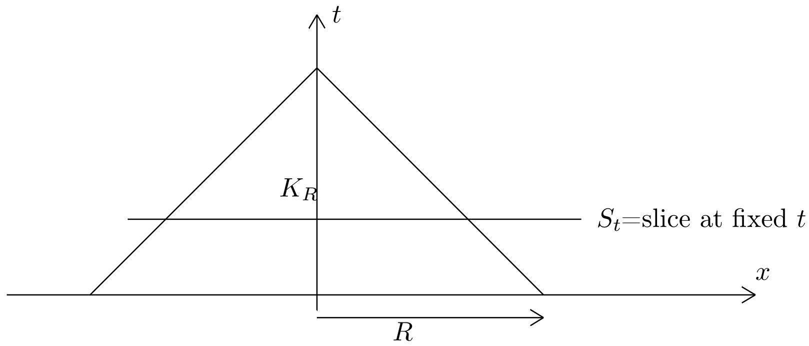

Figure 1.14.

Theorem 1.18. (Kružkov, 1970) *Suppose u, v are entropy solutions to the system such that*

$$\|u\|_{L^\infty(Q)}, \|v\|_{L^\infty(Q)} \leqslant M.$$

Then for almost every $t_1 < t_2$, $t_i \in [0, T]$, we have

$$\int_{S_{t_2}} |u(x, t_2) - v(x, t_2)| \mathrm{d}x \leqslant \int_{S_{t_1}} |u(x, t_1) - v(x, t_1)| \mathrm{d}x.$$

In particular, for a.e. $t \in [0, T]$

$$\int_{S_t} |u(x, t) - v(x, t)| \leqslant \int_{S_0} |u_0(x) - v_0(x)| \mathrm{d}x.$$

Corollary 1.19. *If $u_0 = v_0$, then $u = v$. (I.e. entropy solutions are unique, if they exist.)*

Proof. Two main ideas:

- doubling trick,
- clever choice of test functions.

Recall that if u is an entropy solution for every $\varphi \geqslant 0$ in $C_0^\infty(Q)$ and every $k \in \mathbb{R}$, we have

$$\int_Q [|u(x, t) - k| \varphi_t + \operatorname{sgn}(u - k)(f(u) - f(k)) \cdot D_x \varphi] \mathrm{d}x \, \mathrm{d}t \geqslant 0$$

Fix y, τ such that $v(y, \tau)$ is defined, let $k = v(y, \tau)$.

$$\int_Q [|u(x, t) - v(y, \tau)| \varphi_t + \operatorname{sgn}(u - v)(f(u) - f(v)) \cdot D_x \varphi] \mathrm{d}x \, \mathrm{d}t \geqslant 0.$$

This holds for (y, τ) a.e., so we have

$$\int_Q \int_Q [\text{as above}] \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}y \, \mathrm{d}\tau \geqslant 0.$$

Moreover, this holds for every $\varphi \in C_c^\infty(Q \times Q)$, with $\varphi \geqslant 0$. We also have a symmetric inequality with φ_τ, $D_y \varphi$ instead of φ_t, $D_x \varphi$. Add these to obtain

$$\int_Q \int_Q [|u(x, t) - v(y, \tau)| (\varphi_t + \varphi_\tau) + \operatorname{sgn}(u - v)(f(u) - f(v)) \cdot (D_x \varphi + D_y \varphi)] \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}y \, \mathrm{d}\tau \geqslant 0.$$

This is what is called the *doubling trick*. Fix $\psi \subset C_c^\infty(Q)$ and a "bump" function $\eta\colon \mathbb{R} \to \mathbb{R}$ with $\eta \geqslant 0$, $\int_{\mathbb{R}} \eta \mathrm{d}r = 1$. For $h > 0$, let $\eta_h(r) := 1/h\, \eta(r/h)$. Let

$$\psi(x,t,y,\tau) = \psi\left(\frac{x+y}{2}, \frac{t+\tau}{2}\right)\lambda_h\left(\frac{x-y}{2}, \frac{t-\tau}{2}\right)$$

where

$$\underbrace{\lambda_h(z,s)}_{\text{Approximate identity in } \mathbb{R}^n} = \eta_h(s) \prod_{i=1}^n \eta_h(z_i).$$

$$\begin{aligned} \varphi_t &= \frac{1}{2}\psi_t \cdot \lambda_h + \frac{1}{2}\psi(\lambda_h)_t \\ \varphi_\tau &= \frac{1}{2}\psi_t \lambda_h - \frac{1}{2}\psi(\lambda_h)_t \end{aligned}$$

Adding the two cancels out the last term:

$$\varphi_t + \varphi_\tau = \lambda_h \psi_t.$$

Similarly,

$$D_x\varphi + D_y\varphi = \lambda_h D_x\psi.$$

We then have

$$\int_Q \int_Q \lambda_h\left(\frac{x-y}{2}, \frac{t-\tau}{2}\right)\left[|u(x,t) - v(y,\tau)|\psi_t\left(\frac{x+y}{2}, \frac{t+\tau}{2}\right) + \operatorname{sgn}(u-v)(f(u) - f(v))D_x\psi\right]\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}\tau \geqslant 0$$

λ_h concentrates at $x = y$, $t = \tau$ as $h \to 0$.

Technical step 1. Let $h \to 0$. (partly outlined in homework, Problems 6 & 7)

$$\int_Q [|u(x,t) - v(x,t)|\psi_t + \operatorname{sgn}(u-v)(f(u) - f(v)) \cdot D_x\psi]\mathrm{d}x\,\mathrm{d}t \geqslant 0 \tag{1.5}$$

[To prove this step, use Lebesgue's Differentiation Theorem.]

Claim: (1.5) $\Rightarrow L^1$ stability estimate. Pick two test functions:

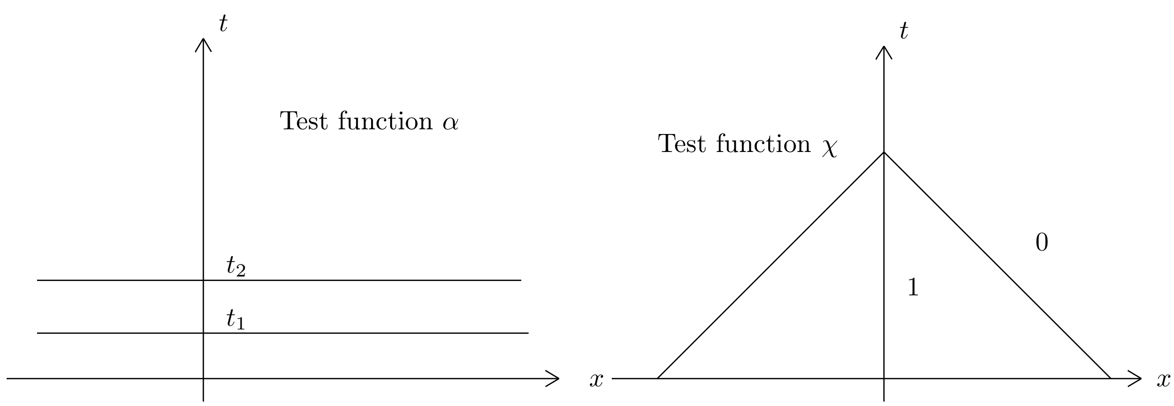

Figure 1.15.

Let

$$\alpha_h(t) = \int_{-\infty}^t \eta_h(r)\mathrm{d}r.$$

Choose

$$\psi(x,t) = (\alpha_h(t - t_1) - \alpha_h(t - t_2))\chi_\varepsilon(x,t).$$

where

$$\chi_\varepsilon = 1 - \alpha_\varepsilon(|x| + c_* t - R + \varepsilon).$$

Observe that

$$(\chi_\varepsilon)_t = -\alpha_\varepsilon'(c_*) \leqslant 0, \quad D_x\chi_\varepsilon = -\alpha_\varepsilon' \cdot \frac{x}{|x|}.$$

Therefore

$$(\chi_\varepsilon)_t + c_*|D_x\chi_\varepsilon| = -\alpha_\varepsilon' c_* + \alpha_\varepsilon' c_* = 0.$$

Drop ε:

$$\begin{aligned} & |u-v|\chi_t + \mathrm{sgn}(u-v)(f(u)-f(v))\cdot D_x\chi \\ = & |u-v|\left[\chi_t + \frac{f(u)-f(v)}{u-v}\cdot D_x\chi\right] \leqslant |u-v|[\chi_t + c_*|D_x\chi|] = 0 \quad (\#\#) \end{aligned}$$

Substitute for ψ and use (##) to find

$$\int_Q (\alpha_h'(t-t_1) - \alpha_h'(t-t_2))|u-v|\chi \,\mathrm{d}x\,\mathrm{d}t \geqslant 0$$

$\Rightarrow L^1$ contraction. □

2 Hamilton-Jacobi Equations

$$u_t + H(x, Du) = 0$$

for $x \in \mathbb{R}^n$ and $t > 0$, with $u(x,0) = u_0(x)$. Typical application: Curve/surface evolution. (Think fire front.)

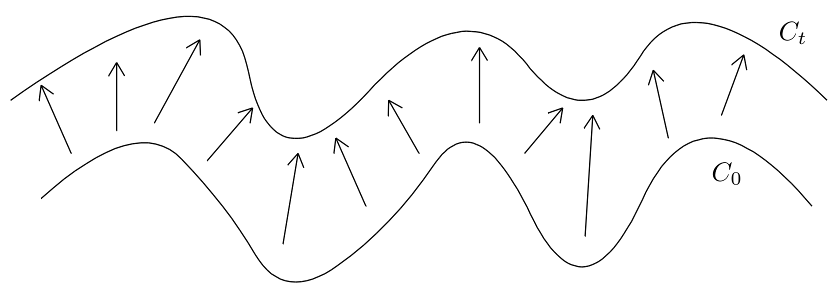

Figure 2.1.

Example 2.1. *(A curve that evolves with unit normal velocity)* If C_t is given as a graph $u(x,t)$. If τ is a tangential vector, then

$$\tau = \frac{(1, u_x)}{\sqrt{1+u_x^2}}.$$

Let $\dot{y} = u_t(x,t)$. So the normal velocity is

$$v_n = (0, \dot{y}) \cdot \nu,$$

where ν is the normal.

$$\nu = \frac{(u_x, -1)}{\sqrt{1+u_x^2}}.$$

Then $v_n = 1 \Rightarrow \dot{y}/\sqrt{1+u_x^2} = -1 \Rightarrow u_t = -\sqrt{1+u_x^2}$.

$$u_t + \sqrt{1+u_x^2} = 0$$

H is the *Hamiltonian*, which in this case is $\sqrt{1+u_x^2}$. In $\mathbb{R}^n$

$$u_t + \sqrt{1+|D_x u|^2} = 0,$$

a graph in $\mathbb{R}^n$.

Other rules for normal velocity can lead to equations with very different character.

Example 2.2. *(Motion by mean curvature)* Here $v_n = -\kappa$ (mean curvature).

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

$v_n = -\kappa$. Then

$$\frac{-u_t}{\sqrt{1+u_x^2}} = \frac{-u_{xx}}{(1+u_x)^{3/2}}.$$

So the equation is

$$u_t = \frac{u_{xx}}{(1+u_x^2)},$$

which is parabolic. Heuristics:

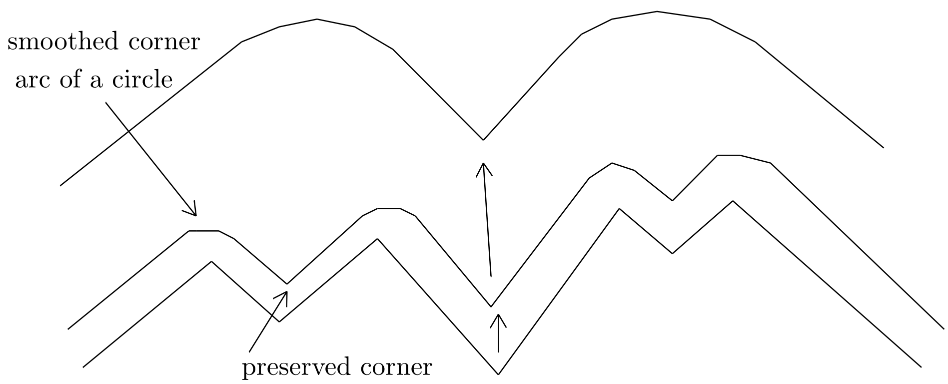

Figure 2.2.

If $(x,y) \in C_t$, then $\mathrm{dist}((x,y), C_0) = t$. Also

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0 \quad \xrightarrow{\text{integrate}} \quad U_t + \frac{U_x^2}{2} = 0.$$

2.1 Other motivation: Classical mechanics/optics

cf. Evans, chapter 3.3

- Newton's second law — $F = m\,a$
- Lagrange's equations
- Hamilton's equations

Lagrange's equations: State of the system $x \in \mathbb{R}^n$ or $\mathcal{M}^n$ (which is the configuration space). Then

$$L(x, \dot{x}, t) = \underbrace{T}_{\text{kinetic}} - \underbrace{U(x)}_{\text{potential}}.$$

Typically, $T = \frac{1}{2} x \cdot M x$, where M is the (pos.def.) mass matrix.

Hamilton's principle: A path in configuration space between fixed states $x(t_0)$ and $x(t_1)$ *minimizes* the action

$$S(\Gamma) = \int_{t_0}^{t_1} L(x, \dot{x}, t) \mathrm{d}t$$

over all paths $x(t) = \Gamma$.

Theorem 2.3. *Assume L is C^2. Fix $x(t_0)$, $x(t_1)$. If Γ is an extremum of S then*

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} = 0.$$

Proof. ("Proof") Assume that there is an optimal path $x(t)$. Then consider a perturbed path that respects the endpoints:

$$x_\varepsilon(t) = x(t) + \varepsilon\varphi(t)$$

with $\varphi(t_0) = \varphi(t_1) = 0$. Sicne $x(t)$ is an extremem of action,

$$\frac{\mathrm{d}S}{\mathrm{d}\varepsilon}(x(t) + \varepsilon\varphi(t))|_{\varepsilon=0} = 0.$$

So

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{t_0}^{t_1} L(x + \varepsilon\varphi, \dot{x} + \varepsilon\dot{\varphi}, t)\mathrm{d}t,$$

which results in

$$\begin{aligned}\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial x}(x,\dot{x},t)\varphi + \frac{\partial L}{\partial \dot{x}}(x,\dot{x},t)\dot{\varphi}\right]\mathrm{d}t &= 0\\ \Rightarrow \int_{t_0}^{t_1}\varphi(t)\left[\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\mathrm{d}t + \underbrace{\frac{\partial L}{\partial \dot{x}}\varphi|_{t_0}^{t_1}}_{=0} &= 0\end{aligned}$$

Since φ was arbitrary,

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} = 0. \qquad \square$$

Typical example: N-body problem

$$x = (y_1, \ldots, y_N), \quad y_i \in \mathbb{R}^3.$$

Then

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i|y_i|^2$$

and $U(x) =$ given potential, $L = T - U$. So

$$m_i\ddot{y}_{i,j} = -\frac{\partial U}{\partial y_{i,j}} \quad i = 1, \ldots, N, \quad j = 1, \ldots, 3.$$

2.1.1 Hamilton's formulation

$$H(x,p,t) = \underbrace{\sup_{y\in\mathbb{R}^n}(p\,y - L(x,y,t))}_{\text{Legendre transform}}$$

Then

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial p},\\ \dot{p} &= -\frac{\partial H}{\partial x},\end{aligned}$$

called *Hamilton's equations*. They end up being $2N$ first-order equations.

Definition 2.4. *Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is convex. Then the* Legendre transform *is*

$$\begin{aligned}f^*(p) &:= \sup_{x\in\mathbb{R}^n}(p\cdot x - f(x))\\ &= \max_{x\in\mathbb{R}^n}(\ldots) \quad \textit{if} \quad \frac{f(x)}{|x|} \to \infty \quad \textit{as} \quad |x| \to \infty.\end{aligned}$$

Example 2.5. $f(x) = \frac{1}{2}m\,x^2$, $m > 0$ and $x \in \mathbb{R}$.

$$(p\,x - f(x))' = 0 \Rightarrow (p - m\,x) = 0 \Rightarrow x = \frac{p}{m}.$$

And

$$f^*(p) = p\cdot\frac{p}{m} - \frac{1}{2}m\left(\frac{p}{m}\right)^2 = \frac{1}{2}\frac{p^2}{m}.$$

Example 2.6. $f(x) = \frac{1}{2}x \cdot M x$, where M is pos.def. Then

$$f^*(p) = \frac{1}{2}p \cdot M^{-1}p.$$

Example 2.7. Suppose $f(x) = x^\alpha/\alpha$ with $1 < \alpha < \infty$.

$$f^*(p) = \frac{p^\beta}{\beta}, \quad \text{where } \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Young's inequality and

$$f^*(p) + f(x) \geqslant p\,x$$

imply

$$\frac{x^\alpha}{\alpha} + \frac{p^\beta}{\beta} \geqslant p\,x.$$

Example 2.8.

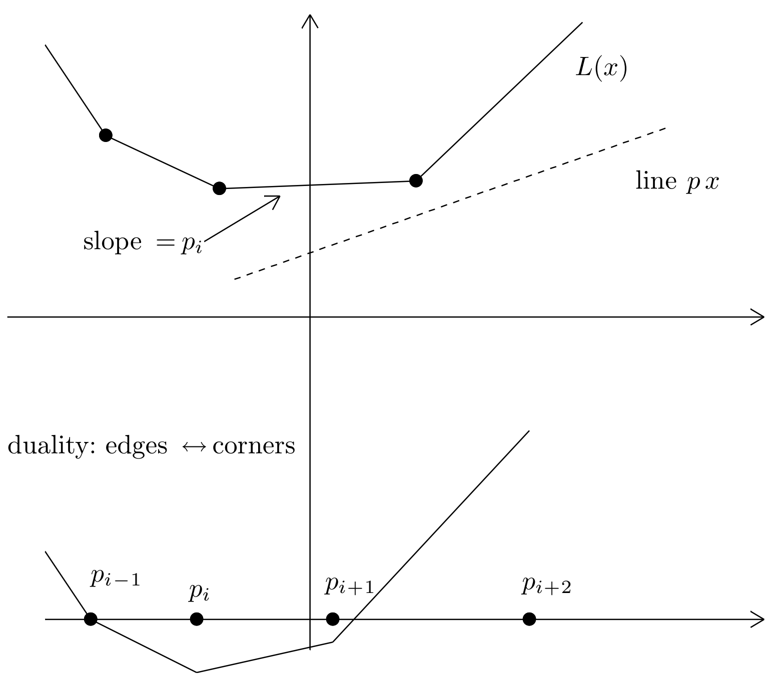

Figure 2.3.

Theorem 2.9. *Assume L is convex. Then $L^{**} = L$.*

Proof. see Evans. Sketch:

- If L_k is piecewise affine, then $L_k^{**} = L_k$ can be verified explicitly.
- Approximation: If $L_k \to L$ locally uniformly, then $L_k^* \to L^*$ locally uniformly. □

Back to Hamilton-Jacobi equations:

$$u_t + H(x, D_x u, t) = 0.$$

H is always assumed to be

- $C^2(\mathbb{R}^n \times \mathbb{R}^n \times [0, \infty))$,
- uniformly convex in $p = D_x u$,

- uniformly superlinear in p.

2.1.2 Motivation for Hamilton-Jacobi from classical mechanics

Principle of least action: For every path connecting $(x_0, t_0) \rightarrow (x_1, t_1)$ associate the 'action'

$$S(\Gamma) = \int_\Gamma L(x, \dot{x}, t)\mathrm{d}t.$$

L Lagrangian, convex, superlinear in $\dot{x}$. Least action $\Rightarrow$ Lagrange's equations:

$$-\frac{\mathrm{d}}{\mathrm{d}t}[D_{\dot{x}}L(x, \dot{x}, t)] + D_x L = 0 \tag{2.1}$$

$x \in \mathbb{R}^n \Rightarrow n$ 2nd order ODE.

Theorem 2.10. *("Theorem") (2.1) is equivalent to*

$$\dot{x} = D_p H, \quad \dot{p} = -D_x H. \tag{2.2}$$

Note that those are $2n$ *first order ODEs.*

Proof. ("Proof")

$$H(x, p, t) = \max_{v \in \mathbb{R}^n} (v p - L(x, v, t)).$$

Maximum is attained when

$$p = D_v L(x, v, t), \tag{2.3}$$

and the solution is unique because of convexity.

$$H(x, p, t) = v(x, p, t) - L(x, v(x, p, t), t),$$

where v solves (2.3).

$$\begin{aligned} D_p H &= v + p\, D_p v - D_v L \cdot D_p v \\ &= v + \underbrace{(p - D_v L)}_{=0 \text{ because of } (2.3)} D_p v \\ &= v. \end{aligned}$$

Thus $\dot{x} = D_p H$. Similarly, we use (2.1)

$$\frac{\mathrm{d}}{\mathrm{d}t}(p) = D_x L$$

Note that

$$\begin{aligned} D_x H &= p\, D_x v - D_x L - D_v L\, D_x v \\ &= -D_x L + \underbrace{[p - D_v L]}_{=0 \text{ because of } (2.3)} D_x v. \end{aligned}$$

Thus, $\dot{p} = -D_x H$. □

Connections to Hamilton-Jacobi:

- (2.2) are characteristics of Hamilton-Jacobi equations.
- If $u = S(\Gamma)$, then $\mathrm{d}u = p\,\mathrm{d}x - H\,\mathrm{d}t$. (cf. Arnold, "Mathematical Methods in Classical Mechanics", Chapter 46)

$$\left\{\frac{\partial u}{\partial t} = -H(x, p, t); \quad D_x u = p\right\} \quad \Rightarrow \quad u_t + H(x, Du, t) = 0.$$

Important special case: $H(x, p, t) = H(p)$.

Example 2.11. $u_t + \sqrt{1 + |D_x u|^2 = 0}$. $H(p) = \sqrt{1 + |p|^2}$.

Example 2.12. $u_t + \frac{1}{2}|D_x u|^2 = 0$. $H(p) = \frac{1}{2}|p|^2$.

$$\begin{cases} \dot{x} = D_p H(p) \\ \dot{p} = 0 \end{cases} \Rightarrow \begin{cases} p(t) = p(0) \\ x(t) = x(0) + D_p H(p(0)) \end{cases} \quad \rightarrow \quad \text{straight line characteristics!}$$

2.2 The Hopf-Lax Formula

$$u_t + H(D_x u) = 0, \quad u(x,0) = u_0(x) \tag{2.4}$$

for $x \in \mathbb{R}^n$, $t > 0$. Always, H is considered convex and superlinear, $L = H^*$. Action on a path connecting $x(t_0) = y$ and $x(t_1) = x$:

$$\int_{t_0}^{t_1} L(x, \dot{x}, t)\mathrm{d}t = \int_{t_0}^{t_1} L(\dot{x}(t))\mathrm{d}t \geqslant (t_1 - t_0) L\left(\frac{x-y}{t_1 - t_0}\right).$$

Using Jensen's inequality:

$$\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} L(\dot{x})\mathrm{d}t \geqslant L\left(\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \dot{x}\,\mathrm{d}t\right) = L\left(\frac{x(t_1) - x(t_0)}{t_1 - t_0}\right).$$

Hopf-Lax formula:

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left[t\,L\left(\frac{x-y}{t}\right) + u_0(y) \right] \tag{2.5}$$

Theorem 2.13. *Assume $u_0: \mathbb{R}^n \to \mathbb{R}$ is Lipschitz with $\mathrm{Lip}(u(\,\cdot\,,t)) \leqslant M$ Then u defined by (2.5) is Lipschitz in $\mathbb{R}^n \times [0,\infty)$ and solves (2.4) a.e.. In particular, u solves (2.4) in $\mathcal{D}'$.*

(Proof exacty follows Evans.)

Lemma 2.14. (Semigroup Property)

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left[(t-s) L\left(\frac{x-y}{t-s}\right) + u(y,s) \right]$$

where $0 \leqslant s < t$.

Proof.

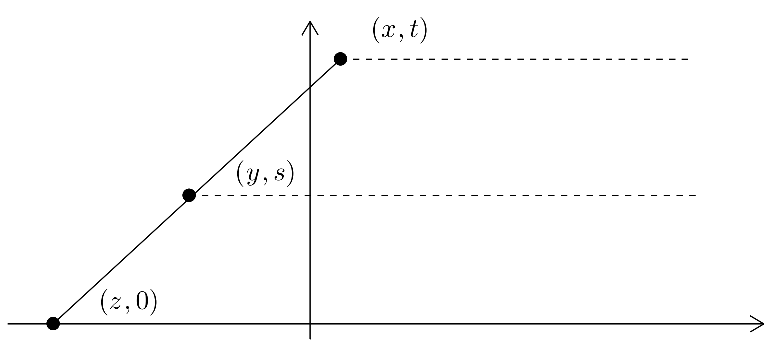

Figure 2.4.

$$\frac{x-z}{t} = \frac{x-y}{t-s} = \frac{y-z}{s}$$

So

$$\frac{x-z}{t} = \left(1 - \frac{s}{t}\right)\left(\frac{x-y}{t-s}\right) + \frac{s}{t}\left(\frac{y-z}{s}\right).$$

Since L is convex,

$$L\left(\frac{x-z}{t}\right) \leqslant \left(1 - \frac{s}{t}\right) L\left(\frac{x-y}{t-s}\right) + \frac{s}{t} L\left(\frac{y-z}{t}\right).$$

Choose z such that

$$u(y,s)=s\,L\left(\frac{y-z}{t}\right)+u_0(z).$$

The minimum is achieved because L is superlinear. Also,

$$\frac{|u_0(y)-u_0(0)|}{|y|}\leqslant M$$

because u_0 is Lipschitz.

$$t\,L\left(\frac{x-z}{t}\right)+u_0(z)\leqslant(t-s)L\left(\frac{x-y}{t-s}\right)+u(y,s).$$

But

$$u(x,t)=\min_{z'}\left[t\,L\left(\frac{x-z'}{t}\right)+u_0(z')\right].$$

Thus

$$u(x,t)\leqslant(t-s)L\left(\frac{x-y}{t-s}\right)+u(y-s)$$

for all $y\in\mathbb{R}^n$. So,

$$u(x,t)\leqslant\min_{y\in\mathbb{R}^n}\left[(t-s)L\left(\frac{x-y}{t-s}\right)+u(y-s)\right].$$

To obtain the opposite inequality, choose z such that

$$u(x,t)=t\,L\left(\frac{x-z}{t}\right)+u_0(z).$$

Let $y=(1-s/t)z+(s/t)x$. Then

$$\begin{aligned}u(y,s)+(t-s)\,L\left(\frac{x-y}{t-s}\right) &= u(y,s)+(t-s)L\left(\frac{x-z}{t}\right)\\ &= u(y,s)-s\,L\left(\frac{y-z}{s}\right)+[u(x,t)-u_0(z)]\\ &= u(y,s)-\left(u_0(z)+s\,L\left(\frac{y-z}{s}\right)\right)+u(x,t)\\ &\leqslant u(x,t).\end{aligned}$$

That means

$$\min_{y\in\mathbb{R}^n}\left[(t-s)L\left(\frac{x-y}{t-s}\right)+u(y-s)\right]\leqslant u(x,t).$$

□

Lemma 2.15. *$u:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ is uniformly Lipschitz. On any slice $t=\mathrm{const}$ we have*

$$\mathrm{Lip}(u(\,\cdot\,,t))\leqslant M.$$

Proof. (1) Fix $x,\hat{x}\in\mathbb{R}^n$. Choose $y\in\mathbb{R}^n$ such that

$$\begin{aligned}u(x,t) &= t\,L\left(\frac{x-y}{t}\right)+u_0(y),\\ u(\hat{x},t) &= t\,L\left(\frac{\hat{x}-y}{t}\right)+u_0(y).\end{aligned}$$

Then

$$u(\hat{x},t)-u(x,t)=\inf_{z\in\mathbb{R}^n}\left[t\,L\left(\frac{\hat{x}-z}{t}\right)+u_0(z)\right]-\left[t\,L\left(\frac{x-y}{t}\right)+u_0(y)\right].$$

Choose z such that

$$\begin{aligned}\hat{x}-z &= x-y\\ \Leftrightarrow z &= \hat{x}-x+y.\end{aligned}$$

Then

$$\begin{aligned} u(\hat{x},t)-u(x,t) &\leqslant u_0(\hat{x}-x+y)-u_0(y) \\ &\leqslant M|\hat{x}-x|, \end{aligned}$$

where $M = \mathrm{Lip}(u_0)$. Similarly,

$$u(x,t)-u(\hat{x},t) \leqslant M|x-\hat{x}|.$$

This yields the Lipschitz claim. In fact, using Lemma 2.14 we have

$$\mathrm{Lip}(u(\,\cdot\,,t)) \leqslant \mathrm{Lip}(u(\,\cdot\,,s))$$

for every $0 \leqslant s < t$, which can be seen as "the solution is getting smoother".

(2) *Smoothness in t:*

$$u(x,t) = \min_y \left[t\,L\left(\frac{x-y}{t}\right) + u_0(y) \right] \leqslant t\,L(0) + u_0(x) \quad \text{(choose } y = x\text{)}. \tag{2.6}$$

Then

$$\frac{u(x,t)-u_0(x)}{t} \leqslant L(0).$$

$$|u_0(y)-u_0(x)| \leqslant M|x-y| \quad \Rightarrow \quad u_0(y) \geqslant u_0(x) - M|x-y|.$$

Thus

$$t\,L\left(\frac{x-y}{t}\right) + u_0(y) \geqslant t\,L\left(\frac{x-y}{t}\right) + u_0(x) - M|x-y|.$$

By (2.6),

$$\begin{aligned} u(x,t)-u_0(x) &\geqslant \min_y \left[t\,L\left(\frac{x-y}{t}\right) - M|x-y| \right] \\ &= -t \max_{z\in\mathbb{R}^n} [M|z| - L(z)] \\ &= -t \max_{z\in\mathbb{R}^n} \left[\max_{\omega\in B(0,M)} \omega\cdot z - L(z) \right] \\ &= -t \max_{\omega\in B(0,M)} \max_{z\in\mathbb{R}^n} [\omega\cdot z - L(z)] \\ &= -t \max_{\omega\in B(0,M)} H(\omega). \end{aligned}$$

Now

$$-\max_{\omega\in B(0,M)} H(\omega) \leqslant \frac{u(x,t)-u_0(x)}{t} \leqslant L(0),$$

where both the left and right term only depend on the equation. $\Rightarrow$ Lipschitz const in time $\leqslant \max\,(L(0), \max_{\omega\in B(0,M)} H(\omega))$. $\square$