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DIFFUSIVE SLOWDOWN IN MISCIBLE VISCOUS FINGERING*

GOVIND MENON† AND FELIX OTTO‡

Abstract. We prove a refined upper bound on the size of the mixing layer in a simplified model of gravity driven miscible fingering that quantifies diffusive slowdown. Mathematically, the system we study is a multi-dimensional system of conservation laws that admits an exact one-dimensional closure for which the Lax entropy condition is not physically appropriate.

Key words. viscous fingering, dynamic scaling, coarsening, hyperbolic conservation laws, Burgers equation, turbulent transport.

MSC subject classifications. 35Q35, 76D27

1. Introduction

Gravity driven flows in a porous medium may be modeled by the system

$$\partial_t s + \mathbf{u}\cdot\nabla s = \triangle s, \quad s\in[0,1] \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{1.2}$$

$$\mathbf{u} = -m(s)[\nabla p - s\mathbf{e}_z]. \tag{1.3}$$

Here $s\in[0,1]$ denotes the concentration of a solute which is transported by convection and diffusion. The domain is $x=(y,z)\in[0,L]^{n-1}\times\mathbb{R}$, $n=2,3$ with periodic boundary conditions in y. Equation (1.3) is Darcy's law: the velocity is linearly proportional to the driving force which comprises a pressure gradient and buoyancy $-s\mathbf{e}_z$. We assume the mobility $m>0$ is a thermodynamically admissible function. L is called the Peclet number. It is the only external parameter, and measures the strength of diffusion. Our interest is in the limit of small diffusion ($L\to\infty$).

We consider initial conditions close to the unstable stratification

$$s_0(z) = \begin{cases} 1, & z\geq 0, \\ 0, & z<0. \end{cases} \tag{1.4}$$

In the absence of diffusion, this is the classical Saffman-Taylor instability for a flat interface [1, 8]. Studies with miscibility are more recent [3, 9]. For typical initial data, the instability develops as follows. For short times ($t\ll 1$ in the scaling of (1.1)–(1.3)) one observes the exponential growth of the maximally unstable mode (determined by L). In the late stage, $t\gg 1$, these sinusoidal perturbations evolve into a mixing layer with a complex mesoscopic network of elongated fingers. The mixing layer has two characteristic scales, a typical finger height, $a(t)$, in the z-direction, and typical finger width, $b(t)$, in the y direction. In experiments, one observes robust scaling laws $a(t)\sim ct$ and $b(t)=O(\sqrt{t})$ [5, 9]. These references also contain vivid photographs and simulations of fingering.

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†Division of Applied Mathematics, Brown University, Providence, RI 02912, USA (menon@dam.brown.edu). Supported by NSF DMS 03-05985.

‡Institute for Applied Mathematics, University of Bonn, Wegelerstr. 10, Bonn 53115, Germany (otto@iam.uni-bonn.de). Partially supported by SFB -611 of the German Science Foundation.

We studied this *dynamic scaling* in a recent article [6] for $m \equiv 1$. A surprising feature is that the rate of growth of the mixing layer is sharply affected by diffusion, even in the limit $L \to \infty$. This *diffusive slowdown* is an experimentally important manifestation of the subtle interplay between nonlinear effects that drive the formation of sharp gradients, and the regularizing effect of diffusion. In this note we reiterate the argument for slowdown in [6] in greater generality to indicate its mathematical interest in the theory of hyperbolic conservation laws. The case of constant mobility $m \equiv 1$ is simpler to analyze, but arises in applications only when one considers a single phase, and small concentrations of a solute (for example, Wooding considered a dilute solution of dye in water [9]). The mobility typically varies strongly with concentration. The assumption $m(s) = c_1 \exp(-c_2 s)$ for constants $c_1, c_2 > 0$ is common in applications [3]. We show here that modest changes in our methods cover this case. Our argument has also been adapted to injection driven spreading by Yortsos and Salin [10].

2. Diffusive slowdown

Our results for diffusive slowdown only apply to the simplified model

$$\partial_t s + \mathbf{u} \cdot \nabla s = \triangle s, \quad s \in [0,1] \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2.2}$$

$$\mathbf{u} = (v, w), \quad w = \alpha(\overline{s})(\overline{s} - s), \tag{2.3}$$

where $\overline{s}(z) = L^{1-n} \int_{[0,L]^{n-1}} s(y,z)\,dy$ is the transverse average, and $\alpha(\overline{s})$ is the modified mobility

$$\alpha(\overline{s}) = \frac{m_h m_l}{m_h \overline{s} + m_l (1 - \overline{s})}. \tag{2.4}$$

The connection to the mobility $m(s)$ in (1.3) is $m(0) = m_l > 0$ and $m(1) = m_h > 0$ (the subscripts denote light and heavy respectively). This system was formally derived from (1.1)–(1.3) for the case $m \equiv 1$ by Wooding [9]. The underlying assumption is a separation of length scales in the y and z directions. The case of general mobilities is not much different. A derivation is outlined at the end of this article. These equations retain two key features of (1.1)–(1.3): (i) the stratification s_0 is unstable, (ii) they yield the same bulk estimates for dissipation of potential energy (see [6, 7]). In particular, the size of the mixing layer for both (1.1)–(1.3) and (2.1)–(2.3) is bounded in a weak, energetic sense by the rarefaction wave solution to a Riemman problem described below.

We will prove the following bound on the size of the mixing layer. We abuse notation here: $s(t,x), x = (y,z)$ is the same as $s(t,y,z)$.

Theorem 2.1. *Let $s(t,x)$ be a classical solution to (2.1)–(2.3) with continuous initial data $s(0,x) : [0,L]^{n-1} \times \mathbb{R} \to [0,1]$ such that*

$$\lim_{z \to -\infty} s(0,y,z) = 0, \quad \lim_{z \to \infty} s(0,y,z) = 1. \tag{2.5}$$

Then for any $c_h > m_h/2$ and $c_l > m_l/2$ we have

$$\lim_{t \to \infty} s(t,y,-c_h t) = 0, \qquad \lim_{t \to \infty} s(t,y,c_l t) = 1. \tag{2.6}$$

Let us explain how this theorem quantifies diffusive slowdown by a factor of 2. To put things in a familiar form, we consider the hyperbolic rescaling

$$t = L\hat{t}, \quad x = L\hat{x}, \quad \hat{s}(\hat{t},\hat{x}) = s(t,x), \quad \hat{u}(\hat{t},\hat{x}) = u(t,x), \quad L\hat{p}(\hat{t},\hat{x}) = p(t,x). \tag{2.7}$$

This rescaling leaves all terms of (1.1)–(1.3) and (2.1)–(2.3) unchanged except for the diffusion. We then have

$$\partial_{\hat{t}}\hat{s} + \hat{\mathbf{u}}\cdot\nabla_{\hat{x}}\hat{s} = \frac{1}{L}\triangle_{\hat{x}}\hat{s}, \tag{2.8}$$

$$\nabla_{\hat{x}}\cdot\hat{\mathbf{u}} = 0 \tag{2.9}$$

$$\hat{\mathbf{u}} = (\hat{v},\hat{w}), \quad \hat{w} = \alpha(\overline{\hat{s}})(\overline{\hat{s}} - \hat{s}). \tag{2.10}$$

The rescaling has the following effect on initial data as $L\to\infty$. Assumption (2.5) implies $\hat{s}(0,\hat{y},\hat{z}) \to s_0(\hat{z})$ pointwise for every $\hat{z}\neq 0$.

In the formal limit $L=\infty$, system (2.8)–(2.10) has an important *closure property*. If initially $\hat{s}\in\{0,1\}$ *a.e*, this constraint is preserved, and we have $\overline{s^2} = \overline{s}$. We average (2.8) to obtain a closed scalar conservation law

$$\partial_t\overline{\hat{s}} + \partial_z\left(f(\overline{\hat{s}})\right) = 0, \qquad f(\overline{\hat{s}}) = \alpha(\overline{\hat{s}})\overline{\hat{s}}(1-\overline{\hat{s}}). \tag{2.11}$$

One may use (2.4) to check that f is strictly concave. Thus, the $L\to\infty$ limit leads naturally to a Riemann problem: solve (2.11) with initial data s_0. This problem has infinitely many weak solutions. It is traditional (motivated by gas dynamics) to single out the rarefaction wave (see for example [2, 4]),

$$\overline{\hat{s}}(z) = s^\sharp\left(\frac{z}{t}\right), \qquad \xi = f'(s^\sharp(\xi)),$$

as the physically appropriate solution. The spread of the rarefaction wave is determined by $f'(0) = -m_h$ and $f'(1) = m_l$, and we have $0 < s^\sharp(\xi) < 1$ for $\xi\in(-m_h,m_l)$. This solution is physically appropriate in gas dynamics as it is the $L\to\infty$ limit of solutions $s_L(t,z)$ to

$$\partial_t s_L + \partial_z(f(s_L)) = \frac{1}{L}\partial_z^2 s_L, \qquad s_L(0,z) = s_0(z). \tag{2.12}$$

However, diffusion has quite a distinct effect on the *system* (2.8)–(2.10).

COROLLARY 2.2. *Let s_k be a sequence of solutions to (2.1)–(2.3) with initial data satisfying (2.5), and Peclet numbers $L_k\to\infty$. If $\hat{s}$ is a subsequential limit of $\{\hat{s}_k\}_{k=1}^\infty$ in the weak-* topology in $L^\infty((0,\infty)\times[0,1]^{n-1}\times\mathbb{R})$,*

$$\hat{s}(\hat{t},\hat{y},\hat{z}) = \begin{cases} 1, & a.e \;\text{ in }\; 2\hat{z} > m_l\hat{t}, \\ 0, & a.e \;\text{ in }\; 2\hat{z} < -m_h\hat{t}. \end{cases} \tag{2.13}$$

In particular, $\overline{\hat{s}}$ cannot be a rarefaction wave.

Proof. Let φ be a continuous test function with compact support in $2\hat{z} > m_l\hat{t}$. Denote a convergent subsequence of the rescaled solutions $\hat{s}_k$ by the same index k. We then have

$$\int_0^\infty\int_Q \varphi(\hat{t},\hat{x})\hat{s}(\hat{t},\hat{x})d\hat{t}d\hat{x} = \lim_{k\to\infty}\int_0^\infty\int_Q \varphi(\hat{t},\hat{x})s_k(Lk\hat{t},L_k\hat{x})d\hat{t}d\hat{x},$$

and $\lim_{k\to\infty} s_k(L_k\hat{t}, L_k\hat{t}, L_k\hat{z}) = 1$ in $2\hat{z} > m_l\hat{t}$ by (2.6). Since $0 \le s_k \le 1$, the bounded convergence theorem allows us to interchange limits. The proof for $2\hat{z} < -m_h\hat{t}$ is similar. ∎

Of course, it is more desirable to prove this for a sequence of solutions to (1.1)–(1.3), but we have been unable to do this. Finally, let us note that there is a self-similar solution $s(t,z) = s^*(z/t)$ to the Riemann problem (2.11) that is not incompatible with Corollary (2.2). It consists of two shocks (inadmissible under the entropy condition):

$$s^*(\xi) = \begin{cases} 1, & m_l < 2\xi, \\ \frac{m_l}{m_h + m_l}, & -m_h < 2\xi < m_l, \\ 0, & 2\xi < -m_h. \end{cases}$$

3. Proofs

3.1. Motivation. As in [6] we construct comparison functions that are viscous shock profiles of Burger's equation. What is perhaps surprising is that our method works with no essential change for a more complicated mobility (2.3). The key heuristic idea is that there must be sharp gradients at the tips of fingers, and diffusion must act here. To derive a comparison function for downward spreading, consider a finger at the leading edge where $\overline{s} \approx 0$. Then (2.3) yields $w \approx -\alpha(0)s = -m_h s$. This suggests comparison with the one-dimensional equation

$$\partial_t s_* - m_h s_* \partial_z s_* = \partial_z^2 s_*. \tag{3.1}$$

This is Burger's equation with concave flux $-m_h s_*^2/2$. It admits downward moving viscous shocks $s_*(t,z) = s_\varepsilon(z + c_\varepsilon t) = s_\varepsilon(\zeta)$ connecting the states $\varepsilon > 0$ and $1+\varepsilon$ at $\mp\infty$ respectively. The speed c_ε is determined by the Rankine-Hugoniot condition

$$c_\varepsilon = \frac{m_h}{2}\left(\frac{(1+\varepsilon)^2 - \varepsilon^2}{1+\varepsilon-\varepsilon}\right) = m_h(\frac{1}{2} + \varepsilon), \tag{3.2}$$

and the shock profiles solve the differential equation

$$\frac{ds_\varepsilon}{d\zeta} = \frac{m_h}{2}(1+\varepsilon - s_\varepsilon)(s_\varepsilon - \varepsilon). \tag{3.3}$$

Thus, s_ε is strictly increasing, and given explicitly by

$$s_\varepsilon(\zeta) = \varepsilon + \frac{1}{2}\left(1 + \tanh\left(\frac{m_h(\zeta - z_0)}{4}\right)\right), \tag{3.4}$$

where z_0 is an arbitrary constant reflecting translation invariance. An analogous calculation yields a comparison function for upward spreading. If we consider a finger at the upper edge where $\overline{s} \approx 1$ we have $w \approx \alpha(1)(1-s) = m_l(1-s)$, and we are led to

$$\partial_t \tilde{s}_* + m_l(1-\tilde{s}_*)\partial_z \tilde{s}_* = \partial_z^2 \tilde{s}_*. \tag{3.5}$$

As before, for any $\varepsilon > 0$ we construct strictly increasing viscous shocks connnecting $-\varepsilon$ and $1-\varepsilon$ at $\mp\infty$. These have speed $m_l(1/2+\varepsilon)$.

3.2. Proof of Theorem 2.1.

LEMMA 3.1. *Assume $s(t,x)$ is a classical solution to (2.1)–(2.3) with continuous initial data $s(0,x)$. There exists $\varepsilon_*>0$ such that for every $\varepsilon\in(0,\varepsilon_*)$ the following comparison principles hold.*

1. *If $s(0,y,z)<s_*(0,z)$ then $s(t,y,z)<s_*(t,z)$ for all $t\geq 0$.*
2. *Similarly, if $s(0,y,z)>\tilde{s}_*(0,z)$ then $s(t,y,z)>\tilde{s}_*(t,z)$ for all $t\geq 0$.*

Proof. [Proof of Theorem 2.1] Fix $c>m_h/2$. Let ε be arbitrary with

$$0<\varepsilon\leq\min\left(\frac{1}{1+m_h}\left(c-\frac{m_h}{2}\right),\varepsilon_*\right).$$

Then, by (3.2)

$$c-c_\varepsilon\geq\varepsilon>0. \tag{3.6}$$

Since $\lim_{z\to-\infty}\max_y s(0,y,z)=1$, we may choose z_0 in (3.4) such that $s(0,x)<s_*(0,x)$ for all x. By Lemma 3.1 we then have

$$s(t,y,-ct)<s_*(t,-ct)=s_\varepsilon((c_\varepsilon-c)t).$$

In view of (3.6) and (3.4), this yields

$$\limsup_{t\to\infty}\max_y s(t,y,-ct)\leq\varepsilon.$$

Since ε was arbitrary, we obtain as desired

$$\lim_{t\to\infty}\max_y s(t,y,-ct)=0.$$

The proof of the lower estimate in (2.6) is similar, and is omitted. ∎

3.3. Proof of Lemma 3.1.

Proof. The proof is a direct application of the maximum principle. We write (2.1) in non-divergence form

$$\partial_t s+v\partial_y s+w\partial_z s-\triangle s=0, \tag{3.7}$$

and compare it with (3.1) rewritten as

$$\partial_t s_*+v\partial_y s_*+w\partial_z s_*-\triangle s_*=(w+m_h s_*)\partial_z s_*. \tag{3.8}$$

Let $\theta=s_*-s$. We subtract (3.7) from (3.8), to obtain

$$\partial_t\theta+v\partial_y\theta+w\partial_z\theta-\triangle\theta=(w+m_h s_*)\partial_z s_*. \tag{3.9}$$

Since $w=\alpha(\overline{s})(\overline{s}-s)$ the coefficient of $\partial_z s_*$ is

$$\alpha(\overline{s})(\overline{s}-s)+m_h s_*=\alpha(\overline{s})\overline{s}+\alpha(\overline{s})\theta+(m_h-\alpha(\overline{s}))s_*, \tag{3.10}$$

so that we may write (3.9) in the form

$$\begin{aligned}&\partial_t\theta+v\partial_y\theta+w\partial_z\theta-\triangle\theta-\alpha(\overline{s})\partial_z s_*\theta\\&=(\alpha(\overline{s})\overline{s}+(m_h-\alpha(\overline{s}))s_*)\,\partial_z s_*.\end{aligned} \tag{3.11}$$

We claim that the right hand side of (3.11) is strictly positive for sufficiently small $\varepsilon > 0$. First, by the strong maximum principle for (2.1) we have $s > 0$ for $t > 0$ and thus also $\overline{s} > 0$ for $t > 0$. Next, $\partial_z s_* > 0$ as can be seen from (3.4). Finally, we use the definition of $\alpha(\overline{s})$ in (2.4) to calculate

$$\alpha(\overline{s})\overline{s} + (m_h - \alpha(\overline{s}))s_* = m_h \overline{s} \frac{m_h s_* + m_l(1-s_*)}{m_h \overline{s} + m_l(1-\overline{s})}. \tag{3.12}$$

We only need check the sign of the numerator for $s_* \in [\varepsilon, 1+\varepsilon]$. If $m_h \geq m_l$ this is clearly positive for every $\varepsilon > 0$. If $m_h < m_l$, then

$$m_h s_* + m_l(1-s_*) \geq m_h(1+\varepsilon) - m_l \varepsilon > 0,$$

for $0 < \varepsilon < m_h/(m_l - m_h) := \varepsilon_*$. In either case,

$$(\alpha(\overline{s})\overline{s} + (m_h - \alpha(\overline{s}))s_*)\partial_z s_* > 0 \qquad \text{for}\, t > 0. \tag{3.13}$$

We now argue by the maximum principle. Assume $\theta \geq 0$ was not true. Since $\theta(0,x) \geq 0$ and $\lim_{z \to \pm\infty} \theta(t,y,z) = \varepsilon$ uniformly in (t,y), there exists a $(t_*, x_*) \in (0,\infty) \times \mathbb{R}^2$ such that

$$\theta(t_*, x_*) = 0 \quad \text{and} \quad \theta(t,x) \geq 0 \; \forall (t,x) \in (0,t_*) \times \mathbb{R}^2.$$

In particular,

$$\partial_t \theta(t_*, x_*) = \partial_y \theta(t_*, x_*) = \partial_z \theta(t_*, x_*) = 0 \quad \text{and} \quad \triangle \theta(t_*, x_*) \geq 0. \tag{3.14}$$

Hence by (3.9) we would obtain

$$(\alpha(\overline{s})\overline{s} + (m_h - \alpha(\overline{s}))s_*)\partial_z s_* \leq 0,$$

contradicting (3.13). The proof of the lower estimate is similar, and is omitted. ☐

3.4. Derivation of reduced model. The elliptic system (2.2)–(2.3) is obtained from (1.2)–(1.3) based on two assumptions: (i) $s \in \{0,1\}$ *a.e*; (ii) $p(y,z) = p(z)$. Let us first verify this, and then comment on the assumptions.

Incompressibility (1.2), and the assumption that the fluid is at rest at infinity imply $\overline{w} = 0$. We average (1.3) to obtain $-\overline{m(s)\partial_z p} = \overline{sm(s)} = m_h \overline{s}$ by assumption (i). Now use assumption (ii), and $\overline{m(s)} = m_h \overline{s} + m_l(1-\overline{s})$ to deduce

$$\partial_z p = \partial_z \overline{p} = -\frac{m_h \overline{s}}{m_h \overline{s} + m_l(1-\overline{s})}.$$

Substitution of this expression in (1.3) yields (2.3).

Heuristically, assumptions (i) and (ii) are motivated by an ideal limit where we have a parallel array of ascending ($s = 0$) and descending ($s = 1$) fingers with a sharp interface in between. Both assumptions are common in applications [10]. Formally, if we neglect diffusion, (1.1) is

$$\partial_t s + \mathbf{u} \cdot \nabla s = 0. \tag{3.15}$$

If $s \in \{0,1\}$ *a.e*, this equation may be interpreted in the sense of distributions as the evolution of the region $\{s = 1\}$. However, the assumption that $s \in \{0,1\}$ *a.e* is incompatible with the presence of diffusion ($L < \infty$ in (2.1)). Ideally, one would like to prove that suitably rescaled solutions to (1.1)–(1.3) converge asymptotically to solutions of (2.2)–(2.3) and (3.15) along with $s \in \{0,1\}$ $a.e$ thus providing a notion of an 'entropy solution' of the sharp interface problem. We have been unable to formulate a rigorous theorem to this effect. However, some evidence that a limiting solution must satisfy $s \in \{0,1\}$ $a.e$ is provided by an argument from [6]. We considered a bulk 'mixing entropy' $H(t) = L^{1-n} \int_{[0,L]^{n-1} \times \mathbb{R}} s(1-s)\, dx$. Observe that H vanishes if and only if $s \in \{0,1\}$ $a.e.$ Heuristically, H measures the size of 'mushy zones' or 'transition layers' around the fingers. These mushy zones act as a drag on bulk transport, and the existence of a lower bound $\liminf_{t\to\infty} t^{-1} H(t) \geq c > 0$ implies diffusive slowdown in an energetic sense as rigorously shown in [6]. Nontrivial analysis is required only when $\lim_{t\to\infty} t^{-1} H(t) = 0$. This suggests we may assume $s \in \{0,1\}$ $a.e$ as a first approximation.

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