A kinetic theory of shock clustering in scalar conservation laws

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We describe a kinetic theory for shock clustering in scalar conservation laws with random initial data. Our main discovery is that for a natural class of random data the shock clustering is described by a completely integrable Hamiltonian system. Thus, the problem is in a precise sense *exactly solvable*. Our results have implications in other areas: mathematical physics (limits of shell models of turbulence, and forced Burgers turbulence); probability theory (explicit computations of laws of excursions); and statistics (limit laws in the vicinity of maxima).

Our work grew out of a study of Smoluchowski's coagulation equation. This is a mean-field model of domain coarsening, first introduced to model coagulation in colloids. Quite remarkably, it also describes the clustering of shocks in Burgers equation for a class of random initial data [3, 7, 12]. This is a particular case of *Burgers turbulence*– the study of shock statistics in Burgers equation with random initial data or forcing. Our goal was to understand if this link between a mean-field model of coalescence and shock clustering was an isolated example, or part of a more general theory. It is in fact, a consequence of the theory outlined below.

The problem. We consider the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \quad x \in \mathbb{R}, \;\; t > 0, \tag{1}$$

with a C^1 convex flux function f and random initial data $u(x,0) = u_0(x)$. The entropy solution to (1) is given by the Hopf-Lax formula. Thus, (1) induces the evolution of the law of u_0. The problem is to determine this evolution.

The main assumption we make is that u_0 is a Markov process (in x) with only downward jumps. This assumption is motivated by some remarkable exact solutions in Burgers turbulence. Burgers considered white noise initial data in his pioneering work on statistical hydrodynamics [4, 5, 6]. The same problem also arose in statistics [9], and was solved in this context by Groeneboom [10]. He showed that for every $t > 0$, the process $u(x,t)$, $x \in \mathbb{R}$ is a stationary Markov process with only downward jumps, and he computed the generator of this process explicitly. There are two remarkable aspects to his solution: the first is the 'structural' fact that the Hopf-Lax formula respects the Markov property of u_0. The second is that the law of $u(x,t)$ can be computed *explicitly*. We now explain how both features hold in generality.

Kinetic theory and Lax equations. This part is joint work with Ravi Srinivasan [13]. The following *closure theorem* holds: if $u(x,0)$ is a Markov processes with only downward jumps, then so is the entropy solution $u(x,t)$, $t > 0$.

Markov processes with some regularity (Feller processes) are characterized by their generators. For example, if $u(x,t)$ is a stationary, spectrally negative Feller process in x, its generator $\mathcal{A}(t)$ acts on test functions $\varphi \in C_c^1(\mathbb{R})$ via

$$\mathcal{A}\varphi(y) = b(y,t)\varphi'(y) + \int_{-\infty}^{y} (\varphi(z) - \varphi(y))\, n(y,dz,t). \tag{2}$$

These terms correspond to the drift and jumps (i.e. rarefactions and shocks) of u.

The closure theorem reduces the problem of evolution of shock statistics to a study of the evolution of the generators. One of our main results is that the evolution of $\mathcal{A}$ is given by the Lax equation

$$\partial_t \mathcal{A} = [\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}. \tag{3}$$

Here $\mathcal{B}$ is defined by its action on test functions as follows:

$$\mathcal{B}\varphi(y) = -f'(y)b(y,t)\varphi'(y) - \int_{-\infty}^{y} \frac{f(y)-f(z)}{y-z} \left(\varphi(z)-\varphi(y)\right) n(y,dz,t). \tag{4}$$

It requires considerable insight to realize that this approach is fruitful, and our work was greatly inspired by Duchon and his co-workers [7, 8]. In particular, (3) simplifies and generalizes their work. The Lax equation (3) expands (using (2)) to yield kinetic equations of shock clustering for b and n. All known exact solutions to Burgers turbulence satisfy (3).

Hamiltonian structure and geodesic flows of Markov operators. Lax pairs are synonymous with completely integrable systems. We also noted other 'integrable properties' in [13] (a Painlevé property, connections with random matrices, and more). Much of our work since the discovery of (3) has been devoted to understanding this structure. The following picture has emerged, though many aspects remain to be pinned down.

To show that (3) is a Hamiltonian system we must introduce a phase space, a symplectic structure and a Hamiltonian. This is done by discretization and a passage to the limit. We restrict $u(x,t)$ to a Markov process on an n-dimensional state space. In this case $\mathcal{A}$ is an $n \times n$ matrix (say A), and (3) yields a matrix evolution equation $\dot{A} = [A, B]$ where $B_{ij} = F_{ij}A_{ij}$ is a natural discretization of (4). This is a Hamiltonian system with the symplectic structure of Kostant and Kirillov and a quadratic Hamiltonian $H(A) = \sum_{ij} A_{ij}B_{ij}$. In the limit $n \to \infty$, we find that (3) is a Hamiltonian flow on a Lie algebra of generators of Markov processes. When $f' > 0$, (3) formally describes geodesic flows on a space of Markov processes with metric determined by f.

The spectral curve and algebraic complete integrability. The fact that (3) is *completely integrable* appears to follow from the following simple observation. Let $\mathcal{M}$ and $\mathcal{N}$ denote multiplication operators acting on the domain of $\mathcal{A}$, defined by

$$\mathcal{M}\varphi(y) = y\varphi(y), \quad \mathcal{N}\varphi(y) = f(y)\varphi(y). \tag{5}$$

It is clear that $\mathcal{M}$ and $\mathcal{N}$ are diagonal operators. We now use the definitions (2), (4) and (5) to find

$$[\mathcal{A}, \mathcal{N}] - [\mathcal{M}, \mathcal{B}] = 0. \tag{6}$$

This observation allows us to introduce a spectral parameter $\mu \in \mathbb{C}$ in the Lax equation. We use (3) and (6) to obtain

$$\partial_t \left(\mathcal{A} - \mu\mathcal{M}\right) = [\mathcal{A} - \mu\mathcal{M}, \mathcal{B} + \mu\mathcal{N}], \quad \mu \in \mathbb{C}. \tag{7}$$

If $\mathcal{A}$, $\mathcal{B}$ were $n \times n$ matrices, it would follow that the spectral curve (Riemann surface)

$$\Gamma = \{(\lambda, \mu) \in \mathbb{C}^2 \,|\det(\mathcal{A} - \lambda \mathrm{id} - \mu \mathcal{M}) = 0\}, \tag{8}$$

is fixed by the evolution. This is the crucial observation that yields the existence of additional integrals for Euler's equations in $so(n)$, $n \geq 4$ in Manakov's treatment of Euler's equations [11]. These integrals are simply the coefficients of the characteristic polynomial above.

This observation shows that the discretizations of (3) describe completely integrable flows on the 'Markov' group $\{A \in gl(n) | \sum_{j=1}^n A_{ij} = 1, i = 1, \dots, n\}$ in precise analogy with Manakov's work. More broadly, it reveals a close relation with a large class of completely integrable systems (including KdV, the Toda lattice, geodesic flows on $so(n)$ and ellipsoids, and the integrable PDEs of random matrix theory). The complete integrability of all these flows may be obtained in a unified way via a general splitting theorem for Lie algebras [1]. This connection also sets the stage for the application of powerful methods from algebraic geometry to integrate (3) explicitly for *every* convex f [2].

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