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Gevrey class regularity for the attractor of the laser equations

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Abstract. Constantin, Foais and Gibbon proved that the laser equations (Lorenz PDE) define a dynamical system in L^2 with a C^∞ attractor. We extend this theorem to show that the attractor is contained in every Gevrey class, G^s, for $1 < s < \infty$. This demonstrates a remarkable smoothing mechanism for this hyperbolic system. We consider the consequences of this theorem for finite-dimensionality of the dynamics.

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1. Introduction

Constantin, Foias and Gibbon proved that the laser equations of Risken and Nummedal define a dynamical system in L^2 with a C^∞ attractor of finite Hausdorff dimension [1, 2]. In this paper we prove the stronger result that for any $1 < s < \infty$ the attractor is contained in the Gevrey class G^s. The methods follow [1]; the new idea is to obtain precise control over the growth of derivatives.

The laser equations are a version of the complex Lorenz equations and we shall consider them in this form. Various physical interpretations and scalings associated to these equations are discussed in [1]. The Lorenz PDE are

$$\partial_t X + \partial_x X = -\sigma X + \sigma Y \tag{1.1}$$

$$\partial_t Y = -ZX - (1 + \mathrm{i}\delta)Y \tag{1.2}$$

$$\partial_t Z = -bZ + \tfrac{1}{2}\left(X^* Y + XY^*\right) - br. \tag{1.3}$$

$X, Y \in \mathbb{C}$ and $Z \in \mathbb{R}$ are periodic on the domain $x \in [0, L]$; σ and b are positive decay constants scaled to the decay rate of Y; r is a positive phenomenological pumping term; δ is a real detuning parameter. In passing we note that when $\delta = 0$ the subspace of real, spatially independent solutions to these equations is the set of solutions to the Lorenz ODE [3].

Equations (1.1)–(1.3) are a semilinear damped hyperbolic system. Thus for finite times we may expect solutions that are only as smooth as the initial data. It is quite surprising then that asymptotically solutions are smoothed out sharply. Recently, Xin and Moloney have considered the laser equations with transverse terms [4]. In their notation, equation (1.1) has a differential operator $\partial_t + \partial_x + a\mathrm{i}\triangle_\perp$ where $\triangle_\perp$ is the Laplacian in (y, z). They prove the existence of global weak solutions and the existence of an attractor with partial smoothness for $(x, y, z) \in \mathbb{T}^1 \times \mathbb{R}^2$ and $(x, y, z) \in \mathbb{T}^3$. The case $(x, y, z) \in \mathbb{T}^3$ is harder and their proof relies on conjectural Strichartz inequalities of Bourgain. Strong asymptotic smoothing seems to be unique to the one-dimensional geometry.

We mention briefly some related results on the existence of analytic and quasianalytic solutions to nonlinear PDE. Kahane proved the spatial analyticity of certain solutions to the Navier–Stokes equations [5]. More recently, Foias and Temam have studied Gevrey regularity for the two-dimensional incompressible Navier–Stokes equations [6]. Doelman and Titi, and Levermore and Oliver, proved the existence of analytic solutions to certain complex Ginzburg–Landau equations [7, 8]. A classical theorem of de La Vallée Poussin asserts that the Gevrey classes are characterised by exponential decay of Fourier coefficients [9]. Levermore and Oliver have proven an elegant decomposition theorem for the Gevrey classes based on this principle. This theorem may be used to study the analyticity of solutions to other parabolic nonlinear PDE. More exhaustive references are contained in their expository article [8]. The Navier–Stokes equations and CGL have a smoothing Laplacian term. Our work differs in that we study a purely hyperbolic problem with linear damping independent of the wavenumber. For such problems, Gevrey regularity may only be expected as an asymptotic property. Moreover, our approach is in the spirit of Kahane: we prove direct estimates on all derivatives rather than estimate the Fourier coefficients. This is facilitated by the special structure of the laser equations, in particular, that equation (1.1) is carried on a different characteristic to equations (1.2) and (1.3).

2. The main theorem

Let $H = (L^2(0, L))^3$ (with the understanding that X, Y are complex and Z is real). Constantin *et al* have shown that the laser equations define a continuous global flow $S : \mathbb{R} \times H \to H$ in the sense of weak solutions [1, theorem 4.1], and for fixed $t \in \mathbb{R}$ the map $S(t) : H \to H$ obtained by restricting S is locally Lipschitz. Furthermore, there exists an *absorbing ball* B_{ρ_0} of radius ρ_0 in H. The radius ρ_0 depends only on b, r, σ and L. The time taken to enter B_{ρ_0} is uniform over bounded sets. The universal attractor $\mathcal{A}$ is defined as $\mathcal{A} = \cap_{t>0} S(t) B_{\rho_0}$. The universal attractor is invariant, that is $S(t)\mathcal{A} = \mathcal{A}$ for all $t \in \mathbb{R}$. This is the key property in the proof of C^∞ regularity. In [1] the authors proved infinite regularity by showing that $\mathcal{A} \subset (W^{n,\infty}_{\text{per}})^3$ for all $n \geqslant 0$. The proof is inductive and at each step one uses the invariance of $\mathcal{A}$ in a bootstrapping argument. In our proof we include more detailed estimates on the $W^{n,\infty}_{\text{per}}$ norm following the methods in [1]. Remarkably, these are sufficient to prove much stronger regularity.

We use the definition of Gevrey classes in [8] as this is most suited to our purpose. There are other equivalent definitions that involve Fourier coefficients [6].

Definition 2.1. *A function $f \in C^\infty_{\text{per}}[0, L]$ is said to belong to the Gevrey class, G^s, if there exist positive numbers M and ρ, such that for every integer $n \geqslant 0$,*

$$\|\partial_x^n f(x)\|_{L^\infty} \leqslant M \left(\frac{n!}{\rho^n} \right)^s .$$

For $s = 1$, the Gevrey class G^1 is identical to the set of periodic, real analytic functions [10, theorem 1, p 3]. For $0 < s_1 < s_2 < \infty$, we have $G^{s_1} \overset{\subset}{\neq} G^{s_2} \overset{\subset}{\neq} C^\infty_{\text{per}}$. Theorem 4.2 in [1] states that $\mathcal{A} \subset (C^\infty_{\text{per}}[0, L])^3$. We prove the stronger result.

Theorem 2.2. *For every $s \in (1, \infty)$, the attractor $\mathcal{A} \subset (G^s[0, L])^3$.*

We let $U = (X, Y, Z)$ denote points in H, and define $a_n = \sup_{U\in\mathcal{A}} \|\partial_x^n X\|_\infty$ and $b_n = \sup_{U\in\mathcal{A}} (\|\partial_x^n Y\|_\infty + \|\partial_x^n Z\|_\infty)$. The proof of theorem 2.2 depends on estimates for a_n and

b_n. The purpose of the following theorem is to quantify the growth of a_n and b_n inductively. The proof is a calculation using the methods in [1]. Since a sufficiently detailed proof has already been provided there for the case $n = 1$, and the generalisation to arbitrary n is natural, but somewhat lengthy, we omit the proof.

Theorem 2.3. *a_n and b_n satisfy the following estimates:*

(a) $a_0, b_0 \leqslant M_0 = 2r(b/\beta)^{1/2} < \infty$ *where* $\beta = \min(1, b/2)$.
(b) For $n \geqslant 1$, $a_n \leqslant C_1 b_{n-1} + \sum_{k=0}^{n-1} \binom{n-1}{k} a_k b_{n-k-1}$ *where* $C_1 = (1+\delta^2)^{1/2} + 3\sigma$.
(c) For $n \geqslant 1$, $b_n \leqslant C_2 \sum_{k=0}^{n-1} \binom{n}{k} b_k a_{n-k}$ *where* $C_2 = \left(2\beta^{-1}(1+b^{-1})\right)^{1/2}$.

The estimates of theorem 2.3 suggest a proof of Gevrey regularity by induction. In any such proof one would need to control the binomial coefficients in some way. We digress briefly to indicate some properties of a combinatorial sum that occurs in our calculations.

Definition 2.4. *Suppose* $0 < p < \infty$. *Define*

(a) $R_n(p) = \sum_{k=0}^{n} \binom{n}{k}^{-p}$.
(b) $R(p) = \sup_{n \geqslant 1} R_n(p)$.

The properties of $R(p)$ needed here are quantified with the following lower and upper bounds. First note that for $p = 0$, the sum $\sum_{k=0}^{n} \binom{n}{k}^{-p} = n+1$, diverges. Suppose $p > 0$, then the elementary estimate

$$\sum_{k=0}^{n} a_k{}^{-p} \geqslant \frac{(n+1)^{p+1}}{(\sum_{k=0}^{n} a_k)^p},$$

for numbers $a_k \geqslant 1$, $k = 0, \ldots n$, applied to $R_n(p)$ gives

$$R(p) \geqslant R_n(p) \geqslant (n+1)^{p+1} 2^{-np}.$$

Maximizing the right hand side in n we have $R(p) \to \infty$ at least as fast as p^{-1}, when $p \to 0$. Next, we derive an upper bound on $R(p)$. Let n_p be an integer depending on p that is chosen as follows: for $0 < p < 1$ it is the integer so that $1 < pn_p \leqslant 1 + p$, and for $1 \leqslant p < \infty$, $n_p = 2$. Each term in the sum $R_n(p)$ is bounded by 1. So $R_n(p) \leqslant 2n_p$ for $1 \leqslant n \leqslant 2n_p - 1$; and for any $n \geqslant 2n_p$

$$R_n(p) = 2\sum_{k=0}^{n_p-1} \binom{n}{k}^{-p} + \sum_{k=n_p}^{n-n_p} \binom{n}{k}^{-p} \leqslant 2n_p + n\binom{n}{n_p}^{-p}.$$

By the choice of n_p, $n\binom{n}{n_p}^{-p}$ is a decreasing function of n in the range $2n_p \leqslant n < \infty$. Thus its maximum occurs when $n = 2n_p$. This gives the upper estimate

$$R(p) = \sup_{n \geqslant 1} R_n(p) \leqslant \left(2 + \binom{2n_p}{n_p}^{-p}\right) n_p.$$

As $p \to 0$, $n_p \to \infty$ and $pn_p \to 1$. Invoking Stirling's approximation

$$\lim_{p \to 0} \binom{2n_p}{n_p}^{-p} = \lim_{p \to 0} \left(\frac{1}{\sqrt{\pi n_p}} 2^{2n_p}\right)^{-p} = \frac{1}{4}.$$

Hence, $R(p) \leqslant 3n_p$, for sufficiently small p. The upper and lower estimates taken together demonstrate that $R(p) = O(1/p)$ as $p \to 0$. One may also show that $R(p)$ is a strictly decreasing continuous function of p that tends to 2 when $p \to \infty$.

Proof (of theorem 2.2). To prove Gevrey regularity we need to show that there are positive constants $M_i, \rho_i, \ i = 1, 2$, so that

$$a_k \leqslant M_1\left(\frac{k!}{\rho_1^k}\right)^s, \qquad b_k \leqslant M_2\left(\frac{k!}{\rho_2^k}\right)^s \qquad \text{for all } \ k \geqslant 0. \tag{2.1}$$

In fact we will show that there are constants M and ρ, so that

$$a_k \leqslant \frac{1}{M}\left(\frac{k!}{\rho^k}\right)^s, \qquad b_k \leqslant M\left(\frac{k!}{\rho^k}\right)^s \qquad \text{for all } \ k \geqslant 1. \tag{2.2}$$

Notice the difference in the starting index for conditions (2.1) and (2.2). Of course, it is sufficient to prove condition (2.2), and then choose $M_1 = M_2 = \max(M_0, M^{-1}, M)$ to prove condition (2.1). We shall suppose at the outset that $M_0 \leqslant M$, so that $b_0 \leqslant M_0 \leqslant M$.

Parts (b) and (c) of theorem 2.3 tell us what conditions M and ρ must satisfy. Suppose that (2.2) is true for $1 \leqslant k \leqslant n-1$. From part (b) of theorem 2.3

$$\begin{aligned} a_n &\leqslant (C_1 + a_0)b_{n-1} + \sum_{k=1}^{n-1}\binom{n-1}{k}a_k b_{n-1-k} \\ &\leqslant (C_1 + M_0)M\left(\frac{(n-1!)}{\rho^{n-1}}\right)^s + \frac{(n-1)!}{\rho^{(n-1)s}}\sum_{k=1}^{n-1}(k!)^{s-1}(n-1-k)!^{s-1} \\ &= \left(\frac{(n-1)!}{\rho^{n-1}}\right)^s\left[(C_1 + M_0)M + R_{n-1}(s-1)\right], \end{aligned}$$

which will be less than $M^{-1}(n!)^s\rho^{-ns}$ if

$$\frac{\rho^s}{n^s}\left[(C_1 + M_0)M + R_{n-1}(s-1)\right] \leqslant \frac{1}{M}.$$

Since $R_{n-1}(s-1) \leqslant n \leqslant n^s$ this condition is satisfied uniformly for $n \geqslant 1$ if we choose

$$\rho^s \leqslant \frac{1}{M^2(C_1 + M_0) + M}\ . \tag{2.3}$$

Next we consider the choice of M, supposing that condition (2.2) is satisfied for $1 \leqslant k \leqslant n-1$, and $M \geqslant M_0$. From part (c) of theorem 2.3

$$b_n \leqslant C_2\frac{n!}{\rho^{ns}}\sum_{k=0}^{n-1}(k!)^{s-1}(n-k)!^{s-1} = C_2\frac{(n!)^s}{\rho^{ns}}\sum_{k=0}^{n-1}\binom{n}{k}^{1-s}.$$

This is less than $M(n!)^s\rho^{-ns}$, if

$$C_2\sum_{k=0}^{n}\binom{n}{k}^{1-s} = C_2R_n(s-1) \leqslant M. \tag{2.4}$$

The only way constraint (2.4) will be satisfied for all $n \geqslant 1$, is if

$$\sup_{n\geqslant 1} C_2R_n(s-1) = C_2R(s-1) \leqslant M.$$

Thus we choose M so large that

$$M = \max(C_2R(s-1), M_0). \tag{2.5}$$

We also need to satisfy $a_1 \leqslant 1/M\rho^s$ to start the proof by induction. But by parts (a) and (b) of theorem 2.3, $a_1 \leqslant (C_1 + M_0)b_0 \leqslant (C_1 + M_0)M_0$. Thus we choose

$$\rho^s = \min\left(\frac{1}{M^2(C_1 + M_0) + M}, \frac{1}{MM_0(C_1 + M_0)}\right) = \frac{1}{M^2(C_1 + M_0) + M}. \tag{2.6}$$

Theorem 2.3 proves that $\mathcal{A} \subset C^\infty_{\text{per}}$. With M and ρ chosen as in conditions (2.5) and (2.6), the inductive step is true for all $n \geqslant 1$. This shows that $\mathcal{A} \subset (G^s)^3$. □

Remark 2.5. *It is natural to enquire about the case $s = 1$. As $s \downarrow 1$, condition (2.5), and the growth estimate of $R(s-1)$, show that M is $O((s-1)^{-1})$. Then condition (2.6) shows that $\rho^s = O((s-1)^2)$. Thus we are faced with a situation where the 'radius of convergence' shrinks to zero. This difficulty may be traced back to the presence of the combinatorial term $\binom{n}{k}$ in part (c) of theorem 2.3. This term comes from using the product rule to estimate the derivatives of the nonlinear terms in equations (1.2) and (1.3). If the combinatorial term were $\binom{n-1}{k}$ as in part (b) an argument similar to the one above shows that solutions are real analytic. Thus the failure of analyticity is quite delicate, and a direct consequence of the nonlinearity.*

3. Remarks on finite-dimensional dynamics

Theorem 2.2 throws new light on the main theorem of [1] that the attractor, $\mathcal{A}$, is of finite Hausdorff dimension. For simplicity, suppose $L = 2\pi$. For every $U \in \mathcal{A}$, the Fourier coefficients satisfy a decay condition

$$|\hat{U}(n)| \leqslant CM\rho^{1/2}|n|^{1/2s} \exp\left(-\rho|n|^{1/s}\right) \qquad n \in \mathbb{Z}, n \neq 0.$$

The constant C depends on the parameters in the laser equations (1.1)–(1.3) and s, but may be taken uniform for s in a finite range, say, $1 < s \leqslant s_0 < \infty$. Thus for s sufficiently close to 1, the restrictions (2.5) and (2.6) imply that there are positive constants A, B so that

$$|\hat{U}(n)| \leqslant A|n|^{1/2s} \exp\left(-B(s-1)^2|n|^{1/s}\right) \qquad n \in \mathbb{Z}, n \neq 0.$$

Thus, finite-dimensional approximations to U (such as Fourier–Galerkin truncations in L^2) will have an exponentially small error. More precisely, the distance in L^2 between the attractor $\mathcal{A}$ and the finite-dimensional subspace $S_N = \mathrm{span}\{\exp(inx) : |n| \leqslant N\}$ decreases exponentially with N. For example, let $s = 2$. Then we have $\mathrm{dist}(\mathcal{A}, S_N) \leqslant CAN^{1/4} \exp\left(-B\sqrt{N}\right)$, where C is a constant depending only on the parameters in the laser equations. Heuristically, this suggests that the asymptotic dynamics of the laser equations are governed by only a finite number of modes. From the computational viewpoint, this result along with the main theorem of [1] justifies finite-dimensional models of the laser system. In particular, finite-dimensional projections of the attractor are exponentially close to the attractor. More than being a technical improvement, theorem 2.2 is intimately related to the dynamics of the system. For CGL, Doelman and Titi proved the stronger result that on finite time intervals the numerical solution obtained from a Galerkin method is exponentially close in H^1 to the true solution (for sufficiently smooth initial data) [7]. No such result can be obtained here since the laser equations are hyperbolic. Infinite regularity is only an asymptotic property, thus the usual tracking arguments of numerical analysis fail. Nevertheless, since any solution approaches the attractor exponentially fast, in a typical numerical simulation one would indeed see rapid smoothing as the laser rises out of noise. For instance, the simulations of Risken and Nummedal show the formation of smooth traveling waves as the laser rises out of noise [2].

We have obtained similar results in two other cases. First, when equation (1.3) has a nonhomogeneous periodic forcing term $f(t,x) = f(t+T,x)$ which is analytic in x and continuous in time, the results of [1] and this paper extend immediately to the natural Poincaré map. We have also studied equation (1.1) with a diffusion term of the form $\alpha\partial_x^2 X$ with $\alpha > 0$ on the right hand side. The laser equations are then a coupled parabolic-hyperbolic system. There are some simplifications because of the parabolic nature of the first equation. Nevertheless, many of the difficulties of the hyperbolic problem remain.

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