Solutions to HW 2

Problem 1. Let K denote the closure of $O_{R_\alpha}(x)$. Let $G = S^1 \backslash K$ denote the complement of K. If G is not empty, then it is a countable union of intervals. Let J be one such interval. We claim that $R_\alpha^k(J)$, $k \in \mathbb{Z}$, form a disjoint family of intervals with the same length as J. This is impossible, because S^1 has finite length. Thus, $K = S^1$.

Let us prove the claim. Since R_α is a rigid rotation, it is clear that $|R_\alpha(I)| = |I|$ for every interval I. Thus, $|R_\alpha^k(J)| = |J|$, $k \in \mathbb{Z}$. Let us show these intervals are disjoint. If $R_\alpha^k(J) \cap J \neq \phi$, the intersection is an interval, and the endpoints of this interval must belong to K, implying $R_\alpha^k(J) \cap J = J$. But then R_α^k has a fixed point. This is impossible because α is irrational. $\square$

Problem 2. (a) Let $f = e^{2\pi ikx}$. If $k = 0$, it is clear that Weyl's theorem holds (here the length of S^1 is 1). If $k \neq 0$, we find

$$\sum_{m=0}^{n-1} e^{2\pi ikR_\alpha^m(x)} = e^{2\pi ikx} \sum_{m=0}^{n-1} e^{2\pi ikm\alpha} = e^{2\pi ikx} \frac{e^{2\pi in\alpha} - 1}{e^{2\pi i\alpha} - 1}.$$

We take absolute values to find

$$\frac{1}{n} \left| \sum_{m=0}^{n-1} e^{2\pi ikR_\alpha^m(x)} \right| \leq \frac{2}{n\, |e^{2\pi i\alpha} - 1|}.$$

The right hand side converges to 0, which is also its integral average.

(b) We take finite linear combinations to see that Weyl's theorem holds for trigonometric polynomials. If f is a continuous function, $\varepsilon > 0$, let P_ε be a trigonometric polynomial such that $\max |f(x) - P_\varepsilon(x)| < \varepsilon$. Then

$$\frac{1}{n} \left| \sum_{m=0}^{n-1} f(R_\alpha^m(x)) - P_\varepsilon(R_\alpha^m(x)) \right| < \varepsilon.$$

Since Weyl's theorem holds for P, we also have

$$\limsup_{n\to\infty} \left| \frac{1}{n} \sum_{m=0}^{n-1} f(R_\alpha^m(x)) - \int_{S^1} f(x)\, dx \right| \leq 2\varepsilon.$$

(c) Let $f = \mathbf{1}_{[a,b]}$. For sufficiently small $\varepsilon > 0$ consider continuous upper and lower approximations $f_{\pm}$ as shown in Figure 0.1. We then have

$$\begin{aligned}\int_{S^1} f(x)dx - \varepsilon &= \int_{S^1} f_-(x)dx = \lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} f_-(R_\alpha^m(x)) \\ &\le \liminf_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} f(R_\alpha^m(x)) \le \limsup_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} f(R_\alpha^m(x)) \\ &\le \lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} f_+(R_\alpha^m(x)) = \int_{S^1} f_+(x)dx = \int_{S^1} f(x)dx + \varepsilon.\end{aligned}$$

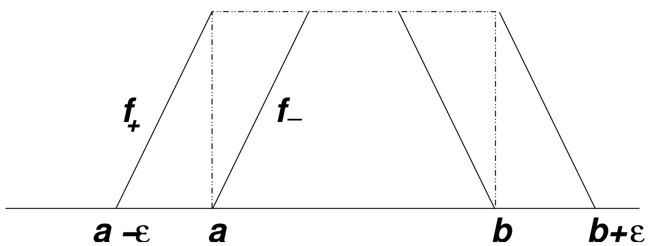

Figure 0.1: Continuous approximations $f\pm$ to $\mathbf{1}_{[a,b]}$

□

Problem 3. Fix a positive integer p. Any integer m can be written as $m = kp + r$ where $0 \le r < p$. The subadditivity assumption implies

$$a_m \le a_{kp} + a_r + c \le ka_p + a_r + (k+1)c.$$

Therefore,

$$\frac{a_m}{m} \le \frac{a_p}{p}\left(\frac{kp}{m}\right) + \frac{a_r + c(k+1)}{m}.$$

Let $m \to \infty$ to obtain

$$\limsup_{m\to\infty} \frac{a_m}{m} \le \frac{a_p + c}{p}.$$

Since p was arbitrary, we now let $p \to \infty$ to obtain

$$\limsup_{m\to\infty} \frac{a_m}{m} \le \liminf_{p\to\infty} \frac{a_p}{p}.$$

□

Problem 4. (a) This is immediate from the definition of N.

(b) We may write $N(y,m) = \sum_{k=0}^{m-1} \mathbf{1}(y_k)$ where $\mathbf{1}$ denotes the indicator function of $[x, f(x))$. Since the function $\mathbf{1}(y)$ is continuous except at the endpoints x and $f(x)$, the function $N(y,m)$ is continuous unless $y_k = x$ or $y_k = f(x)$ for some k. That is, the only possible discontinuities of N are at the preimages $f^{-j}(x)$, $-1 \le j < m$. However, for $0 \le j \le m-2$ there is no discontinuity in N as the sum 'compensates' in the following sense: if $f^j(y) = x$, then for z_1,z_2 sufficiently close to y and $z_1 < y < z_2$ we have $f^j(z_1) \in I_1$ and $f^{j+1}(z) \in I_0$ and $f^j(z_2) \in I_0$ and $f^{j+1}(z_2) \in I_1$.

Therefore, the only possible discontinuities of $N(y,m)$ are at the points $f(x)$ and $f^{-m+1}(x)$. If these two points are distinct, they partition the circle into two intervals on which $N(y,m)$ is constant, such that the value in both intervals differs at most by 1. If the two points are identical, then $N(y,m)$ is the same for all $y \neq f(x)$ and differs by 1 from $N(f(x,m))$.

(c) We have the upper and lower estimates

$$N(y,m) + N(y,n) - 1 \le N(y,m+n) \le N(y,m) + N(y,n) + 1.$$

Apply problem (3) with $c_m = N(y,m)$ to see that $\rho_y(f) = \lim_{m\to\infty} N(y,m)/m$ exists and is a number in $[0,1]$. This also yields the estimate

$$\left|\rho_y(f) - \frac{N(y,n)}{n}\right| \le \frac{1}{n}.$$

Thus, we also have

$$\left|\rho_y(f) - \frac{N(z,n)}{n}\right| \le \frac{2}{n},$$

which shows that $\rho_y(f) = \rho_z(f)$. □

Problem 5. To make the x-dependence explicit, let N_x denote the counting function of Problem 4. We saw that $N_x(y,m)$ has discontinuities only at $y = f^{-m+1}(x)$ and $y = f(x_i)$. If y is not either of these points, we see that $N_x(y,m) = N_z(y,m)$ for all z near x. At a point of discontinuity, the value changes at most by 1 and we have $|N_x(y,m) - N_z(y,m)| \le m$. □

Proof. A very intuitive geometric proof of this assertion is attached (from Arnold's 'Geometric methods in the theory of ordinar differential equations'). A more analytic proof relies on the following recurrence relations. As we have seen, the closest return times satisfy

$$q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad n \ge 0, \quad q_{-1} = 0, q_0 = 1.$$

It is also true that p_n satisfy the same recurrence relation with different initial conditions

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, \quad n \geq 0, \quad p_0 = 0, p_{-1} = 1.$$

(You can prove this inductively, look up a proof online, or see the attached note). Now one can show analytically that $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$ which yields

$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^k}{q_k q_{k-1}}.$$

The rest of the proof is as in Arnol'd. □

Problem 7. It will suffice to prove the assertion for $\alpha \in (0,1)$. Let $F \subset (0,1)$ denote the set of irrational numbers that are badly-approximated in the sense that there exists $K(\alpha, \varepsilon)$ such that $|\alpha - p/q| \geq Kq^{-(2+\varepsilon)}$ for positive integers p, q. Let G denote the complement of F. We show that $|G| = 0$.

If $\alpha \in G$ then $|\alpha - p/q| < q^{-(2+\varepsilon)}$ infinitely often. Let G_q denote the set of intervals of width $2q^{-(2+\varepsilon)}$ centered at p/q, $p = 1, \ldots, q-1$. Then

$$G \subset \bigcap_{m=1}^{\infty} \bigcup_{q=m}^{\infty} G_q,$$

and we have for every m

$$|G| \leq \sum_{q=m}^{\infty} |G_q| = \sum_{q=m}^{\infty} q(2q^{-(2+\varepsilon)}) \leq Cm^{-\varepsilon}.$$

□