

HW2, due Oct. 10

All measures we consider are non-negative. A measure μ is invariant under f if $\mu(f^{-1}(A)) = \mu(A)$ for every Borel set A . A finite measure is called a probability measure if its total mass is 1.

1. Suppose $K \subset S^1$ is a nonempty, Cantor set (a perfect and totally disconnected set). Construct a circle map that has K as its minimal set.

2. *Herman's identity remarquable.* Suppose f is a C^3 circle diffeomorphism. Show that

$$D^2 \log(Df^n) - \frac{1}{2} (D \log Df^n)^2 = \sum_{k=0}^{n-1} \varphi \circ f^k \cdot (Df^k)^2,$$

where

$$\varphi = D^2 \log Df - \frac{1}{2} (D \log Df)^2.$$

3. As a consequence establish the estimate

$$|D \log Df^n|_0 \leq (2n|\varphi|_0)^{1/2} \max_{0 \leq k < n} |Df^k|_0.$$

Here $|f|_0$ denotes $\max_{x \in S^1} |f(x)|$.

4. Prove that every circle map has an invariant probability measure (Hint: consider the measures μ_n defined by

$$\mu_n(A) = \frac{1}{n} \sum_{k=0}^{n-1} \lambda(g^{-k}(A))$$

where λ is some probability measure on S^1 and use compactness).

5. Suppose g is a circle map without periodic points.

- (a) Show that the support of any invariant measure μ is equal to the minimal set of g .
- (b) Show that there is at most one invariant probability measure, by verifying that for any interval I and any $x \in S^1$, μ satisfies

$$\mu(I) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_I(f^k(x)).$$

6. Consider again the renormalization of rotations (§1.4) and the Gauss map G . Let $\alpha_0 \in (0, 1)$ be an irrational rotation, and let $\alpha_n \in (0, 1)$, $n \geq 1$ be the renormalized rotations generated by equations (1.14) and (1.16) in the notes with initial condition α_0 . Let a_n be the sequence of first return times defined in §1.5. Show that

$$a_n = \left\lfloor \frac{1}{G^{n-1}(\alpha_0)} \right\rfloor, \quad n \geq 1.$$