HW2, due Sept. 26

This homework deals with the classical theory of rotation numbers. The answers to most questions can be founds in many books on dynamical systems, but try to do them without looking up the results. Notation: $S^1 = \mathbb{R}/\mathbb{Z}$. $R_\alpha : S^1 \to S^1$ denotes rotation through α, ie. $R_\alpha(x) = x + \alpha \mod 1$; $O_{R_\alpha}(x)$ denotes the orbit $\{x, R_\alpha(x), \ldots, R_\alpha^k(x) \ldots\} = \{x, x_1, x_2, \ldots\}$.

1. Suppose $\alpha \in (0,1)$ is irrational. Prove that $O_{R_\alpha(x)}$ is dense in S^1.

2. Let (a,b) denote an interval in S^1. Weyls' equidistribution theorem asserts that for irrational α

$$\lim_{n\to\infty} \frac{\#\{x_k \in (a,b),\ 0 \leq k \leq n-1\}}{n} \to b - a.$$

This is an example of an ergodic theorem: an assertion that a 'time average' equals a 'space average', and may be rewritten in the form

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} f(R_\alpha^k(x))}{n} = \int_{S^1} f(x)\, dx, \tag{1}$$

where $f(x) = \mathbf{1}_{(a,b)}(x)$ is the indicator (or characteristic) function of the set (a,b). Prove Weyl's theorem as follows:

(a) Prove (1) when f is e^{ikx}, $k \in \mathbb{Z}$.

(b) Prove (1) when f is a continuous function on S^1.

(c) Prove (1) when $f = \mathbf{1}_{(a,b)}$.

3. A non-negative sequence $\{a_n\}_{n\geq 0}$is *subadditive* if

$$a_{m+n} \leq a_m + a_n + c, \quad m, n \geq 0,$$

for some fixed $c \geq 0$. Prove that $\lim_{n\to\infty} a_n/n$ exists.

4. The classical definition of the rotation number is as follows. Suppose f is a circle map and $x_0 \in S^1$ an arbitrary point. We partition S^1 into two arcs $I_0 = [x, f(x))$ and $I_1 = [f(x), x)$. For any $y \in S^1$, let $\{y_k\}_{k\geq 0}$ denote its orbit. We define

$$\rho(f) = \lim_{n\to\infty} \frac{\#\{y_k \in I_0,\ 0 \leq k \leq n-1\}}{n}$$

It is necessary to prove that $\rho(f)$ is well-defined. This is done as follows:

(a) Let $N(y,k) = \#\{y_k \in I_0,\ 0 \le k \le n-1\}$. Prove that

$$N(y, m+n) = N(y,m) + N(f^m(y), n).$$

(b) Prove that $|N(y,m) - N(z,m)| \le 1$ for $y, z \in S^1$, $m \ge 0$.

(c) Use subadditivity.

5. The proof above shows that the rotation number is independent of y. Prove that it is also independent of the choice of x.

6. The basic theorem on rational approximation of irrational numbers is as follows. For any irrational α there exists a sequence of rational approximations p_n/q_n with $q_n \to \infty$ such that

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}.$$

Show that the continued fraction expansion (obtained via first return maps) proves this theorem.

7. Show that almost every irrational number cannot be approximated any better than this. Precisely, let $\varepsilon > 0$. Then for almost every $\alpha \in \mathbb{R}$ there exists $K = K(\alpha, \varepsilon)$ such that

$$\left| \alpha - \frac{p}{q} \right| \ge \frac{K}{|q|^{2+\varepsilon}},$$

for all integers p and $q \ne 0$.