1. Suppose $f : [0, 1] \rightarrow [0, 1]$ is $C^2$ and has a fixed point $x_\ast$. Suppose $|f'(x_\ast)| < 1$. Prove that $x_\ast$ is attracting—that is, there exists an interval $J$ containing $x_\ast$ such that $x_0 \in J$ implies $x_n \rightarrow x_\ast$.

The following questions apply to the logistic map.

2. Prove that the origin is attracting for $0 < r < 1$, and undergoes a transcritical bifurcation at $r = 1$.

3. Show that the fixed point $x = 1 - 1/r$ undergoes a first period-doubling bifurcation at $r = 3$, and a second period-doubling bifurcation at $r = 1 + \sqrt{6}$.

4. Compute the first two superstable orbits and parameter values $R_0$ and $R_1$.

5. There are several smooth dark curves running through the orbit diagram of the logistic map (Fig. 1.1). What are these curves?

6. Consider the Feigenbaum-Cvitanović functional equation. Compute a 2nd order approximate solution by assuming $g(x) \approx 1 + c_2 x^2$. If you are more ambitious, try a fourth-order approximation.