PDE, HW 5 solutions

1. Define the logarithm in $\mathbb{C}$ with a branch cut along the negative real axis. In polar coordinates, we then have

$$u = \mathrm{Re}\left(\frac{z}{\log z}\right) = \frac{r(\cos\theta \log r + \theta \sin\theta)}{(\log r)^2 + \theta^2}.$$

The boundary $\partial\Omega = \{u = 0\}$ is given parametrically by,

$$x(\theta) = r(\theta)\cos\theta, \quad y(\theta) = r(\theta)\sin\theta, \quad r(\theta) = e^{-\theta\tan(\theta)}, \quad \theta \in (-\pi/2, \pi/2).$$

As $\theta \to \pm\pi/2$, $r(\theta) \to 0$, thus $\partial\Omega$ is a closed curve defined on $[-\pi/2, \pi/2]$. Observe also that the tangent vector is continuous at $\pm\pi/2$ since

$$\begin{aligned}\frac{dx}{dy} = \frac{dx/d\theta}{dy/d\theta} &= \frac{\cos\theta\frac{dr}{d\theta} - r\sin\theta}{\sin\theta\frac{dr}{d\theta} + r\cos\theta} \\ &= \frac{-2\tan\theta - \theta\sec^2\theta}{-\tan^2\theta - \theta\tan\theta\sec^2\theta + 1} \to 0, \quad \text{as} \quad \theta \to \pm\frac{\pi}{2}.\end{aligned}$$

Finally, since u is harmonic in the interior, we have

$$u(x,0) = \frac{x}{\log x}, \quad \text{and} \quad u_x(0,0) = 0.$$

$\square$

2. Let us denote the linear operator by

$$L = \triangle + b\frac{x_i x_j}{|x|^2}D_{ij}u, \quad b = -1 + \frac{n-1}{1-\lambda}.$$

First check that L is uniformly elliptic. For any $\xi \in \mathbb{R}^n$ we have

$$\xi^t A(x)\xi = |\xi|^2 + b\frac{x\cdot\xi}{|x|^2} \geq |\xi|^2,$$

since

$$b = -1 + \frac{n-1}{1-\lambda} = \frac{n+\lambda}{1-\lambda} \geq 0,$$

provided $\lambda < 1$ as assumed. This may also be written as the condition $2(2-\lambda) > 2$. Now verify that u_2 is a solution. Differentiate to find

$$D_{ij}u_2 = \lambda(\lambda-2)|x|^{\lambda-4}x_i x_j + \lambda|x|^{\lambda-2}\delta_{ij}, \quad \triangle u = \lambda(\lambda - 2 + n)|x|^{\lambda-2},$$

and substitute to see that $Lu_2 = 0$. In order that $u_2 \in W^{2,2}$ we require (see HW 4, #3) $\lambda - 2 > -n/2$ or $n > 2(2-\lambda)$.

This does not contradict the uniqueness theorem, because the equation $Lu = 0$ is *not* in divergence form with L^∞ coefficients. $\square$

6, Evans, 1 ed., p. 346. A warning: Evan's definition of L is really $-L$ in Gilbarg and Trudinger's definition.

Some smoothness assumption on $\partial\Omega$ is necessary to make sense of terms such as $\partial u/\partial\nu$. It will suffice to assume $\partial\Omega$ is C^1 (C^2 would imply an interior ball condition, which is too strong). The main observation is that a barrier yields both a sub and supersolution. Let $M = \|f\|_\infty$ and consider $v_\pm = u \pm Mw$. We then have $Lv_+ \geq 0$ in Ω, and $v_+ \geq 0$ on $\partial\Omega$. Similarly $Lv_- \leq 0$, and $v_- \leq 0$ on $\partial\Omega$. By the weak maximum and minimum principles, this implies $v_+ \geq 0$, and $v_- \leq 0$ in Ω. Since the maximum and minimum are attained at x^0, we also have the inequalities

$$\frac{\partial v_+}{\partial\nu} \leq 0, \quad \frac{\partial v_-}{\partial\nu} \geq 0, \quad \text{or} \quad M\frac{\partial w}{\partial\nu} \leq \frac{\partial u}{\partial\nu} \leq -M\frac{\partial w}{\partial\nu}.$$

We may also apply the weak minimum principle to w to obtain $w \geq 0$ in Ω and $\partial w/\partial\nu \leq 0$. Finally, since $\partial\Omega$ is C^1 and $u = 0$ on $\partial\Omega$ we have $|Du| = |\partial u/\partial\nu|$. □

7, Evans, 1ed, p. 346. This is routine. A proof may be found in the notes on Laplace equation from last semester. □

4. Here is the proof of (c), which is the most interesting. Without loss of generality, suppose $|\Omega| = 1$ and $\int_\Omega |u|^{p_0}\,dx < \infty$, some $p_0 > 0$. The function

$$p \mapsto \int_\Omega |u|^p\,dx = \int_\Omega e^{p\log|u|}$$

is then an analytic function of p on $(-\infty, p_0)$ (the convention is $e^{-\infty} = 0$). It is only necessary to justify the first derivative but this follows by taking finite differences and the dominated convergence theorem. We have

$$\frac{d\int_\Omega |u|^p\,dx}{dp} = \frac{1}{\int_\Omega |u|^p\,dx}\int_\Omega |u|^p \log|u|\,dx.$$

We now consider $\Phi_p(u)$ as a function of p defined on the interval $(-\infty, p_0]$. We write

$$\Phi_p(u) = \exp\left(\frac{1}{p}\log\int_\Omega |u|^p\,dx\right),$$

and observe that the limit $p \to 0$ is the same as evaluating the derivative at $p = 0$. □

5. Fix $r > 0$. The oscillation $\omega(r) = \max_{\theta_1,\theta_2} u(r,\theta_1) - u(r,\theta_2)$. As always the basic inequality uses the fundamental theorem of calculus,

$$|u(r,\theta_1) - u(r,\theta_2)| \leq \int_{\theta_1}^{\theta_2} \left|\frac{\partial u}{\partial \theta}\right| \, d\theta.$$

Since the maximum distance between two points on the circle is π, we have

$$\omega(r) \leq \sup_{\theta_1} \int_{\theta_1}^{\theta_1+\pi} \left|\frac{\partial u}{\partial \theta}\right| \, d\theta \leq \sqrt{\pi} \left(\int_0^{2\pi} \left|\frac{\partial u}{\partial \theta}\right|^2 \, d\theta \right)^{1/2}.$$

This inequality holds for every $0 < r < R$ and may be integrated to yield

$$\int_r^R \frac{\omega(r')^2}{r'} \, dr' \leq \pi \int_r^R \int_0^{2\pi} \frac{1}{r'} \left|\frac{\partial u}{\partial \theta}\right|^2 \, d\theta dr' \leq \pi \int_{B_R} |Du|^2 \, dx := \pi D(R).$$

If we further assume that $\omega(r)$ is non-decreasing, the left-hand side is no greater than $\omega(r)^2 \log(R/r)$ and we obtain the desired inequality.
(b) Let $a(r) = \min_\theta u(r,\theta)$, $b(r) = \max_\theta u(r,\theta)$. By the maximum principle $b(r)$ is non-decreasing, and $a(r)$ is non-increasing so that $\omega(r) = b(r) - a(r)$ is non-decreasing. Suppose $M(R) := \int_{B_R} |D \log u|^2 \, dx < \infty$ (redefine R as $R/2$ if necessary) . Then

$$M(R) = \int_{B_R} \frac{|Du|^2}{u^2} \, dx \geq \frac{D(R)}{b(R)^2} \geq \frac{\omega(r) \log(R/r)}{\pi b(R)},$$

for any $0 < r < R$ by part(a). Since $\omega(r) = b(r) - a(r)$ we obtain after simplifying and using $0 < a(R) < a(r)$

$$\frac{b(r)}{a(r)} \leq 1 + \frac{\pi M(R)}{\log(R/r)} \frac{b(R)}{a(R)}, \quad 0 < r < R$$

which implies the Harnack inequality. The bound on $M(R)$ is obtained by Moser's method, and I refer to equation (8.53) in Gilbarg and Trudinger. $\square$