PDE, HW 3 solutions

7, p.163. Suppose g is C^1. The Hopf-Lax formula implies

$$Dg(y) \in \partial L\left(\frac{x-y}{t}\right),$$

at a inverse Lagrangian point y. By problem 6, this is is equivalent to

$$\frac{x-y}{t} \in \partial H(Dg(y)),$$

which implies $y \in B(x, Rt)$. $\square$

8,p.163. The Hamiltonian is $H(p) = |p|^2$, thus $L = H* = |q|^2/4$. The Hopf-Lax formula says

$$u(x,t) = \inf_{y \in \mathbb{R}^n} \left(\frac{|x-y|^2}{4t} + u_0(y)\right).$$

If $y \in \mathbb{R}^n \backslash E$, the right hand side is $+\infty$. Thus, it suffices to consider the infimum over E where $u_0(y) = 0$. Hence,

$$u(x,t) = \inf_{y \in E} \frac{|x-y|^2}{4t} = \frac{1}{4t}\mathrm{dist}(x,E)^2.$$

$\square$

13, p. 291. We compute

$$|Du(x)| = \frac{1}{\log\left(1 + 1/|x|\right)} \frac{1}{1+|x|} \frac{1}{|x|},$$

so that

$$\int_{B(0,1)} |Du(x)|^n \, dx = \omega_n \int_0^1 \frac{1}{\log(1+1/r)} \frac{1}{1+r} \frac{1}{r} \, dr \leq \omega_n \int_{\log 2}^{\infty} \frac{1}{t^n} \, dt$$

after the change of variables $t = \log(1 + 1/r)$. $\square$

18, p. 291. Let $\hat{u}(\xi)$ denote the Fourier transform of $u(x)$. By the Fourier inversion formula $\|u\|_{L^\infty} \leq C_n \|\hat{u}\|_{L^1}$, which is controlled by the H^s norm,

$$\int_{\mathbb{R}^n} |\hat{u}| \, d\xi \leq \left(\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^s} \, d\xi\right)^{1/2} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s \, |\hat{u}|^2 \, d\xi\right)^{1/2}.$$

The first term is finite if $s > n/2$, the second is $\|u\|_{H^s}$ by Parseval's equality. $\square$

2,p.563. This is hard work. The problem requires that you absorb Kruzkov's proof for conservations laws, and adapt it to viscosity solutions. While this is a worthy goal, I am not going to do it here. The proof may be found in §V.2, Crandall and Lions, Trans. AMS, Vol. 277, (1983), pp.1-42. □

3,p.563. Suppose $u - v$ has a global maximum at (x_0, t_0). Without loss of generality, we may suppose $u(x_0, t_0) = v(x_0, t_0)$. Define

$$e_\varepsilon = \|u^\varepsilon - u\|_\infty, \eta_\varepsilon^2 = 4e_\varepsilon, v^\varepsilon(x,t) = v(x,t) + \eta_\varepsilon(|x - x_0|^2 + |t - t_0|^2).$$

Observe that if $|x - x_0|^2 + |t - t_0|^2 = r^2$ with $r^2 \geq \eta_\varepsilon$ we have

$$(u^\varepsilon - v^\varepsilon)(x,t) = (u^\varepsilon - u)(x,t) + (v - v^\varepsilon)(x,t) + (u - v)(x,t)$$
$$\leq e_\varepsilon - \eta_\varepsilon r^2 \leq e_\varepsilon - \eta_\varepsilon^2 = -3e_\varepsilon.$$

On the other hand, $(u^\varepsilon - v^\varepsilon)(x_0, t_0) = u^\varepsilon - u(x_0, t_0) \geq -e_\varepsilon$. Therefore, $u^\varepsilon - v^\varepsilon$ has a global maximum $(x_\varepsilon, t_\varepsilon)$ in the ball $B((x_0, t_0), \sqrt{\eta_\varepsilon})$. At this point, $u_t^\varepsilon = v_t^\varepsilon$ and $Du^\varepsilon = Dv^\varepsilon$ and since

$$u_t^\varepsilon + H(Du^\varepsilon, x^\varepsilon) = \varepsilon a_{ij} D_{ij} u^\varepsilon \leq 0,$$

we have the inequality

$$v_t^\varepsilon + H(Dv^\varepsilon, x^\varepsilon) \leq 0.$$

Now let $\varepsilon \to 0$ using $\eta_\varepsilon \to 0$, $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$, $v_t^\varepsilon \to v_t$, $DV^\varepsilon \to DV$. □

4, p. 563. (a) It will suffice to check the definition of viscosity solution at $x_0 = 0$. Suppose $u - v$ has a maximum at 0. Then, $(u - v)(x) \leq (u - v)(0)$ for all $x \in (-1, 1)$ which implies

$$1 - |x| - \big(v(0) + v'(0)x + o(x)\big) \leq 1 - v(0).$$

First suppose $x > 0$. We simplify, the expression and let $x \to 0$ to obtain $-v'(0) \leq 1$. Considering $x < 0$ yields $v'(0) \leq 1$, or taken together $|v'(0)| \leq 1$.
(b) Suppose $u = |x| - 1$. Let v be a smooth function such that $u - v$ has a maximum at zero. An argument as above yields $v'(0) \geq 1$ and $v'(0) \leq -1$, that is $|v'(0)| \geq 1$, contradicting the definition of viscosity solution.
(c) Here the Hamiltonian is $H(p) = -|p| + 1$, and the requirement is $|v'(x_0)| \geq 1$ when $u - v$ has a maximum at x_0. This is clearly satisfied as seen in (b).
(d) The Hamiltonian in (a) is $H(p) = |p| - 1$, not $1 - |p|$. □