PDE, HW 2 solutions

1. For brevity, let $a_\pm = a(x_\pm, t)$,$u_\pm = u(x_\pm, t)$, so that $u_\pm = (x - a_\pm)/t$. We also have $G(x, a_-, t) = G(x, a_+, t)$ which may be rewritten as the equation

$$\begin{aligned}\int_{a_-}^{a_+} u_0(y)\, dy &= \frac{(x-a_-)^2 - (x-a_+)^2}{2t} \\ &= (a_+ - a_-)\frac{2x - a_+ - a_-}{2t} = (a_+ - a_-)\frac{u_- + u_+}{2}.\end{aligned}$$

□

2. Let $U_0(y) = \int_{-\infty}^{y} u_0(y)dy$, and $M = \int_{\mathbb{R}} u_0(y)dy$. Observe that

$$\sqrt{t}u(x\sqrt{t}, t) = x - \frac{a(x\sqrt{t}, t)}{\sqrt{t}} := x - \hat{a}(x, t),$$

where $\hat{a}(x, t)$ is the argmin of the functional

$$\hat{G}(x, y, t) = \frac{(x-y)^2}{2} + U_0(\sqrt{t}y).$$

The rescaled functions $U_0(\sqrt{t}y) \to \{0, M\}$ pointwise for $y < 0$ and $y > 0$ respectively. This suggests that there may be a limiting Cole-Hopf functional as $t \to \infty$. However, some care is needed with the value at zero. The Cole-Hopf procedure requires only that U_0 is lower semicontinuous, and graphically (see Figure 0.1) it is clear that the limiting functional should be

$$U_\infty(y) = \begin{cases} 0, & y < 0 \\ N, & y = 0 \\ M, & y > 0, \end{cases}$$

where $N = \min_y U_0(y) \le \min(0, M)$. If we consider the associated Hamilton-Jacobi equation for $U(x, t) = \int_{-\infty}^{x} u(y, t)dy$, then it is easy to see from the Hopf-Lax formula that $N = \min_x U(x, t)$ for every $t \ge 0$.

Recall that we may solve for $a(x, t)$ graphically by moving a parabola centered at x upwards until it touches U_0. It is then clear that $\hat{a}(x, t)$ converges pointwise to $a_\infty(x)$, the argmin of $(x - y)^2/2 + U_\infty(y)$. Here, we have $x = a_\infty(x)$ except for points where the parabola may touch $(0, N)$ first. This occurs in the range $x \in [x_1, x_2]$ where

$$x_1 = -\sqrt{-2N}, \quad x_2 = \sqrt{2(M-N)}, \quad a(x) = 0.$$

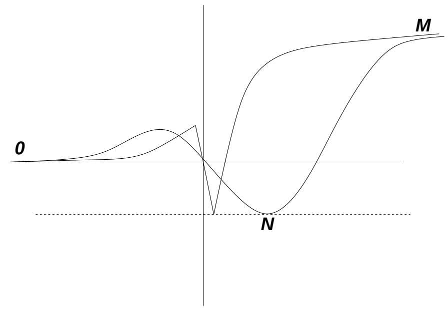

Figure 0.1: The effect of rescaling U_0

Thus, the limiting solution is

$$\lim_{t\to\infty} \sqrt{t}u(x\sqrt{t},t) = \begin{cases} 0, & x < x_1, \\ x, & x \in (x_1,x_2), \\ 0, & x > x_2. \end{cases}$$

$\square$

3. A viscous shock is a traveling wave connecting the end states $u_\pm$. That is, a solution $u^\varepsilon(x-ct) = v((x-ct)/\varepsilon)$ where

$$-c(v-u_-) + f(v) - f(u_-) = v', \quad v(\pm\infty) = u_\pm.$$

The dissipation is

$$\begin{aligned} \lim_{\varepsilon\to 0} \frac{d}{dt}\int_{\mathbb{R}} (u^\varepsilon)^2\,dx &= -\lim_{\varepsilon\to 0} 2\varepsilon \int_{\mathbb{R}} (u_x^\varepsilon)^2\,dx = -2\int_{\mathbb{R}} (v')^2\,dx \\ &= -2\int_{u_-}^{u_+} v'\,dv = -2\int_{u_-}^{u_+} \left(-c(v-u_-) + f(v) - f(u_-)\right)\,dv. \end{aligned}$$

In general, one cannot integrate this without further assumptions. One such assumption is that of a *weak shock*, ie. $u_+ \approx u_-$. By Taylor's theorem

$$\begin{aligned} 2\int_{u_-}^{u_+} v'\,dv &\approx 2\int_{u_+}^{u_-} (v-u_-)\left[f'(u_-) - c + \frac{1}{2}f''(u_-)(v-u_-)^2\right] dv \\ &= (f'(u_-)-c)^2(u_+-u_-)^2 + f''(u_-)\frac{(u_+-u_-)^3}{3} \approx \frac{f''(u_-)}{6}(u_- - u_+)^3. \end{aligned}$$

$\square$

4. Let $M = \|u\|_{L^\infty(Q)}$. Choose $k = \pm M$ in Kružkov's definition to obtain

$$\int_0^\infty \int_{\mathbb{R}^n} [u\varphi_t + f(u) \cdot D_x\varphi] \, dx \, dt = 0, \tag{0.1}$$

for every $\varphi \in C_c^\infty(Q)$ with $\varphi \geq 0$. Any $\varphi \in C_c^\infty(Q)$ can be written as $\varphi = \varphi_+ - \varphi_-$, however $\varphi_\pm$ are not C^∞ in general. Pick a positive mollifier ψ and consider $\varphi^\varepsilon = \varphi_+^\varepsilon - \varphi_-^\varepsilon$, where $\varphi_\pm^\varepsilon = \psi_\varepsilon \star \varphi_\pm$. Then $\varphi_\pm^\varepsilon$ are C_c^∞ and non-negative, so that we have

$$\int_0^\infty \int_{\mathbb{R}^n} [u\varphi_t^\varepsilon + f(u) \cdot D_x\varphi^\varepsilon] \, dx \, dt = 0,$$

for every $\varepsilon > 0$. Since φ^ε converges uniformly to φ along with all derivatives, and $\|u\|_\infty, \|f(u)\| < \infty$ we may pass to the limit by the dominated convergence theorem. $\square$

5. An apology: what I had in mind was that $u_- > u_+$ holds for shocks when there is a well-defined normal. This requires that u be BV, which was not mentioned in the hypotheses. We also need strict convexity of f. More to the point, what I had in mind was only the simple computation below that shows that Kružkov's condition implies Oleinik's condition. Proceeding as in Evans, p. 139, if $\nu = (\nu_t, \nu_x)$ is the normal to the shock curve in space-time with $\nu_t > 0$ we find

$$\begin{aligned} &|u_+ - k|\nu_t + \operatorname{sgn}(u_+ - k)\,(f(u_+) - f(k))\,\nu_x \leq \\ &\qquad |u_- - k|\nu_t + \operatorname{sgn}(u_+ - k)\,(f(u_-) - f(k))\,\nu_x, \quad k \in \mathbb{R}. \end{aligned} \tag{0.2}$$

Choosing $k > \max(u_-, u_+)$ we find

$$0 \leq (u_+ - u_-)\nu_t + (f(u_+) - f(u_-))\nu_x.$$

Similarly, $k < \min(u_-, u_+)$ yields the opposite inequality, and we have

$$(f(u_-) - f(u_+))\nu_t + (u_- - u_+)\nu_x = 0.$$

Now suppose $u_- < u_+$. We choose $u_- < k < u_+$ to obtain

$$(u_+ - k)\nu_t + (f(u_+) - f(k))\nu_x \leq (k - u_-)\nu_t + (f(k) - f(u_-))\nu_x,$$

which may be simplified to the inequality

$$2\nu_t \left(\frac{k - u_-}{u_+ - u_-} f(u_+) + \frac{u_+ - k}{u_+ - u_-} f(u_-) - f(k) \right) \leq 0.$$

Since $\nu_t > 0$, the left hand side is positive by the strict convexity of f, contradicting the inequality. The only other possibility is $u_- > u_+$. $\square$

6. Let us denote mean values by $(f)_{x,r} = \fint_{B(x,r)} f(y)\,dy$. A standard mollifier has compact support, so without loss we may assume that $f \in L^1(\mathbb{R}^n)$ (as opposed to $L^1_{loc}(\mathbb{R}^n)$). By Lebesgue's differentiation theorem, $\lim_{r\to 0}(f)_{x,r} = f(x)$ for a.e $x \in \mathbb{R}^n$. Since

$$|f(x) - f_\varepsilon(x)| \leq |f(x) - (f)_{x,\varepsilon}| + |(f)_{x,\varepsilon} - f_\varepsilon(x)|,$$

it will suffice to show that the second term goes to zero at all Lebesgue points x. This term may be rewritten as

$$\begin{aligned} |(f)_{x,\varepsilon} - f_\varepsilon(x)| &= \left| \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} (f(y) - (f)_{x,\varepsilon})\, \psi\left(\frac{x-y}{\varepsilon}\right) dy \right| \\ &\leq \frac{\omega_n \|\psi\|_\infty}{n\varepsilon^n} \fint_{B(x,\varepsilon)} |f(y) - (f)_{x,\varepsilon}|\, dy \to 0, \end{aligned}$$

as $\varepsilon \to$ since x is a Lebesgue point. $\square$

7. Change coordinates via the transformation

$$a = \frac{1}{2}(x-y), b = \frac{1}{2}(x+y), \quad x = a+b, y = b-a.$$

We may then write V_ε as the integral

$$V_\varepsilon = \frac{1}{2^n} \int_{|b|\leq\rho} \int_{|a|\leq\varepsilon} |v(b+a) - v(b-a)|\, da\, db.$$

The proof of problem (6) shows that Lebesgue's differentiation theorem also applies to averages over concentric cubes, thus

$$\lim_{\varepsilon\to 0} \frac{1}{2^n} \int_{|a|\leq\varepsilon} |v(b+a) - v(b-a)|\, da = 0,$$

for a.e $b \in \mathbb{R}^n$. Moreover, for any $\varepsilon > 0$ we have the uniform bound,

$$\left| \frac{1}{2^n} \int_{|a|\leq\varepsilon} |v(b+a) - v(b-a)|\, da \right| \leq 2\|v\|_\infty.$$

We apply the dominated convergence theorem to obtain $\lim_{\varepsilon\to 0} V_\varepsilon = 0$. $\square$