PDE, HW 2 solutions

1. For brevity, let \( a_\pm = a(x_\pm, t), u_\pm = u(x_\pm, t) \), so that \( u_\pm = (x - a_\pm)/t \). We also have \( G(x, a_-, t) = G(x, a_+, t) \) which may be rewritten as the equation

\[
\int_{a_-}^{a_+} u_0(y) \, dy = \frac{(x - a_-)^2 - (x - a_+)^2}{2t} = \frac{(a_+ - a_-)(2x - a_+ - a_-)}{2t} = (a_+ - a_-) \frac{a_- + u_+}{2}.
\]

2. Let \( U_0(y) = \int_{-\infty}^{y} u_0(y) \, dy \), and \( M = \int_{\mathbb{R}} u_0(y) \, dy \). Observe that

\[
\sqrt{t} u(x\sqrt{t}, t) = x - \frac{a(x\sqrt{t}, t)}{\sqrt{t}} := x - \hat{a}(x, t),
\]

where \( \hat{a}(x, t) \) is the argmin of the functional

\[
\hat{G}(x, y, t) = \frac{(x - y)^2}{2} + U_0(\sqrt{t}y).
\]

The rescaled functions \( U_0(\sqrt{t}y) \to \{0, M\} \) pointwise for \( y < 0 \) and \( y > 0 \) respectively. This suggests that there may be a limiting Cole-Hopf functional as \( t \to \infty \). However, some care is needed with the value at zero. The Cole-Hopf procedure requires only that \( U_0 \) is lower semicontinuous, and graphically (see Figure 0.1) it is clear that the limiting functional should be

\[
U_\infty(y) = \begin{cases} 
0, & y < 0 \\
N, & y = 0 \\
M, & y > 0,
\end{cases}
\]

where \( N = \min_y U_0(y) \leq \min(0, M) \). If we consider the associated Hamilton-Jacobi equation for \( U(x, t) = \int_{-\infty}^{x} u(y, t) \, dy \), then it is easy to see from the Hopf-Lax formula that \( N = \min_x U(x, t) \) for every \( t \geq 0 \).

Recall that we may solve for \( a(x, t) \) graphically by moving a parabola centered at \( x \) upwards until it touches \( U_0 \). It is then clear that \( \hat{a}(x, t) \) converges pointwise to \( a_\infty(x) \), the argmin of \( (x - y)^2/2 + U_\infty(y) \). Here, we have \( x = a_\infty(x) \) except for points where the parabola may touch \( (0, N) \) first. This occurs in the range \( x \in [x_1, x_2] \) where

\[
x_1 = -\sqrt{-2N}, \quad x_2 = \sqrt{2(M - N)}, \quad a(x) = 0.
\]
Figure 0.1: The effect of rescaling $U_0$

Thus, the limiting solution is

$$
\lim_{t \to \infty} \sqrt{t}u(x\sqrt{t}, t) = \begin{cases} 
0, & x < x_1, \\
, & x \in (x_1, x_2), \\
0, & x > x_2.
\end{cases}
$$

3. A viscous shock is a traveling wave connecting the end states $u_\pm$. That is, a solution $u^\varepsilon(x - ct) = v((x - ct)/\varepsilon)$ where

$$
-c(v - u_-) + f(v) - f(u_-) = v', \quad v(\pm \infty) = u_\pm.
$$

The dissipation is

$$
\lim_{\varepsilon \to 0} \frac{d}{dt} \int_{\mathbb{R}} (u^\varepsilon)^2 \, dx = -2 \lim_{\varepsilon \to 0} \varepsilon \int_{\mathbb{R}} (u^\varepsilon)^2 \, dx = -2 \int_{\mathbb{R}} (v')^2 \, dx \\
= -2 \int_{u_-}^{u_+} \! \! \! v' \, dv = -2 \int_{u_-}^{u_+} \left( -c(v - u_-) + f(v) - f(u_-) \right) \, dv.
$$

In general, one cannot integrate this without further assumptions. One such assumption is that of a weak shock, i.e. $u_+ \approx u_-$. By Taylor’s theorem

\begin{align*}
2 \int_{u_-}^{u_+} \! \! \! v' \, dv & \approx 2 \int_{u_-}^{u_+} \left( f'(u_-) - c + \frac{1}{2} f''(u_-)(v - u_-)^2 \right) \, dv \\
& = \left( f'(u_-) - c \right)^2 (u_+ - u_-)^2 + f''(u_-) \frac{(u_+ - u_-)^3}{3} \approx \frac{f''(u_-)}{6} (u_- - u_+)^3.
\end{align*}
4. Let $M = \|u\|_{L^\infty(Q)}$. Choose $k = \pm M$ in Kružkov’s definition to obtain

$$\int_0^\infty \int_{\mathbb{R}^n} [u \varphi_t + f(u) \cdot D_x \varphi] \, dx \, dt = 0,$$

(0.1)

for every $\varphi \in C_c^\infty(Q)$ with $\varphi \geq 0$. Any $\varphi \in C_c^\infty(Q)$ can be written as $\varphi = \varphi_+ - \varphi_-$, however $\varphi_{\pm}$ are not $C^\infty$ in general. Pick a positive mollifier $\psi$ and consider $\varphi^\varepsilon = \varphi_+^\varepsilon - \varphi_-^\varepsilon$, where $\varphi_{\pm}^\varepsilon = \psi^\varepsilon \ast \varphi_{\pm}$. Then $\varphi_{\pm}^\varepsilon$ are $C_c^\infty$ and non-negative, so that we have

$$\int_0^\infty \int_{\mathbb{R}^n} [u \varphi_t^\varepsilon + f(u) \cdot D_x \varphi^\varepsilon] \, dx \, dt = 0,$$

for every $\varepsilon > 0$. Since $\varphi^\varepsilon$ converges uniformly to $\varphi$ along with all derivatives, and $\|u\|_{L^\infty}, \|f(u)\| < \infty$ we may pass to the limit by the dominated convergence theorem. \qed

5. An apology: what I had in mind was that $u_- > u_+$ holds for shocks when there is a well-defined normal. This requires that $u$ be BV, which was not mentioned in the hypotheses. We also need strict convexity of $f$. More to the point, what I had in mind was only the simple computation below that shows that Kružkov’s condition implies Oleinik’s condition. Proceeding as in Evans, p. 139, if $\nu = (\nu_t, \nu_x)$ is the normal to the shock curve in space-time with $\nu_t > 0$ we find

$$|u_- - k| \nu_t + \text{sgn}(u_+ - k) (f(u_+) - f(k)) \nu_x \leq (0.2)$$

$$|u_+ - k| \nu_t + \text{sgn}(u_- - k) (f(u_-) - f(k)) \nu_x, \quad k \in \mathbb{R}.$$ 

Choosing $k > \max(u_-, u_+)$ we find

$$0 \leq (u_+ - u_-) \nu_t + (f(u_+) - f(u_-)) \nu_x.$$ 

Similarly, $k < \min(u_-, u_+)$ yields the opposite inequality, and we have

$$(f(u_-) - f(u_+)) \nu_t + (u_- - u_+) \nu_x = 0.$$ 

Now suppose $u_- < u_+$. We choose $u_- < k < u_+$ to obtain

$$(u_+ - k) \nu_t + (f(u_+) - f(k)) \nu_x \leq (k - u_-) \nu_t + (f(k) - f(u_-)) \nu_x,$$

which may be simplified to the inequality

$$2\nu_t \left( \frac{k - u_-}{u_+ - u_-} f(u_+) + \frac{u_+ - k}{u_+ - u_-} f(u_-) - f(k) \right) \leq 0.$$ 

Since $\nu_t > 0$, the left hand side is positive by the strict convexity of $f$, contradicting the inequality. The only other possibility is $u_- > u_+$. \qed
6. Let us denote mean values by \((f)_{x,r} = \frac{1}{B(x,r)} \int f(y) \, dy\). A standard mollifier has compact support, so without loss we may assume that \(f \in L^1(\mathbb{R}^n)\) (as opposed to \(L^1_{loc}(\mathbb{R}^n)\)). By Lebesgue’s differentiation theorem, \(\lim_{r \to 0} (f)_{x,r} = f(x)\) for a.e \(x \in \mathbb{R}^n\). Since
\[
|f(x) - f_\varepsilon(x)| \leq |f(x) - (f)_{x,\varepsilon}| + |(f)_{x,\varepsilon} - f_\varepsilon(x)|,
\]
it will suffice to show that the second term goes to zero at all Lebesgue points \(x\). This term may be rewritten as
\[
|(f)_{x,\varepsilon} - f_\varepsilon(x)| = \left| \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} (f(y) - (f)_{x,\varepsilon}) \psi \left( \frac{x - y}{\varepsilon} \right) \, dy \right| \\
\leq \frac{\omega_n \|\psi\|_{\infty}}{n \varepsilon^n} \int_{B(x,\varepsilon)} |f(y) - (f)_{x,\varepsilon}| \, dy \to 0,
\]
as \(\varepsilon \to 0\) since \(x\) is a Lebesgue point. \(\Box\)

7. Change coordinates via the transformation
\[
a = \frac{1}{2}(x - y), b = \frac{1}{2}(x + y), \quad x = a + b, y = b - a.
\]
We may then write \(V_\varepsilon\) as the integral
\[
V_\varepsilon = \frac{1}{2^n} \int_{|b| \leq \rho} \int_{|a| \leq \varepsilon} |v(b + a) - v(b - a)| \, da \, db.
\]
The proof of problem (6) shows that Lebesgue’s differentiation theorem also applies to averages over concentric cubes, thus
\[
\lim_{\varepsilon \to 0} \frac{1}{2^n} \int_{|a| \leq \varepsilon} |v(b + a) - v(b - a)| \, da = 0,
\]
for a.e \(b \in \mathbb{R}^n\). Moreover, for any \(\varepsilon > 0\) we have the uniform bound,
\[
\left| \frac{1}{2^n} \int_{|a| \leq \varepsilon} |v(b + a) - v(b - a)| \, da \right| \leq 2 \|v\|_{\infty}.
\]
We apply the dominated convergence theorem to obtain \(\lim_{\varepsilon \to 0} V_\varepsilon = 0\). \(\Box\)