PDE, HW 1 solutions

1. Fix $x \in \Omega$ such that $B(x,R) \subset \Omega$. Let $M = \max_{y \in \overline{B(x,R)}} |f(y)|$. Suppose $0 < r < R$ and $\omega \in S^{n-1}$. By convexity, we have

$$f(x + r\omega) \leq (1 - \frac{r}{R})f(x) + \frac{r}{R}f(x + R\omega),$$

which implies the upper bound

$$\frac{f(x+r\omega) - f(x)}{r} \leq \frac{f(x+R\omega) - f(x)}{R} \leq \frac{2M}{R}. \tag{0.1}$$

Similarly, convexity implies

$$f(x) \leq \left(1 - \frac{r}{r+R}\right) f(x + r\omega) + \frac{r}{r+R} f(x - R\omega),$$

which implies the lower bound

$$\frac{f(x+r\omega) - f(x)}{r} \geq \frac{f(x+r\omega) - f(x-R\omega)}{r+R} \geq -\frac{2M}{R}.$$

□

2. First assume that $f = \sup_k L_k$. Suppose $x = \theta y + (1-\theta)z$, $\theta \in [0,1]$. Let $\varepsilon > 0$, and choose k such that $f(x) \leq L_k(x) + \varepsilon$. But then

$$\begin{aligned} &f(x) - \varepsilon \leq L_k(x) = \theta L_k(y) + (1-\theta)L_k(z) \\ &\leq \theta \sup_m L_m(y) + (1-\theta) \sup_n L_n(z) = \theta f(y) + (1-\theta) f(z). \end{aligned}$$

The other direction requires more work. The first inequality in (0.1) shows that the difference quotient $(f(x + r\omega) - f(x))/r$ is an increasing function on $(0,\infty)$ for every direction $\omega \in S^{n-1}$. Thus, it has a limit as $r \to 0$, and we may use linearity to deduce that

$$\lim_{t \to 0_+} \frac{f(x+tz) - f(x)}{t} = A(x, \frac{z}{|z|})|z|, \quad z \in \mathbb{R}^n.$$

One may now use the standard basis of $\mathbb{R}^n$ and the fact that $(f(x + r\omega) - f(x))/r$ is increasing to deduce that for every $x \in \Omega$ there is at least one $a(x) \in \mathbb{R}^n$ such that

$$f(y) - f(x) \geq a(x) \cdot (y - x), \quad y \in \mathbb{R}^n.$$

The set of such $a(x)$ is the *subdifferential* of f at x. We construct L_k as follows. Let x_k be a countable dense subset of $\mathbb{R}^n$, and choose a_k from the subdifferential of f at x. Let $L_k(x) = a_k \cdot (x - x_k) + f(x_k)$. The choice of a_k ensures $f(x) \geq L_k(x)$ for every k, thus $f(x) \geq \sup_k L_k(x)$. The opposite inequality is proven as follows. Fix $x \in \mathbb{R}^n$ and a subsequence $x_{k_l} \to x$. Observe that a_{k_l} are uniformly bounded by Problem 1. Thus, we have

$$L_{k_l}(x) = f(x_{k_l}) + a_{k_l}(x - x_{k_l}) \to f(x).$$

$\square$

3. We may suppose $|\Omega| = 1$. Let $\bar{u} = \int u\,dx$. It is clear that

$$L_k(\bar{u}) = \int L_k(u(x))\,dx \leq \int f(u(x))\,dx.$$

Choose a sequence L_{k_l} such that $f(\bar{u}) = \lim_{l\to\infty} L_{k_l}(\bar{u})$. Since the bound above is uniform, $f(\bar{u}) \leq \int f(u(x))\,dx$. $\square$

4. For smooth solutions $u(x,t) = u_0(x_0) = u_0(x - tu(x,t))$ and

$$\partial_x u(x,t) = \frac{1}{1 + tu_0'(x_0)}.$$

Thus the maximal time of existence is $T = -1/\min(u_0')$. $\square$

5. The traveling wave ansatz $u(x,t) = u^\varepsilon(x - ct) := u^\varepsilon(\xi)$ yields

$$-c(u^\varepsilon)' + f(u^\varepsilon)' = \varepsilon(u^\varepsilon)''. \tag{0.2}$$

If we rescale $\zeta = \xi/\varepsilon$ and set $v(\zeta) = u^\varepsilon(\xi)$ we have

$$-cv' + f(v)' = v''. \tag{0.3}$$

For any $\varepsilon > 0$ we see that (0.2) has a solution with the right conditions at $\pm\infty$ if and only (0.3) does.
(b) The basic observation is that traveling wave profiles correspond to *heteroclinic orbits* of the ODE obtained by integrating (0.3). The conditions at $\pm\infty$ imply that $u_\pm$ are equilibria and that

$$v' = -c(v - u_-) + f(v) - f(u_-), \quad c = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \tag{0.4}$$

The speed of a traveling wave (if it exists) is determined by the jump condition alone. Since we have a flow on the line, orbits connecting $u_\pm$ exist if and only if there are no equilibria in between these points. Suppose first that $u_- < u_+$. Then we require $v' > 0$ for $v \in (u_-, u_+)$. However, by convexity

$$f(v) \leq \frac{v - u_-}{u_+ - u_-} f(u_+) + \frac{u_+ - v}{u_+ - u_-} f(u_-),$$

which when substituted in (0.4) yields $v' \leq 0$. Thus, traveling waves cannot exist in this case. On the other hand, if $u_- > u_+$ we observe that for $v \in (u_+, u_-)$

$$f(v) \leq \frac{v - u_+}{u_- - u_+} f(u_-) + \frac{u_- - v}{u_- - u_+} f(u_+),$$

and we have $v' \leq 0$. In this case, one does need *strict convexity* even if this wasn't stated as such in the problem.
(c) The assumption of convexity is not necessary to ensure the existence of traveling waves. All that is a required of f is that $\mathrm{sgn}(v') = \mathrm{sgn}(u_+ - u_-)$ for v between $u_\pm$. To be concrete, suppose $u_- > u_+$. We may weaken the assumption of convexity to a *chord condition*: the graph of f lies below the chord connecting $(u_+, f(u_+))$ and $(u_-, f(u_-))$. This is necessary and sufficient. □

7. This may be checked by differentiation if u is differentiable, but let us work directly with the Cole-Hopf functional. Denote the inverse Lagrangian functionals by a and $a^{(b)}$ so that

$$u(x,t) = \frac{x - a(x,t)}{t}, \quad u^{(b)}(x,t) = \frac{x - a^{(b)}(x,t)}{t}.$$

We then see that

$$u^{(b)}(x,t) = \frac{1}{1+bt} u\left(\frac{x}{1+bt}, \frac{t}{1+bt}\right) + \frac{bx}{1+bt},$$

if and only if

$$a^b(x,t) = a\left(\frac{x}{1+bt}, \frac{t}{1+bt}\right). \tag{0.5}$$

By definition, $a(x/1+bt, t/(1+bt))$ is the argmin of the Cole-Hopf functional

$$\frac{(x/(1+bt) - y)^2}{2t/(1+bt)} + U_0(y) = \frac{(x-y)^2}{2t} + U_0(y) + \frac{by^2}{2} + \frac{btx^2}{2t(1+bt)},$$

after some algebra. On the other hand, $a^{(b)}(x,t)$ is the argmin of

$$\frac{(x-y)^2}{2t} + U_0(y) + \frac{by^2}{2}.$$

The two functionals differ only by the term $btx^2/(2t(1+bt))$. Since we are minimizing in y, this term cannot change the argmin, and (0.5) holds. □

8. We have shown that if $u_0 = -\Delta \mathbf{1}_{x>0}$, $\Delta > 0$ then

$$u(x,t) = -\Delta \mathbf{1}_{x>x(t)}, \quad x(t) = -\frac{\Delta}{2} t.$$

That is, shock paths are lines if the end states are constant. An intuitive picture is to think of the shock as a particle with mass m traveling at velocity $-\Delta/2$. This basic solution is enough to understand the problem.

First suppose $b = 0$. We then have N downward jumps, and for a short time, each of these moves on a straight line path

$$x_k(t) = y_k + v_k t, \quad v_k = -\sum_{j=1}^{k-1} \Delta_j - \frac{\Delta_k}{2}.$$

Clearly $v_{k+1} < v_k$, $k = 1, \dots, N-1$, and collisions between nearest neighbors are inevitable. The shocks at x_k and x_{k+1} meet at time

$$t_k = \frac{y_{k+1} - y_k}{v_{k+1} - v_k}, \quad k = 1, \dots, N-1.$$

Let $t_* = \min_k t_k = t_{k_*}$. Immediately after this collision, the left and right states are

$$u_- = -\sum_{j=1}^{k_*-1} \Delta_j, \quad u_+ = -\sum_{j=1}^{k_*+1} \Delta_j.$$

Thus, the velocity of the new shock is simply

$$v_{k_*} = -\sum_{j=1}^{k_*-1} \Delta_j - \frac{\Delta_{k_*} + \Delta_{k_*+1}}{2}.$$

But now we are back to a system of the kind we started with, and the process may be repeated. An efficient way of keeping track of the system is as follows. At any time t we have $N(t)$ shocks, at locations $x_k(t), k = 1 \dots N(t)$, of magnitude $\Delta_k(t)$. The solution is then

$$u(x,t) = -\sum_{k=1}^{N(t)} \Delta_k(t) \mathbf{1}_{x>x_k(t)}, \tag{0.6}$$

where

$$\frac{dx_k}{dt} = v_k(t) = -\sum_{j=1}^{k-1} \Delta_j(t) - \frac{\Delta_k(t)}{2}, \tag{0.7}$$

in between collisions. If a collision occurs at time t then $\Delta_j(t_+) = \Delta_j(t_-) + \Delta_{j+1}(t_-)$. This corresponds to a physical picture of *sticky particles* or *ballistic aggregation*. If we set $m_k = \Delta_k$, and v_k as above, then the rule of evolution is that the particles move linearly until they meet, and when they meet they conserve mass and momentum, ie.

$$m_k(t_+) = m_k(t_-)+m_{k+1}(t_-), \quad v_k(t_+) = \frac{m_k(t_-)v_k(t_-) + m_{k+1}(t_-)v_{k+1}(t_-)}{m_k(t_-) + m_{k+1}(t_-)}.$$

If $b \neq 0$ we simply apply Problem 7 to (0.6). $\square$