PDE, HW 1 solutions

1. Fix \( x \in \Omega \) such that \( B(x, R) \subset \Omega \). Let \( M = \max_{y \in B(x, R)} |f(y)| \). Suppose \( 0 < r < R \) and \( \omega \in S^{n-1} \). By convexity, we have

\[
f(x + r \omega) \leq (1 - \frac{r}{R}) f(x) + \frac{r}{R} f(x + R \omega),
\]

which implies the upper bound

\[
\frac{f(x + r \omega) - f(x)}{r} \leq \frac{f(x + R \omega) - f(x)}{R} \leq \frac{2M}{R}.
\]

Similarly, convexity implies

\[
f(x) \leq \left(1 - \frac{r}{r + R}\right) f(x + r \omega) + \frac{r}{r + R} f(x - R \omega),
\]

which implies the lower bound

\[
\frac{f(x + r \omega) - f(x)}{r} \geq \frac{f(x + r \omega) - f(x - R \omega)}{r + R} \geq -\frac{2M}{R}.
\]

\[\square\]

2. First assume that \( f = \sup_k L_k \). Suppose \( x = \theta y + (1 - \theta) z, \theta \in [0, 1] \). Let \( \varepsilon > 0 \), and choose \( k \) such that \( f(x) \leq L_k(x) + \varepsilon \). But then

\[
f(x) - \varepsilon \leq L_k(x) = \theta L_k(y) + (1 - \theta) L_k(z)
\]

\[
\leq \theta \sup_m L_m(y) + (1 - \theta) \sup_n L_n(z) = \theta f(y) + (1 - \theta) f(z).
\]

The other direction requires more work. The first inequality in (0.1) shows that the difference quotient \( (f(x + r \omega) - f(x))/r \) is an increasing function on \((0, \infty)\) for every direction \( \omega \in S^{n-1} \). Thus, it has a limit as \( r \to 0 \), and we may use linearity to deduce that

\[
\lim_{t \to 0^+} \frac{f(x + tz) - f(x)}{t} = A(x, \frac{z}{|z|}|z|), \quad z \in \mathbb{R}^n.
\]

One may now use the standard basis of \( \mathbb{R}^n \) and the fact that \( (f(x + r \omega) - f(x))/r \) is increasing to deduce that for every \( x \in \Omega \) there is at least one \( a(x) \in \mathbb{R}^n \) such that

\[
f(y) - f(x) \geq a(x) \cdot (y - x), \quad y \in \mathbb{R}^n.
\]
The set of such \( a(x) \) is the subdifferential of \( f \) at \( x \). We construct \( L_k \) as follows. Let \( x_k \) be a countable dense subset of \( \mathbb{R}^n \), and choose \( a_k \) from the subdifferential of \( f \) at \( x \). Let \( L_k(x) = a_k \cdot (x - x_k) + f(x_k) \). The choice of \( a_k \) ensures \( f(x) \geq L_k(x) \) for every \( k \), thus \( f(x) \geq \sup_k L_k(x) \). The opposite inequality is proven as follows. Fix \( x \in \mathbb{R}^n \) and a subsequence \( x_{k_l} \to x \). Observe that \( a_{k_l} \) are uniformly bounded by Problem 1. Thus, we have
\[
L_{k_l}(x) = f(x_{k_l}) + a_{k_l}(x - x_{k_l}) \to f(x).
\]

3. We may suppose \( |\Omega| = 1 \). Let \( \bar{u} = \int u \, dx \). It is clear that
\[
L_k(\bar{u}) = \int L_k(u(x)) \, dx \leq \int f(u(x)) \, dx.
\]
Choose a sequence \( L_{k_l} \) such that \( f(\bar{u}) = \lim_{l \to \infty} L_{k_l}(\bar{u}) \). Since the bound above is uniform, \( f(\bar{u}) \leq \int f(u(x)) \, dx \). □

4. For smooth solutions \( u(x, t) = u_0(x_0) = u_0(x - tu(x, t)) \) and
\[
\partial_x u(x, t) = \frac{1}{1 + tu_0'(x_0)}.
\]
Thus the maximal time of existence is \( T = -1/\min(u_0') \). □

5. The traveling wave ansatz \( u(x, t) = u^\varepsilon(x - ct) := u^\varepsilon(\xi) \) yields
\[
-c(u^\varepsilon)' + f(u^\varepsilon)' = \varepsilon(u^\varepsilon)''.
\]
If we rescale \( \zeta = \xi/\varepsilon \) and set \( v(\zeta) = u^\varepsilon(\xi) \) we have
\[
-cv' + f(v)' = v''.
\]
For any \( \varepsilon > 0 \) we see that (0.2) has a solution with the right conditions at \( \pm \infty \) if and only (0.3) does.

(b) The basic observation is that traveling wave profiles correspond to heteroclinic orbits of the ODE obtained by integrating (0.3). The conditions at \( \pm \infty \) imply that \( u_\pm \) are equilibria and that
\[
v' = -c(v - u_-) + f(v) - f(u_-), \quad c = \frac{f(u_+ - f(u_-))}{u_+ - u_-}.
\]
The speed of a traveling wave (if it exists) is determined by the jump condition alone. Since we have a flow on the line, orbits connecting \( u_\pm \) exist if and
only if there are no equilibria in between these points. Suppose first that $u_- < u_+$. Then we require $v' > 0$ for $v \in (u_-, u_+)$. However, by convexity

$$f(v) \leq \frac{v - u_-}{u_+ - u_-} f(u_+) + \frac{u_+ - v}{u_+ - u_-} f(u_-),$$

which when substituted in (0.4) yields $v' \leq 0$. Thus, traveling waves cannot exist in this case. On the other hand, if $u_+ > u_-$ we observe that for $v \in (u_+, u_-)$

$$f(v) \leq \frac{v - u_+}{u_+ - u_-} f(u_-) + \frac{u_- - v}{u_+ - u_-} f(u_+),$$

and we have $v' \leq 0$. In this case, one does need strict convexity even if this wasn’t stated as such in the problem.

(c) The assumption of convexity is not necessary to ensure the existence of traveling waves. All that is required of $f$ is that $\text{sgn}(v') = \text{sgn}(u_+ - u_-)$ for $v$ between $u_\pm$. To be concrete, suppose $u_- > u_+$. We may weaken the assumption of convexity to a chord condition: the graph of $f$ lies below the chord connecting $(u_+, f(u_+))$ and $(u_-, f(u_-))$. This is necessary and sufficient.

7. This may be checked by differentiation if $u$ is differentiable, but let us work directly with the Cole-Hopf functional. Denote the inverse Lagrangian functionals by $a$ and $a^{(b)}$ so that

$$u(x, t) = \frac{x - a(x, t)}{t}, \quad u^{(b)}(x, t) = \frac{x - a^{(b)}(x, t)}{t}.$$

We then see that

$$u^{(b)}(x, t) = \frac{1}{1 + bt} u(\frac{x}{1 + bt}, \frac{t}{1 + bt}) + \frac{bx}{1 + bt},$$

if and only if

$$a^{b}(x, t) = a(\frac{x}{1 + bt}, \frac{t}{1 + bt}). \quad (0.5)$$

By definition, $a(x/1+bt, t/(1+bt))$ is the argmin of the Cole-Hopf functional

$$\frac{(x/(1 + bt) - y)^2}{2t/(1 + bt)} + U_0(y) = \frac{(x - y)^2}{2t} + U_0(y) + \frac{by^2}{2} + \frac{bt x^2}{2t(1 + bt)},$$

after some algebra. On the other hand, $a^{(b)}(x, t)$ is the argmin of

$$\frac{(x - y)^2}{2t} + U_0(y) + \frac{by^2}{2}.$$ 

The two functionals differ only by the term $bt x^2/(2t(1 + bt))$. Since we are minimizing in $y$, this term cannot change the argmin, and (0.5) holds.
8. We have shown that if \( u_0 = -\Delta \mathbf{1}_{x>0} \), \( \Delta > 0 \) then
\[
 u(x,t) = -\Delta \mathbf{1}_{x>x(t)}, \quad x(t) = -\frac{\Delta}{2} t.
\]

That is, shock paths are lines if the end states are constant. An intuitive picture is to think of the shock as a particle with mass \( m \) traveling at velocity \(-\Delta/2\). This basic solution is enough to understand the problem.

First suppose \( b = 0 \). We then have \( N \) downward jumps, and for a short time, each of these moves on a straight line path
\[
 x_k(t) = y_k + v_k t, \quad v_k = -\sum_{j=1}^{k-1} \Delta_j - \frac{\Delta_k}{2}.
\]

Clearly \( v_{k+1} < v_k, \ k = 1, \ldots, N-1 \), and collisions between nearest neighbors are inevitable. The shocks at \( x_k \) and \( x_{k+1} \) meet at time
\[
 t_k = \frac{y_{k+1} - y_k}{v_{k+1} - v_k}, \quad k = 1, \ldots, N-1.
\]

Let \( t_\ast = \min_k t_k = t_{k\ast} \). Immediately after this collision, the left and right states are
\[
 u_- = -\sum_{j=1}^{k\ast-1} \Delta_j, \quad u_+ = -\sum_{j=1}^{k\ast+1} \Delta_j.
\]

Thus, the velocity of the new shock is simply
\[
 v_{k\ast} = -\sum_{j=1}^{k\ast-1} \Delta_j - \frac{\Delta_{k\ast} + \Delta_{k\ast+1}}{2}.
\]

But now we are back to a system of the kind we started with, and the process may be repeated. An efficient way of keeping track of the system is as follows. At any time \( t \) we have \( N(t) \) shocks, at locations \( x_k(t), k = 1 \ldots N(t) \), of magnitude \( \Delta_k(t) \). The solution is then
\[
 u(x,t) = -\sum_{k=1}^{N(t)} \Delta_k(t) \mathbf{1}_{x>x_k(t)}, \quad (0.6)
\]

where
\[
 \frac{dx_k}{dt} = v_k(t) = -\sum_{j=1}^{k-1} \Delta_j(t) - \frac{\Delta_k(t)}{2}, \quad (0.7)
\]
in between collisions. If a collision occurs at time $t$ then $\Delta_j(t_+) = \Delta_j(t_-) + \Delta_{j+1}(t_-)$. This corresponds to a physical picture of sticky particles or ballistic aggregation. If we set $m_k = \Delta_k$, and $v_k$ as above, then the rule of evolution is that the particles move linearly until they meet, and when they meet they conserve mass and momentum, ie.

$$m_k(t_+) = m_k(t_-) + m_{k+1}(t_-), \quad v_k(t_+) = \frac{m_k(t_-)v_k(t_-) + m_{k+1}(t_-)v_{k+1}(t_-)}{m_k(t_-) + m_{k+1}(t_-)}.$$

If $b \neq 0$ we simply apply Problem 7 to (0.6). \qed