PDE, HW 5, due Wednesday 4/19/06

1. Consider Hopf's lemma as stated in Evans p. 340. The following counterexample shows that the interior ball condition is essential. Let $L = \triangle$, and let Ω be the region of the right half plane where $u = \mathrm{Re}(z/\log z) < 0$. Show that $\partial\Omega$ is C^1 near $z = 0$ and $u_x(0,0) = 0$.

2. *Nonuniqueness for the Dirichlet problem* Consider the following elliptic equation with L^∞ coefficients

$$\triangle u + b\frac{x_i x_j}{|x|^2} D_{ij} u = 0, \quad b = -1 + \frac{n-1}{1-\lambda}, \quad 0 < \lambda < 1, \quad x \in B(0,1),$$

where $B(0,1)$ denotes the unit ball in $\mathbb{R}^n$. Show that both $u_1(x) = 1$ and $u_2(x) = |x|^\lambda$ are $W^{2,2}(B)$ solutions to the Dirichlet problem agreeing on ∂B for $2 < 2(2-\lambda) < n$. Why doesn't this contradict the uniqueness theorem we proved?

3. Problems 6 and 9 p. 346-347, Evans.

4. Suppose $\Omega \subset \mathbb{R}^n$ is bounded and $|u|^p \in L^1(\Omega)$ for some $p \in \mathbb{R}$. Define

$$\Phi_p(u) = \left[\frac{1}{|\Omega|}\int_\Omega |u|^p\, dx\right]^{1/p}.$$

Show that (a) $\lim_{p\to\infty} \Phi_p(u) = \sup_\Omega |u|$, (b) $\lim_{p\to-\infty} \Phi_p(u) = \inf_\Omega |u|$, (c) $\lim_{p\to 0} \Phi_p(u) = \exp\left(\fint_\Omega \log|u|\, dx\right)$.

5. *The Harnack inequality in* $\mathbb{R}^2$ This is problem 8.5, p. 217 in Gilbarg and Trudinger. You may need to read the proof of Thm. 8.18 there to solve this problem. It is stated here for completeness.
(a) In this problem B_r denotes $B(0,r) \subset \mathbb{R}^2$. Let $u \in C^1(B_R)$, and write for $0 < r < R$,

$$\omega(r) = \mathrm{osc}_{\partial B_r} u, \quad D(r) = \int_{B_r} |Du|^2\, dx.$$

If ω is nondecreasing, show that for $0 < r < R$ we have

$$\omega(r) \le \sqrt{\frac{\pi D(r)}{\log(R/r)}}.$$

(b) Let $Lu = D_i\,(a_{ij} D_j u) + b_i D_i u = 0$ where we sum over $i, j = 1, 2$. Suppose that A, b satisfy the standard hypotheses (E_1) and (E_2). Prove the Harnack inequality as follows. Assuming $u > 0$ in B_R, show that the Dirichlet integral of $v = \log u$ is bounded in every disk $B_r, 0 < r < R$, in terms of $\Lambda/\lambda, \nu, r, R$. Apply the weak maximum principle and part (a) to obtain

$$C^{-1}u(0) \leq u(x) \leq Cu(0), \quad x \in B_{R/2}, \quad C = C(\lambda, \Lambda, R).$$