PDE, HW 4, due Friday 3/24/06

1. Show that $W^{1,p}(\mathbb{R}^n) = W_0^{1,p}(\mathbb{R}^n), 1 \le p < \infty$.

2. Show that Campanato's theorem characterizes $C^{0,\alpha}(\overline{\Omega})$ by proving the following converse: if $u \in C^{0,\alpha}(\overline{\Omega})$ then there is $M > 0$ such that

$$\fint_{B(x,r)} |u(y) - \overline{u}_B| \, dy \le M r^\alpha, \quad B(x,r) \subset \Omega.$$

3. Let B denote the unit ball centered at the origin. When is $|x|^\alpha, \alpha \in \mathbb{R}$ in $W^{1,p}(B)$, $1 \le p \le \infty$?

4. Show that $\log |x|$ is in $BMO(\mathbb{R}^n)$.

5. Show that Rademacher's theorem cannot be extended to finite p in the range $n < p < \infty$. That is, there are functions in $C^{0,1-n/p}(\overline{\Omega})$ that are not in $W^{1,p}(\Omega)$.

6. The total variation of a function $u \in L^1(\Omega)$ is defined by

$$\int_\Omega |Du| \, dx = \sup \left\{ \int_\Omega u \operatorname{div} \mathbf{v} \,\middle|\, \mathbf{v} \in C_c^1(\Omega; \mathbb{R}^n), |\mathbf{v}| \le 1 \right\}.$$

Show that the space $BV(\Omega)$ of functions of finite total variation is a Banach space under the norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + \int_\Omega |Du|,$$

and that $W^{1,1}(\Omega)$ is a closed subspace.

7. Let Ω be bounded and $u \in BV(\Omega)$. Show that there exists a sequence $\{u_k\} \subset C^\infty(\Omega) \bigcap W^{1,1}(\Omega)$ such that $u_m \to u$ in $L^1(\Omega)$ and

$$\int_\Omega |Du_m| \, dx \to \int_\Omega |Du| \, dx.$$