PDE, HW 2, due Wednesday 2/22/06

In problems 1 and 2, $u(x,t)$ denotes the Cole-Hopf solution to Burgers equation with initial data u_0. In problems 4 and 5, $u(x,t)$ is the entropy solution (in the sense of Kružkov) to

$$u_t + D_x \cdot (f(u)) = 0, \quad x \in \mathbb{R}^n, t > 0, \quad u(x,0) = u_0(x). \tag{0.1}$$

Problems 6 and 7 are technical steps in Kružkov's proof, best resolved with Lebesgue's differentiation theorem.

1. Suppose there is a shock at (x,t). Show that we have conservation of momentum in the sense that

$$\frac{1}{2}\left(u(x_-,t) + u(x_+,t)\right) = \frac{1}{a_+(x,t) - a_-(x,t)} \int_{a_-(x,t)}^{a_+(x,t)} u_0(y)\,dy.$$

2. *The N-wave:* Suppose $u_0 \in L^1(\mathbb{R})$. For fixed $x \in \mathbb{R}$ show that $\bar{u}(x) = \lim_{t\to\infty} \sqrt{t}u(x\sqrt{t},t)$ exists. Determine $\bar{u}$ explicitly.

3. Suppose $u^\varepsilon(x,t)$ is a viscous shock connecting the states $u_- > u_+$ for a scalar conservation law with convex flux f, ie.

$$u_t^\varepsilon + (f(u^\varepsilon))_x = \varepsilon u_{xx}^\varepsilon.$$

Compute the energy dissipation at shocks, that is $\lim_{\varepsilon\to 0} \varepsilon \int_{\mathbb{R}} (\partial_x u^\varepsilon)^2\,dx$.

4. Show that an entropy solution to (0.1) is a weak solution.

5. Let $n = 1$, and suppose $f : \mathbb{R} \to \mathbb{R}$ is convex. Show that a piecewise continuous entropy solution with left and right limits satisfies $u(x_-,t) > u(x_+,t)$ at points of discontinuity.

6. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is locally integrable. Let ψ be a standard mollifier, $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon)$, and $f_\varepsilon = \psi_\varepsilon \star f$. Show that $\lim_{\varepsilon\to 0} f_\varepsilon = f$ a.e.

7. Suppose $v \in L^\infty(\mathbb{R}^n)$, and $\rho, \varepsilon > 0$. Let

$$\Omega = \left\{ (x,y) \,\middle|\, \frac{|x+y|}{2} \le \rho, \frac{|x-y|}{2} \le \varepsilon. \right\},$$

and consider the integral

$$V_\varepsilon = \varepsilon^{-n} \int_\Omega |v(x) - v(y)|\; dx\,dy.$$

Show that $\lim_{\varepsilon\to 0} V_\varepsilon = 0$.