PDE, HW 2, due Wednesday 2/22/06

In problems 1 and 2, \( u(x,t) \) denotes the Cole-Hopf solution to Burgers equation with initial data \( u_0 \). In problems 4 and 5, \( u(x,t) \) is the entropy solution (in the sense of Kružkov) to

\[
u_t + D_x \cdot (f(u)) = 0, \quad x \in \mathbb{R}^n, t > 0, \quad u(x,0) = u_0(x). \quad (0.1)
\]

Problems 6 and 7 are technical steps in Kružkov’s proof, best resolved with Lebesgue’s differentiation theorem.

1. Suppose there is a shock at \((x,t)\). Show that we have conservation of momentum in the sense that

\[
\frac{1}{2} (u(x-,t) + u(x+,t)) = \frac{1}{a_+(x,t) - a_-(x,t)} \int_{a_-(x,t)}^{a_+(x,t)} u_0(y) \, dy.
\]

2. The N-wave: Suppose \( u_0 \in L^1(\mathbb{R}) \). For fixed \( x \in \mathbb{R} \) show that \( \bar{u}(x) = \lim_{t \to \infty} \sqrt{t} u(x\sqrt{t},t) \) exists. Determine \( \bar{u} \) explicitly.

3. Suppose \( u^\varepsilon(x,t) \) is a viscous shock connecting the states \( u_- > u_+ \) for a scalar conservation law with convex flux \( f \), ie.

\[
u_t^\varepsilon + (f(u^\varepsilon))_x = \varepsilon u_{xx}^\varepsilon.
\]

Compute the energy dissipation at shocks, that is \( \lim_{\varepsilon \to 0} \varepsilon \int_{\mathbb{R}} (\partial_x u^\varepsilon)^2 \, dx \).

4. Show that an entropy solution to (0.1) is a weak solution.

5. Let \( n = 1 \), and suppose \( f : \mathbb{R} \to \mathbb{R} \) is convex. Show that a piecewise continuous entropy solution with left and right limits satisfies \( u(x-,t) > u(x+,t) \) at points of discontinuity.

6. Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is locally integrable. Let \( \psi \) be a standard mollifier, \( \psi_\varepsilon(x) = \varepsilon^{-n} \psi(x/\varepsilon) \), and \( f_\varepsilon = \psi_\varepsilon \ast f \). Show that \( \lim_{\varepsilon \to 0} f_\varepsilon = f \) a.e.

7. Suppose \( v \in L^\infty(\mathbb{R}^n) \), and \( \rho, \varepsilon > 0 \). Let

\[
\Omega = \left\{ (x,y) \mid \frac{|x + y|}{2} \leq \rho, \frac{|x - y|}{2} \leq \varepsilon \right\},
\]

and consider the integral

\[
V_\varepsilon = \varepsilon^{-n} \int_\Omega |v(x) - v(y)| \, dx \, dy.
\]

Show that \( \lim_{\varepsilon \to 0} V_\varepsilon = 0 \).