PDE, HW 1, due Wednesday 2/8/06

In the following problems, you will need the following definitions. An open set \( \Omega \subset \mathbb{R}^n \) is convex if the line segment \((1 - t)x + ty, t \in [0, 1]\) is contained in \( \Omega \) when \( x, y \in \Omega \). A measurable function \( f : \Omega \to \mathbb{R} \) is convex if

\[
f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y), \quad t \in [0, 1].
\] (0.1)

\( f \) is strictly convex, if the inequality in (0.1) is strict for \( t \in (0, 1) \). It is not necessary to assume that \( f \) is continuous, but you may do so for simplicity.

1. Suppose \( f : \Omega \to \mathbb{R} \) is convex. Show that \( f \) is locally Lipschitz.

2. Show that \( f : \mathbb{R}^n \to \mathbb{R} \) is convex if and only if there exists a countable family of affine maps \( L_k(x) = a_k \cdot x + b_k, \ a_k \in \mathbb{R}^n, \ b_k \in \mathbb{R} \) such that

\[
f(x) = \sup_k L_k(x).
\]

3. Jensen's inequality. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is convex and bounded below. Use problem 2 to prove Jensen's inequality. If \( \Omega \subset \mathbb{R}^n \) is bounded, and \( u \in L^1(\Omega) \) then

\[
f \left( \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx \right) \leq \frac{1}{|\Omega|} \int_{\Omega} f(u(x)) \, dx,
\] (0.2)

where \( |\Omega| \) is the \( n \)-dimensional volume of \( \Omega \).

4. Let \( u_0 \in C^\infty_c(\mathbb{R}) \) be smooth initial data for the inviscid Burgers that is not identically zero. In particular, \( u_0 \) is smooth and not monotone. Determine the maximal time of existence for classical solutions.

5. Oleinik's entropy condition. Consider the scalar conservation law with convex flux \( f : \mathbb{R} \to \mathbb{R} \)

\[
\frac{\partial u}{\partial t} + (f(u))_x = 0.
\] (0.3)

We are interested in the admissibility of shocks connecting the states \( u_- \) and \( u_+ \). To this end, we add a viscous term to (0.3)

\[
\frac{\partial u}{\partial t} + (f(u))_x = \varepsilon u_{xx},
\] (0.4)

and look for traveling waves \( u(x, t) = u^\varepsilon(x - ct) \) with \( \lim_{x \to \pm \infty} u^\varepsilon(x) = u_\pm \).

(a) Show that traveling waves exist for some \( \varepsilon > 0 \) if and only if they exist for every \( \varepsilon > 0 \).

(b) Traveling waves exist if and only if \( u_- > u_+ \).
(c) Is the assumption of convexity of $f$ necessary? Yes, this is a little vague, but be creative.

6. **Nonuniqueness of weak solutions.** Consider the initial value problem for the inviscid Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

(0.5)

with initial data

$$u_0(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0. \end{cases}$$

(0.6)

Show that for any $\alpha \in [0,1]$ there is a weak solution

$$u_\alpha(x,t) = \begin{cases} -1, & -\infty < x \leq -t, \\ \frac{t}{\alpha}, & -t < x \leq -\alpha t, \\ -\alpha, & -\alpha t < x \leq 0, \\ \frac{x}{\alpha}, & 0 < x \leq \alpha t, \\ \alpha, & \alpha t < x \leq t, \\ x, & t < x < \infty. \end{cases}$$

7. **Removing the drift in Burgers equation.** Suppose $u(x,t)$ is the entropy (ie. Cole-Hopf) solution to (0.5) with initial data $u_0(x)$ where $u_0(x) = o(|x|)$ as $|x| \to \infty$. Suppose $b \neq 0$. Show that the entropy solution $u^{(b)}(x,t)$ for initial data $u_0^{(b)}(x) = u_0(x) + bx$ is given by

$$u^{(b)}(x,t) = \frac{1}{1 + bt} u \left( \frac{x}{1 + bt}, \frac{t}{1 + bt} \right) + \frac{bx}{1 + bt}, \quad t \in [0, T_b),$$

where $T_b = -b^{-1}$ if $b < 0$ and $T_b = \infty$ otherwise.

8. **Shock interaction in Burgers equation.** Fix $b > 0$, $\Delta_k > 0$, $k = 1, \ldots, N$ and $y_1 < \ldots < y_N$. Describe the entropy solution to (0.5) with 'sawtooth' initial data

$$u_0(x) = bx - \sum_{k=1}^{N} \Delta_k 1_{x \geq y_k}.$$

(The problem simplifies considerably if you use Problem 7 to get rid of the drift).