PDE, HW 1, due Wednesday 2/8/06

In the following problems, you will need the following definitions. An open set $\Omega \subset \mathbb{R}^n$ is convex if the line segment $(1-t)x + ty, t \in [0,1]$ is contained in Ω when $x, y \in \Omega$. A measurable function $f : \Omega \to \mathbb{R}$ is convex if

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y), \quad t \in [0,1]. \tag{0.1}$$

f is strictly convex, if the inequality in (0.1) is strict for $t \in (0,1)$. It is not necessary to assume that f is continuous, but you may do so for simplicity.

1. Suppose $f : \Omega \to \mathbb{R}$ is convex. Show that f is locally Lipschitz.

2. Show that $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if there exists a countable family of affine maps $L_k(x) = a_k \cdot x + b_k$, $a_k \in \mathbb{R}^n$, $b_k \in \mathbb{R}$ such that $f(x) = \sup_k L_k(x)$.

3. *Jensen's inequality*. Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex and bounded below. Use problem 2 to prove Jensen's inequality. If $\Omega \subset \mathbb{R}^n$ is bounded, and $u \in L^1(\Omega)$ then

$$f\left(\frac{1}{|\Omega|}\int u(x)\,dx\right) \leq \frac{1}{|\Omega|}\int_\Omega f(u(x))\,dx, \tag{0.2}$$

where $|\Omega|$ is the n-dimensional volume of Ω.

4. Let $u_0 \in C_c^\infty(\mathbb{R})$ be smooth initial data for the inviscid Burgers that is not identically zero. In particular, u_0 is smooth and not monotone. Determine the maximal time of existence for classical solutions.

5. *Oleinik's entropy condition.* Consider the scalar conservation law with convex flux $f : \mathbb{R} \to \mathbb{R}$

$$u_t + (f(u))_x = 0. \tag{0.3}$$

We are interested in the admissibility of shocks connecting the states u_- and u_+. To this end, we add a viscous term to (0.3)

$$u_t + (f(u))_x = \varepsilon u_{xx}, \tag{0.4}$$

and look for traveling waves $u(x,t) = u^\varepsilon(x - ct)$ with $\lim_{x\to\mp\infty} u^\varepsilon(x) = u_\mp$.

(a) Show that traveling waves exist for some $\varepsilon > 0$ if and only if they exist for every $\varepsilon > 0$.

(b) Traveling waves exist if and only if $u_- > u_+$.

(c) Is the assumption of convexity of f necessary? Yes, this is a little vague, but be creative.

6. *Nonuniqueness of weak solutions.* Consider the initial value problem for the inviscid Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \tag{0.5}$$

with initial data

$$u_0(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0. \end{cases} \tag{0.6}$$

Show that for any $\alpha \in [0,1]$ there is a weak solution

$$u_\alpha(x,t) = \begin{cases} -1, & -\infty < x \leq -t, \\ \frac{x}{t}, & -t < x \leq -\alpha t, \\ -\alpha, & -\alpha t < x \leq 0, \\ \alpha, & 0 < x \leq \alpha t, \\ \frac{x}{t}, & \alpha t < x \leq t, \\ 1, & t < x < \infty. \end{cases}$$

7. *Removing the drift in Burgers equation.* Suppose $u(x,t)$ is the entropy (ie. Cole-Hopf) solution to (0.5) with initial data $u_0(x)$ where $u_0(x) = o(|x|)$ as $|x| \to \infty$. Suppose $b \neq 0$. Show that the entropy solution $u^{(b)}(x,t)$ for initial data $u_0^{(b)}(x) = u_0(x) + bx$ is given by

$$u^{(b)}(x,t) = \frac{1}{1+bt} u\left(\frac{x}{1+bt}, \frac{t}{1+bt}\right) + \frac{bx}{1+bt}, \quad t \in [0, T_b),$$

where $T_b = -b^{-1}$ if $b < 0$ and $T_b = \infty$ otherwise.

8. *Shock interaction in Burgers equation.* Fix $b > 0$, $\Delta_k > 0$, $k = 1, \ldots, N$ and $y_1 < \ldots < y_N$. Describe the entropy solution to (0.5) with 'sawtooth' initial data

$$u_0(x) = bx - \sum_{k=1}^{N} \Delta_k \mathbf{1}_{x \geq y_k}.$$

(The problem simplifies considerably if you use Problem 7 to get rid of the drift).