PDE, Final exam. Due by 5 pm, Monday, May 15, 2006

Notes

1. You may use any reference sources (books, notes, internet), provided you cite these sources appropriately.
2. Do not discuss the exam with anyone.
3. If stuck, try to formulate a partial answer for special cases. Do not be discouraged if you cannot solve all problems.
4. Please inform me of any errors you find in the exam, I will post them immediately.
5. Ω is always an open subset of $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, $|x|$ always denotes the Euclidean norm $\sqrt{x_1^2 + \ldots + x_n^2}$. If $A \in \mathbb{M}^{m\times n}$, then $|A|$ denotes the Euclidean norm $\sup_{x\in\mathbb{R}^n\setminus\{0\}} |Ax|/|x|$. $B(x,r)$ denotes an open ball of radius r. $S(x,r)$ denotes its boundary.

Problems

1. Consider the following discrete approximation to a scalar conservation law

$$u_k^{n+1} = \frac{\left(u_{k-1}^n + u_{k+1}^n\right)}{2} - \left(f(u_{k+1}^n) - f(u_{k-1}^n)\right)\frac{\Delta t}{2\Delta x}. \tag{0.1}$$

Here u_k^n is the discrete approximation to the value of u at the point $(x,t) = (k\Delta x, n\Delta t)$ in space-time $\mathbb{R}\times[0,\infty)$. Focus on the case $f(u) = \log(a + be^u)$ where a and b are fixed.

(a) Show that (0.1) can be solved explicitly by a discrete version of the Cole-Hopf transformation that yields a linear difference equation.

(b) Analyze the limit $\Delta x, \Delta t \to 0$ and show that the limit yields the Hopf-Lax formula for solutions to $u_t + f(u)_x = 0$.

2. Let A denote the closure of *divergence-free* $\mathbf{u} \in C_c^\infty(\Omega;\mathbb{R}^n)$ in the L^2 norm $(\int_\Omega |\mathbf{u}|^2\,dx)^{1/2}$. Show that there is a set of divergence-free $C_c^\infty(\Omega;\mathbb{R}^n)$ vector fields $\{a^\nu\}_{\nu=1}^\infty$ that forms an orthonormal basis for A.

3. Consider the Hamilton-Jacobi equation $u_t + \sqrt{1+u_x^2} = 0$ modeling evolution of a curve. Show explicitly that the viscosity solution is not reversible. That is, find initial data u_0 such that the equations $u_t + \sqrt{1+u_x^2} = 0$ and $u_t - \sqrt{1+u_x^2} = 0$ have distinct solutions. Here $x \in \mathbb{R}$, $t \geq 0$.

4. This is from Evans (problem 6, p.488, 1st ed.) Let $\Sigma \subset \mathbb{R}^3$ denote the graph of a smooth function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^2$. Then

$$I[u] = \int_\Omega (1 + |Du|^2)^{-3/2} \det(D^2 u)\, dx$$

is the integral of the Gauss curvature over Σ. Prove that this expression depends only on Du restricted to $\partial\Omega$. You may assume $\partial\Omega$ is C^∞.

5. Also from the same page on Evans. Give an example of a nonconvex function $F : \mathbb{M}^{m\times n} \to \mathbb{R}$ that satisfies the Legendre-Hadamard condition

$$\sum_{i,j=1}^{n} \sum_{k,l=1}^{m} \frac{\partial^2 F(B)}{\partial A_{ki} A_{lj}} \eta_k \eta_l \xi_i \xi_j \geq 0,$$

for every $B \in \mathbb{M}^{m\times n}$, $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^m$.

6. *Rigidity of rotations.* You are led through the proof of the following 'rigidity' theorem. If $u : \Omega \to \mathbb{R}^n$ is a Lipschitz map such that Du is a rotation $a.e$ (that is $Du \in SO(n)$ $a.e$) then Du is a constant and $u = Qx + b$ for some fixed rotation $Q \in SO(n)$ and translation $b \in \mathbb{R}^n$.

(a) Use the fact that $\mathrm{div}(\mathrm{cof}Du) = 0$ in $\mathcal{D}'$ to deduce that u is harmonic (ie. u_i is harmonic, $i = 1, \dots, n$).

(b) Show that $D^2 u \equiv 0$ (i.e $D_{jk} u_i = 0$ for every $i, j, k = 1, \dots, n$).

7. Rotations are important as they are the simplest examples of isometries (maps that preserve length). An interesting generalization of isometry is the following. A map $u : \Omega \to \mathbb{R}^n$ has bounded distortion if:

1. u is continuous.
2. $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ for some $p \geq 1$.
3. $\det(Du) \geq 0$ and there is $\kappa \geq 1$ such that $|Du|^n \leq \kappa \det(Du)$.

In what follows, you are led through a proof of the following surprising fact. Suppose $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ is of bounded distortion, then u is $C^{0,1/\kappa}$. More precisely, there is a constant $C > 0$ such that

$$|u(x_2) - u(x_1)| \leq C \|Du\|_{L^n} |x_2 - x_1|^{1/\kappa}, \quad x_1, x_2 \in \Omega.$$

You may assume u is smooth while deriving the following estimates. The final assertion then follows by mollification.

(a) Use the isoperimetric inequality to prove that

$$\int_{B(x,r)} \det(Du)\,dy \leq \frac{r}{n} \int_{S(x,r)} |Du|^n dS_y, \quad B(x,r) \subset \Omega.$$

(b) Use (c) in the definition of bounded distortion and step (a) to show

$$\int_{B(x,r)} |Du|^n\,dy \leq \frac{\kappa r}{n} \int_{S(x,r)} |Du|^n\,dS_y \quad B(x,r) \subset \Omega.$$

(c) Integrate step (b) to obtain the bound

$$\int_{B(x,r)} |Du|^n\,dy \leq \left(\frac{r}{R}\right)^{n/\kappa} \int_{B(x,R)} |Du|^n\,dy, \;\; r \in (0,R),\; B(x,R) \subset \Omega.$$

(d) Deduce Hölder continuity from Morrey's inequality.