PDE, HW 6 solutions

(1) Problem 1, p. 45, John. Suppose f and g have support within $[-M.M]$. If $|x| > M + ct$ we have

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy = 0.$$

Suppose F and G have compact support. We then use (4.12) to find

$$0 = \lim_{x\to\infty} F(x) = \frac{1}{2c}\int_0^{\infty} g(y)\,dy + \delta.$$

Similarly,

$$0 = \lim_{x\to\infty} F(x) = \frac{1}{2c}\int_0^{-\infty} g(y)\,dy + \delta.$$

Thus, $\int_{-\infty}^{\infty} g(y)\,dy = 0$. A similar calculation holds for G. □

(2) Problem 1, p. 132, John. (a) If u is radial and continuous, we have $M_u(0,r,t) = u(r,t)$. Thus, we only need apply the general solution formula with $x = 0$. As for the 1-D wave equation the solution may be written as

$$u(r,t) = \frac{F(r+ct) + G(r-ct)}{r}, \quad r = |x|,$$

where

$$F(x) = \frac{1}{2}xf(x) + \frac{1}{2c}\int_0^x yg(y)\,dy + c, \quad G(x) = -\frac{1}{2}xf(x) - \frac{1}{2c}\int_0^x yg(y)\,dy - c,$$

for a constant $c \in \mathbb{R}$. Here we use $M_f(0,r) = f(r)$ and $M_g(0,r) = g(r)$. We may suppose $c = 0$ if we are only interested in u.

(b). If $f = 0$ we have immediately

$$u(r,t) = \frac{1}{2cr}\int_{r-ct}^{r+ct} \rho g(\rho) d\rho.$$

(c). Here $g_\rho = \mathbf{1}_{|\rho|\le a}$ so we have

$$u(r,t) = \frac{1}{2cr}\int_{\max(r-ct,-a)}^{\min(r+ct,a)} \rho d\rho = \frac{1}{4cr}\left((\min(r+ct,a))^2 - (\max(r-ct,-a))^2\right).$$

More explicitly, we compute the solution in two different time ranges. First suppose, $0 < t < a/c$. Then we have

$$u(r,t) = \begin{cases} (2cr)^{-1} \int_{r-ct}^{r+ct} \rho d\rho = t, \quad r \in [0, a-ct), \\ (2cr)^{-1} \int_{r-ct}^{a} \rho d\rho = (4cr)^{-1} \left(a^2 - (r-ct)^2\right), \quad r \in [a-ct, a+ct), \\ 0, \quad r \in [a+ct, \infty). \end{cases}$$

For $t > a/c$ since $\rho g(\rho)$ is odd, we have

$$u(r,t) = \begin{cases} 0, \quad r \in [0, ct-a), \\ (2cr)^{-1} \int_{ct-r}^{a} \rho d\rho = (4cr)^{-1} \left(a^2 - (ct-r)^2\right), \quad r \in [ct-a, ct+a), \\ 0, \quad r \in [ct+a, \infty). \end{cases}$$

In particular, we have the discontinuity

$$u(0,t) = \begin{cases} t, & t < a/c \\ 0, & t > a/c. \end{cases}$$

□

(3) Problem 4, p. 133, John. (a) The problem is false as stated. We do have $|u(x,t)| \leq Kt^{-1}$ for every $t > 0$, but K is not independent of the initial data. The estimate stated does hold for large t however. We follow the hint and write u as a volume integral,

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi c^2 t^2} \int_{S(x,ct)} \sum_{i=1}^{n} \left[(tg(y) + f(y)) \frac{y_i - x_i}{ct} + ct f_{y_i} \right] \xi_i \, dS_y \\ &= \frac{1}{4\pi c^2 t^2} \int_{B(x,ct)} \left[\sum_{i=1}^{n} \partial_{y_i} \left((tg(y) + f(y)) \frac{(y_i - x_i)}{ct} \right) + ct \triangle f \right] dx. \end{aligned}$$

Now take absolute values, and replace the domain $B(x,ct)$ by $\mathbb{R}^n$ to obtain

$$|u(x,t)| \leq \frac{1}{4\pi c^2 t} \int_{\mathbb{R}^n} \left[|\triangle f| + |Dg| + \frac{1}{ct} \left(c\,|Df| + n\,|g| \right) + \frac{n}{c^2 t^2} |f| . \right] dx.$$

For $ct > 1$ we then have a universal constant K such that

$$|u(x,t)| \leq \frac{K}{t} U(0).$$

(b) This is a very useful trick. Since the wave equation is invariant under reflection in time $t \to -t$, if $u(x,t)$ is a solution for $t > 0$, then for any fixed $T > 0$, $v(x,t,T) = u(x,T-t)$ is a solution for $t \in [0,T]$. Initially $v(x,0,T) = u(x,T)$, therefore for any $t \in (0,T)$,

$$|u(x,t)| = |v(x,T-t,T)| \leq \frac{K}{T-t} U(T).$$

We let $T \to \infty$ to obtain $|u(x,t)| = 0$. $\square$

(4) Problem, p. 139, John. This is a direct, but worthwhile, computation. We may as well suppose that the problem is one dimensional to simplify notation. If $v(x',t') = u(x,t)$ then we have

$$\begin{aligned}
\partial_t^2 u &= \partial_{x'}^2 v \left(\frac{\partial x'}{\partial t}\right)^2 + \partial_{t'}^2 v \left(\frac{\partial t'}{\partial t}\right)^2 + 2\partial_{t'}\partial_{x'} v \frac{\partial x'}{\partial t}\frac{\partial t'}{\partial t} \\
\partial_x^2 u &= \partial_{x'}^2 v \left(\frac{\partial x'}{\partial x}\right)^2 + \partial_{t'}^2 v \left(\frac{\partial t'}{\partial x}\right)^2 + 2\partial_{t'}\partial_{x'} v \frac{\partial x'}{\partial x}\frac{\partial t'}{\partial x}.
\end{aligned}$$

Now compute

$$\begin{aligned}
\frac{\partial x'}{\partial t} &= \frac{-\gamma c^2}{\sqrt{1-\gamma^2 c^2}}, & \frac{\partial x'}{\partial x} &= \frac{1}{\sqrt{1-\gamma^2 c^2}}, \\
\frac{\partial t'}{\partial t} &= \frac{1}{\sqrt{1-\gamma^2 c^2}}, & \frac{\partial t'}{\partial x} &= \frac{-\gamma}{\sqrt{1-\gamma^2 c^2}}.
\end{aligned}$$

Finally, we substitute these expressions to find

$$\left(\partial_{t'}^2 - c^2 \partial_{x'}^2\right) v = \left(\partial_t^2 - c^2 \partial_x^2\right) u = 0.$$

$\square$

(5) Problem 1, Rauch p. 162. Assume $c = 1$ for simplicity. We may use the solution computed in Problem 2 (even if it is not in $\S(\mathbb{R}^n)$). We have

$$\|u_t(0)\|_{L^p}^p = \omega_3 \int_0^\infty \rho^2 g(\rho)\, d\rho = \frac{\omega_3}{3} a^3.$$

Observe that $u(r,t)$ given in Problem 2,(c) is piecewise differentiable. We hold t fixed and consider solutions with $0 < a \ll t$. We then have

$$u_t(r,t) = \frac{(t-r)}{2r} \mathbf{1}_{t-a<r<a+t},$$

which may be integrated to yield

$$\|u_t(t)\|_{L^p}^p = \omega_3 \int_{a-t}^{a+t} \left|\frac{t-r}{2r}\right|^p r^2 dr.$$

If $p = 2$, the integral may be computed exactly. It is

$$\frac{\omega_3}{12}\left(a^3 - (a-2t)^3\right).$$

If $p \neq 2$ we have the lower estimate.

$$\begin{aligned}\|u_t(t)\|_{L^p}^p &\geq \omega_3 2^{-p}\left|\frac{t}{t-a} - 1\right|^p \int_{t-a}^{a+t} r^2 dr \\ &= \frac{\omega_3}{3} 2^{-p} a^p (t-a)^{3-p}\left((1+\frac{2a}{t-a})^3 - 1\right) = a^{p+1}t^{2-p}\left(2^{1-p}\omega_3 + o(1)\right).\end{aligned}$$

Therefore, the ratio

$$\frac{\|u_t(t)\|_{L^p}}{\|u_t(0)\|_{L^p}} \geq a^{p-2}t^{2-p}\left(\frac{2^{1-p}}{3} + o(1)\right).$$

If $p < 2$, we consider a sequence of solutions with initial data $\mathbf{1}_{|\rho|\leq a_k}$ such that $a_k \to 0$. It then follows that

$$\lim_{k\to\infty} \frac{\|u_t^{(k)}(t)\|_{L^p}}{\|u_t^{(k)}(0)\|_{L^p}} = +\infty.$$

If $p > 2$ we have $a_k^{p-2} \to 0$ for the same sequence of initial data. This still suffices, since the wave equation is reversible. A calculation as above shows that we also have the upper estimate

$$\|u_t(t)\|_{L^p}^p \leq \omega_3 2^{-p}\left|\frac{t}{t+a} - 1\right|^p \int_{t-a}^{a+t} r^2 dr = a^{p+1}t^{2-p}\left(2^{1-p}\omega_3 - o(1)\right).$$

We then have

$$\lim_{k\to\infty} \frac{\|u_t^{(k)}(0)\|_{L^p}}{\|u_t^{(k)}(t)\|_{L^p}} = +\infty,$$

and since the wave equation is reversible in time, we may treat $u_t(t)$ as initial data, to complete the proof. (If the bound held in $\mathcal{S}(\mathbb{R}^n)$ the above example would not be possible). $\square$