PDE, HW 5 solutions

Problem 1. Change variables to $p = |x|^2/4t$. Then

$$\begin{aligned}\int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}\, dt &= \frac{1}{4\pi^{n/2}}|x|^{2-n}\int_0^\infty e^{-p}p^{\frac{n}{2}-2}\, dp \\ &= \frac{1}{4\pi^{n/2}}|x|^{2-n}\Gamma\left(\frac{n}{2}-1\right) = \frac{1}{4\pi^{n/2}}|x|^{2-n}\frac{2}{n-2} = \frac{|x|^{2-n}}{\omega_n(n-2)}.\end{aligned}$$

Here the Γ-function identity $\Gamma(z+1) = z\Gamma(z)$ has been used, along with $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ (see HW 1). □

Problem 2. There are two standard proofs. First, we may use uniqueness of solutions to the heat equation. Fix $t > 0$. Consider the solution of the heat equation $u_s = \triangle u$ with initial data $u(x,0) = k(x, t/2)$. Then we have

$$u(x, \frac{t}{2}) = \int_{\mathbb{R}^n} k(x-y, \frac{t}{2})k(y, \frac{t}{2})\, dy.$$

On the other hand, $k(x, s+t/2)$ also solves $k_s = \triangle k$ with the same initial data at $s = 0$. Therefore, $u(x, t/2) = k(x,t)$ and we obtain the convolution identity for $m = 2$. Now use induction. Suppose $m > 2$. The uniqueness argument above yields

$$k(x,t) = \int_{\mathbb{R}^n} k(y, \frac{m-1}{m}t)k(x-y, \frac{t}{m})\, dy,$$

and we may write the first term as a convolution of $m-1$ terms by the induction hypothesis.

One may also use the Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}f(x)\, dx$. The Fourier transform of a Gaussian is Gaussian

$$\hat{k}(\xi, t) = e^{-t|\xi|^2} = e^{-t|\xi|^2/m}\dots e^{-t|\xi|^2/m}.$$

Now use the convolution property of Fourier transforms. □

Problem 3. (a) This is an advertisement for the probabilistic approach. We have $u(x,t) = \mathbb{E}(f(x+W_t))$ where W_t is a Brownian motion started at 0. Fix $x, y \in \mathbb{R}^n$ and $\theta \in (0,1)$, then by the definition of convexity

$$\begin{aligned}u(\theta x + (1-\theta)y, t) &= \mathbb{E}\left(f(\theta(x+W_t) + (1-\theta)(y+W_t))\right) \\ &\le \mathbb{E}\left(\theta f(x+W_t) + (1-\theta)f(y+W_t)\right) = \theta u(x,t) + (1-\theta)u(y,t).\end{aligned}$$

(b) For any $t > 0$ we have

$$u(x,t) = \mathbb{E}f(x+W_t) \geq f(\mathbb{E}(x+W_t)) = f(x),$$

using Jensen's inequality and $\mathbb{E}W_t = 0$. More generally, for any $t_2 \geq t_1$, we may use the semigroup property, part (a) and Jensen's inequality to obtain

$$u(x,t_2) = \mathbb{E}(u(x+W_{t_2-t_1},t_1)) \geq u\left(\mathbb{E}(x+W_{t_2-t_1}),t_1\right) = u(x,t_1).$$

$\square$

Problem 4. Let V be the cylinder $U \times (0,T)$. Consider continuous boundary data $f : \partial V \to \mathbb{R}$ and suppose $u \in C^2_1(V) \bigcap C(\bar{V})$ solves the Dirichlet problem $\partial_t u = \triangle u$ in V and $u = f$ on ∂V. The weak maximum principle implies $\max_{\overline{V}} u(x,t) \leq \max_{\partial_1 V} f(x,t)$. In particular, $f(x,T) \leq \max_{\partial_1 V} f$ for every $x \in U$. Thus, f cannot be arbitrary. $\square$

Problem 5. Appell's transformation. This is a computation:

$$v_t = k_t u - \frac{xk}{t} u_x + \frac{k}{t^2} u_t,$$

$$v_x = k_x u + \frac{k}{t} u_x, \quad v_{xx} = k_{xx} u + \frac{2k_x u_x}{t^2} + \frac{k}{t^2} u_{xx}.$$

Since $k_t = k_{xx}$ and $u_t = u_{xx}$, we see that

$$v_t - v_{xx} = -\frac{u_x}{t}\left(\frac{xk}{t} + 2k_x\right) = 0,$$

as may be verified by a computation, or by noting that this is the ODE used to define k. $\square$

Problem 6. (a). Let us assume that $E[\mu] \geq 0$ with equality if and only if $\mu = 0$. For any $c \neq 0$, since $0 \leq E[\mu - c\nu]$ we have

$$2E[\mu,\nu] \leq \frac{1}{c} E[\mu] + cE[\nu].$$

The bound is best when we choose $c = \sqrt{E[\mu]/E[\nu]}$ to obtain

$$E[\mu,\nu] \leq \sqrt{E[\mu]E[\nu]}.$$

Since the right hand side is unchanged under $\mu \to -\mu$, we have $|E[\mu,\nu]| \le \sqrt{E[\mu]E[\nu]}$ as desired. The inequality is sharp unless $\mu = c\nu$ for some $c \in \mathbb{R}$. Part (2) of Thm. 1.56 is similar. If $\mu \neq \nu$

$$\begin{aligned} E[t\mu + (1-t)\nu] &= t^2 E[\mu] + (1-t)^2 E[\nu] + 2t(1-t)E[\mu,\nu] \\ &< t^2 E[\mu] + (1-t)^2 E[\nu] + 2t(1-t)\sqrt{E[\mu]E[\nu]} \\ &\le t^2 E[\mu] + (1-t)^2 E[\nu] + t(1-t)\left(E[\mu] + E[\nu]\right) = tE[\mu] + (1-t)E[\nu]. \end{aligned}$$

(b) Problem (2) yields

$$\begin{aligned} k(x-y,t) &= \int_{\mathbb{R}^n} k(x-y-z', \frac{t}{2}) k(z', \frac{t}{2})\, dz' \\ &= \int_{\mathbb{R}^n} k(x-z, \frac{t}{2}) k(z-y, \frac{t}{2})\, dz = \int_{\mathbb{R}^n} k(z-x, \frac{t}{2}) k(z-y, \frac{t}{2})\, dz, \end{aligned}$$

changing variables $z = z' + y$, and using the symmetry $k(x-z,t) = k(z-x,t)$. Now integrate in time, and use problem (1) to obtain the desired identity

$$|x-y|^{2-n} = \omega_n(n-2) \int_0^\infty \int_{\mathbb{R}^n} k(z-x, \frac{t}{2}) k(z-y, \frac{t}{2})\, dz\, dt.$$

(c) Let $u(z,t) = \int_{\mathbb{R}^n} k(z-x,t)\mu(dx)$. Observe that u is a solution to the heat equation with the measure μ as initial data. The identity (b) implies

$$\begin{aligned} &\frac{1}{\omega_n(n-2)} E[\mu] = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^\infty \int_{\mathbb{R}^n} k(z-x, \frac{t}{2}) k(z-y, \frac{t}{2})\, dz\, dt\, \mu(dx)\, \mu(dy) \\ &= \int_0^\infty \int_{\mathbb{R}^n} \left| u(z, \frac{t}{2}) \right|^2 dz\, dt \ge 0. \end{aligned}$$

The interchange of limits is justified by Fubini's theorem. The assumption $E[|\mu|] < \infty$ is used here.

(d) One approach to the uniqueness question is the following. If $E[\mu] = 0$ we must have $u(x,t) = 0$ *a.e.* Therefore, one may integrate and use Fubini's theorem to obtain that for *a.e* $x \in \mathbb{R}^n$

$$0 = \int_0^\infty u(x,t)\, dt = \int_0^\infty \int_{\mathbb{R}^n} k(x-y,t)\mu(dy)\, dt = \omega_n(n-2) \int_{\mathbb{R}^n} |x-y|^{2-n} \mu(dy).$$

Therefore, the potential of the measure μ vanishes *a.e*, and uniqueness of potentials (Thm 1.48) implies $\mu \equiv 0$. $\square$