PDE, HW 3 solutions

Problem 1. No. If a sequence of harmonic polynomials on $[-1,1]^n$ converges uniformly to a limit f then f is harmonic. $\square$

Problem 2. By definition $U_r \subset U$ for every $r > 0$. Suppose w is a barrier at y for U. Then the restriction of w to U_r is also a barrier. Therefore, if y is regular for ∂U, it is also regular for ∂U_r.

Conversely, if w is a barrier at y for ∂U_r it may be extended to a barrier on U_r as follows. Let $A_r = \{x \in U_r | r/2 \leq |x-y| \leq r\}$, and let $M = \max_{x \in A_r} w(x)$. By the definition of a barrier, $M < 0$. For $x \in U_r$ let $\tilde{w}(x) = \max\{w(x), M\}$. This is the maximum of two subharmonic functions, and is therefore subharmonic. Moreover, since $w \leq M$ on A_r it also follows that $\tilde{w}(x) = M$ for every $x \in A_r$. We extend $\tilde{w}$ to U by setting $\tilde{w} = M < 0$ for all $x \in U, |x-y| \geq r$. This extension is a barrier at y for U. $\square$

First proof for Problem 3. I hope you did not struggle too much with the calculation. After some experimentation, I realized that the calculation is simpler if one works with $\xi = x_n/|x|$ rather than θ as defined in the problem statement. Assume $u = |x|^\lambda h(\xi)$. Then $D\xi$ is given in coordinates by

$$\xi_{,i} = -\frac{x_n x_i}{|x|^3} + \frac{\delta_{ni}}{|x|}.$$

A nice cancellation is

$$x \cdot D\xi = x_i \xi_{,i} = -\frac{x_n x_i x_i}{|x|^3} + \frac{x_n}{|x|} = 0.$$

The derivatives of u are obtained by the chain rule

$$Du = \lambda |x|^{\lambda-2} x h + |x|^\lambda h' D\xi,$$

or in coordinates

$$u_{,i} = x_i |x|^{\lambda-2} \left(\lambda h - \xi h'\right) + |x|^{\lambda-1} h' \delta_{ni} := (a) + (b).$$

Evaluate each term in turn. By using the cancellation above we have

$$(a)_{,i} = |x|^{\lambda-2}(\lambda - 2 + n)(\lambda h - \xi h'),$$

and

$$(b)_{,i} = (\lambda - 1)|x|^{\lambda-2}\xi h' + |x|^{\lambda-2}(1-\xi^2)h''.$$

Summing up, we have the differential equation

$$(1-\xi^2)h'' + (1-n)\xi h' + \lambda(\lambda - 2 + n)h = 0. \tag{0.1}$$

If we change variables according to $\cos\theta = \xi$ and $g(\theta) = h(\xi)$ we have

$$g'' + (n-2)\cot\theta g' + \lambda(\lambda + n - 2)g = 0.$$

The analysis is simplest (and unecessary) when $n = 2$. However, this is a good check that the calculation is OK. Here we have

$$g'' + \lambda^2 g = 0,$$

with an even solution $g = \cos(\lambda\theta)$. This solution is positive for $|\theta| < \alpha$ if we choose $\lambda\alpha < \pi/2$. More generally, our task is to find a positive solution to (0.1) on any interval $(-a, a)$ for $0 < a = \cos\alpha < 1$. One way to do this is to look at a classical reference on Legendre's differential equation or hypergeometric functions (Abramowitz and Stegun for example), but a crude analysis will also suffice. Fix $\cos\alpha < 1$. Observe that $g \equiv 1$ is an even positive solution on $(-a, a)$ when $\lambda = 0$. Now consider solutions with $g(0) = 1$ and $g'(0) = 0$ for small $\lambda > 0$. The only problem is at 0, but here $\lim_{\theta\to 0} g'' + (n-2)\cot\theta g' = (n-1)g''(0) = -\lambda(\lambda+n-2)g(0) = -\lambda(\lambda+n-2)$. Therefore, for sufficiently small $\delta > 0$ one may use a series expansion to construct an analytic solution g, convergent in the neighborhood $(-\delta, \delta)$ for all $0 \leq \lambda < 1$. In particular,

$$g_\lambda = 1 - \lambda(\lambda + n - 2)\frac{\theta^2}{2} + o(\theta^2),$$

and the perturbation is continuous in λ. For fixed $\delta > 0$, the equation for g is regular on $[\delta, a]$ and one may use continuous dependence of the solution of an ODE on parameters to say that for sufficiently small $\lambda > 0$ there is a solution g_λ that is positive such that $g_\lambda(x) \to 1$ as $\lambda \to 0$. □

Second proof of Problem 3. I can see why the proof above may be unpalatable. Here is a classical argument that avoids the ODE, and relies only on a direct construction of a barrier. Let U denote the domain $U = B(0,1) \bigcap C$. Consider the Perron function for boundary data $u = |x|$ on ∂U. Then every point of ∂U is regular except the origin. Since $u(x) \geq 0, x \in U$ we also have

$$\liminf_{x\to 0} u(x) \geq 0.$$

We only need show $c = \limsup_{x\to 0} u(x) = 0$. It is clear that $c \geq 0$. We use the scaling invariance of the cone: for any $k > 1$ consider the scaled function $v(x) = u(kx)$. Then on all boundary points $|x| \neq 0$ we have

$$v(x) = k|x| > u(x), \quad \text{and} \quad v(0) = u(0).$$

Thus, there is a constant $a < 1$ such that

$$u(x) \leq av(x), \quad x \in \partial U, x \neq 0.$$

We claim that this inequality also holds at all points in the interior. If so, by taking $x \to 0$ we obtain,

$$\limsup_{x\to 0} u(x) = c \leq ac = a \limsup_{x\to 0} u(kx) < c.$$

Thus, $c = 0$ and u is a barrier.

It remains to show $u \leq av$ for all x in the interior. We cannot use the maximum principle directly since we do not know a priori that u and v are continuous on the boundary. However, this is an easy problem to fix. For any $1 > \varepsilon > 0$ consider the domain $U_\varepsilon = U \bigcap \{|x| > \varepsilon\}$. Consider a harmonic function on the annulus $\{\varepsilon \leq |x| \leq 1\}$ with boundary values $w_\varepsilon = 0$ on $|x| = 1$ and $w_\varepsilon = \sup_{x\in\partial U, |x|\geq\varepsilon}(u - av) < 0$ on $|x| = \varepsilon$. By the maximum principle, we then have $u - av \leq w_\varepsilon \leq 0$ for all $x \in U_\varepsilon$. Let $\varepsilon \to 0$ to find $u \leq av$, $x \in U$ as desired. □

Problem 4. Let ν denote the given measure $\mu \mathbf{1}_{|y|<a} dy$. The potential is

$$u_\nu(x) = \int_{\mathbb{R}^3} |x - y|^{-1} \nu(dy).$$

Since ν is invariant under rotations, we may evaluate the integral by assuming x is along the x_1-axis. If θ is the polar angle measured from the axis, and we denote $|x| = \rho$ and $|y| = r$ we have

$$|x - y|^2 = \rho^2 - 2\rho r \cos\theta + r^2.$$

First consider the case where $|x| > a$. In this case,

$$u_\nu(x) = \mu \int_0^a \int_0^\pi (\rho^2 - 2\rho r \cos\theta + r^2)^{-1/2} (2\pi r \sin\theta) r \, d\theta \, dr.$$

Evaluate the inner integral via the substitution $p = \rho^2 - 2\rho r \cos\theta + r^2$:

$$\begin{aligned} u_\nu(x) &= 2\pi\mu \int_0^a \int_{(\rho-r)^2}^{(\rho+r)^2} p^{-1/2} \frac{dp}{2\rho} r \, dr \\ &= \frac{2\pi\mu}{\rho} \int_0^a (|\rho + r| - |\rho - r|) \, r \, dr = \frac{2\pi\mu}{\rho} \int_0^a 2r^2 dr = \frac{4\pi\mu a^3}{3\rho}. \end{aligned}$$

If $|x| \leq a$, then we may use the same calculation to evaluate the integral on the shell $0 < r < \rho$ to obtain a contribution $4\pi\mu\rho^2/3$. However, on the shell $\rho < r < a$ we have $|\rho + r| - |\rho - r| = 2\rho$ and we have the integral

$$\frac{2\pi\mu}{\rho} \int_\rho^a 2\rho r \, dr = 2\pi\mu(a^2 - \rho^2).$$

Summing these two contributions we have

$$u_\nu(x) = 2\pi\mu(a^2 - \frac{\rho^2}{3}).$$

The attraction $F = Du$ is found by differentiating u (with left and right derivatives at $\rho = a$ that are equal). We have

$$F(x) = \begin{cases} -\frac{4\pi\mu x}{3}, & |x| < a, \\ -\frac{4\pi\mu a^3 x}{3|x|^3}, & |x| \geq a \end{cases}$$

If we define F as Du this calculation is entirely legitimate. On the other hand, if we define F as the integral

$$F(x) = (2-n) \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} \nu(dy),$$

we must show this equals Du. This can be done using finite differences. To clarify ideas, let us do this in generality. Fix ω with $|\omega| = 1$. Let

$$D_{h\omega} u = \frac{u(x + h\omega) - u(x)}{h} = h^{-1} \int_{\mathbb{R}^n} \left(|x + h\omega - y|^{2-n} - |x - y|^{2-n} \right) \nu(dy).$$

If we know a priori that u is differentiable (for example, as in this problem) we may use the mean value theorem to obtain a number $\tau \in (0, 1)$ such that

$$D_{h\omega} u = (2-n) \int_{\mathbb{R}^n} \frac{(x + \tau h\omega - y) \cdot \omega}{|x + \tau h\omega - y|^n} \nu(dy).$$

We use the definition of F to obtain

$$D_{h\omega} u - F(x) \cdot \omega = (2-n) \int_{\mathbb{R}^n} \left(\frac{(x + \tau h\omega - y)}{|x + \tau h\omega - y|^n} - \frac{x-y}{|x-y|^n} \right) \cdot \omega \, \nu(dy).$$

As $h \to 0$, the integrand converges to 0 pointwise. All that is needed is to justify the interchange of limits. This is best done through the dominated convergence theorem. For any $h \neq 0$, the integrand is bounded by

$$(n-2) \int_{\mathbb{R}^n} (|x + \tau h\omega - y|^{1-n} + |x - y|^{1-n}) \nu(dy).$$

If this term is finite, we are done. We only need consider the case $|x| \leq a$. Since ν is absolutely continuous with respect to Lebesgue measure, the singularity is integrable. Indeed,

$$\int_{\mathbb{R}^n} |x-y|^{1-n}\nu(dy) = \mu \int_{B(0,a)} |x-y|^{1-n}dy \leq \mu \int_{B(0,2a)} |y|^{1-n}dy = 2a\mu\omega_n.$$

□

Problem 5. First suppose $x \in \mathbb{R}$. Fix $p > 0$ and let $g(x) = e^{-x^{-p}}, x > 0$ and $g = 0, x \leq 0$. As you have shown, g is C^∞. To construct a bump function, choose $f(x) = g(1/2+x)g(1/2-x)$. Then $\mathrm{supp}(f) \subset\subset (-1,1)$. In $\mathbb{R}^n$ one may choose the product

$$\psi(x) = f(\frac{x_1}{\sqrt{n}})f(\frac{x_2}{\sqrt{n}})\ldots f(\frac{x_n}{\sqrt{n}}),$$

and obtain a normalized bump function by defining

$$\varphi(x) = \frac{\psi(x)}{\int_{\mathbb{R}^n}\psi(y)dy}.$$

The support of φ is compactly contained within $(-n^{-1/2}, n^{-1/2})^n$ which is contained within $B(0,1)$.

Now consider g_δ as defined in the problem. Suppose $\mathrm{dist}(x, \partial U) > \delta$. Let $V = \mathbb{R}^n \backslash \overline{U}$. We then have either $B(x,\delta) \subset U$ or $B(x,\delta) \subset V$. To see this, suppose x is not in U. Observe that $\mathrm{dist}(x,\overline{U}) = \mathrm{dist}(x,\partial U)$ since $\partial U = \overline{U}\backslash U$ and $\inf_{y\in\overline{U}} |x-y|$ cannot be attained in U. Therefore, $\mathrm{dist}(x,\overline{U}) > \delta$ which implies $B(x,\delta) \subset V$. All we have used here is that U is open. Since V is also open, if $x \in U$ we have similarly $B(x,\delta) \subset U$. In either case, $\mathbf{1}_U(x-y) = \mathbf{1}_U(x)$ if $|y| < \delta$. □

Problem 6. Let U_l and $\tilde{U}_m$ be two sequences of increasing, bounded domains used in the definition of p_F. That is, $U_l \subset U_{l+1} \subset\subset U$ and $\bigcup_{l=1}^\infty U_l = U$, and similarly for $\tilde{U}_m$. We define a sequence of Perron functions p_l by solving the Dirichlet problem with boundary data $p_l = 1, x \in F$, $p_l = 0$ on the 'outer boundary' of U_l (and similarly for $\tilde{p}_m$). We have shown that p_l and $\tilde{p}_m$ converge to functions harmonic on $U = \mathbb{R}^n \backslash F$. Denote these by p_F and $\tilde{p}_F$ respectively. We must show that $p_F = \tilde{p}_F$.

Fix l. Since U_l is a bounded domain, we must have $U_l \subset \tilde{U}_{m_l}$ for sufficiently large m_l. By the maximum principle $1 > \tilde{p}_{m_l} > p_l$ on U_l. Passing to the limit, we find $\tilde{p}_F \geq p_F$, $x \in U$. A similar argument with U_l replaced by $\tilde{U}_m$ yields $p_F \geq \tilde{p}_F, x \in U$. □

Problem 7. p_F is generated by a charge μ_F with support in ∂F

$$p_F(x) = \int_F |x-y|^{2-n} \mu_F(dy).$$

If $x \in U$, we have $r(x) \le |x-y| \le R(x)$, and we have the desired estimate

$$R(x)^{2-n} \text{cap}(F) \le p_F(x) \le r(x)^{2-n} \text{cap}(F).$$

□

Remark 0.1. The virtue of the above estimate is that it gives us the leading order asymptotics of p_F as $x \to \infty$. Conversely, if we are able to determine $p_F(x_n)$ along some sequence $x_n \to \infty$, we may determine the capacity. For example, to compute the capacity of a planar disc in $\mathbb{R}^3$ we can use symmetry to determine p_F exactly along the axis of the disk, and let $x \to \infty$ to find the capacity (one needs elliptic integrals away from the axis).

Problem 8. First suppose ∂F is smooth. In this case, observe that $p_\beta(x) = p_F(\beta x)$ is the potential associated to F_β (use uniqueness of potentials here). Now change variable to say that $\text{cap}(F_\beta) = \beta^{n-2}\text{cap}(F)$ in this case. If ∂F is not smooth, use the approximation theorem.

I apologize for an error in equation (1.62), p.38 of the notes. I hope you were able to fix this. It should read

$$\sum_{k=0}^{\infty} \lambda^{k(2-n)} \text{cap}(F_k) = \infty.$$

The proof of the exterior cone condition using Wiener's criterion goes as follows. Suppose $y \in \partial U$ satisfies an exterior cone condition. Let $\lambda = 1/2$ and let F_k be the compact sets in Theorem 1.50. Observe that F_k contains a ball of radius $c\lambda^k$ for some $c > 0$ (independent of k). Therefore, using the scaling property just proved, and the fact that balls of radius R have capacity R^{n-2} we find

$$\text{cap}(F_k) \ge (c\lambda^k)^{n-2}.$$

Therefore, the sum in Wiener's criterion

$$\sum_{k=0}^{\infty} \lambda^{k(2-n)} \text{cap}(F_k) \ge c^{n-2} \sum_{k=0}^{\infty} 1 = \infty.$$

□

Problem 9. In case, you were confused about the notation for $|Df|$, what was intended was the following uniform Lipschitz estimate

$$|f(x+y) - f(x)| \le |y|, \tag{0.2}$$

which is all that is needed in this problem. If f is C^1 this bound is derived as follows. By the fundamental theorem of calculus

$$f(x+y) - f(x) = \int_0^1 \frac{d}{dt} f(x+ty)\, dt = \int_0^1 Df(x+ty)y\, dt.$$

If $|Df|$ denotes the (Euclidean) norm of the matrix Df we obtain (0.2).

The estimate $\mathrm{cap}(f(F)) \le \mathrm{cap}(F)$ is intuitively believable, but quite subtle to prove. It is easy to fall into the following trap: Since f contracts distances, $f(F) \subset F$ and therefore $\mathrm{cap}(f(F)) \le \mathrm{cap}(F)$. The inclusion $f(F) \subset F$ works for balls (after a suitable translation), but not in general.

The key property of capacity needed here is the following. Consider a finite collection of balls $B(a_k, r_k), k = 1, \dots N$. If we move the centers of all balls further apart, then the capacity increases. Precisely, if a_k, b_k are such that $|a_k - a_l| \le |b_k - b_l|$ for every $k, l, k \ne l$, then

$$\mathrm{cap}(\bigcup_{k=1}^N \overline{B(a_k, r_k)}) \le \mathrm{cap}(\bigcup_{k=1}^N \overline{B(b_k, r_k)}). \tag{0.3}$$

Let us first show how (0.3) implies $\mathrm{cap}(f(F)) \le \mathrm{cap}(F)$. We use the approximation theorem to say that for any $\varepsilon > 0$ there is a finite cover $\bigcup_{k=1}^N B(x_k, r_k) \supset F$ such that

$$\mathrm{cap}(F) \le \mathrm{cap}(\bigcup_{k=1}^N \overline{B(x_k, r_k)}) \le \mathrm{cap}(F) + \varepsilon.$$

By the contraction property, $f(B(x_k, r_k)) \subset B(f(x_k), r_k)$ which implies

$$f(\bigcup_{k=1}^N B(x_k, r_k)) = \bigcup_{k=1}^N f(B(x_k, r_k)) \subset \bigcup_{k=1}^N \overline{B(f(x_k), r_k)}.$$

The centers of all the balls have moved closer together since $|f(x_k) - f(x_l)| \le |x_k - x_l|$. It now follows from (0.3) that

$$\begin{aligned} \mathrm{cap}(f(F)) &\le \mathrm{cap}(f(\bigcup_{k=1}^N \overline{B(x_k, r_k)})) \\ &\le \mathrm{cap}(\bigcup_{k=1}^N \overline{B(f(x_k), r_k)}) \le \mathrm{cap}(\bigcup_{k=1}^N \overline{B(x_k, r_k)}) \le \mathrm{cap}(F) + \varepsilon. \end{aligned}$$

Let us now prove (0.3). To fix intuition, consider the situation where we have two balls with centers at a_1 and a_2. Suppose the ball at a_2 is moved to b_2 as shown in Figure 0.1. Observe that translated points y_i satisfy $|x-y_2| \geq |x-y_1|$ (if the balls intersect, this only applies to points y_i not in the intersection, which is all we need). Recall that for any compact F

$$\operatorname{cap}(F) = \sup\{\mu(F) \,|\, \mu \geq 0, \quad u_\mu \leq 1, \quad \operatorname{supp}(\mu) \subset F\} \tag{0.4}$$

Let μ be the charge that achieves this supremum for the balls centered at a_1 and a_2. Now move the ball centered at a_2 to b_2. Consider the measure ν obtained from μ as follows: on $S(a_1, r_1)$ it agrees with μ and on $S(b_2, r_2)$ it is obtained by translating the restriction of μ on $S(a_2, r_2)$ to $S(b_2, r_2)$. The claim is that ν is admissible for the balls centered at a_1 and b_2. If so, criterion (0.4) shows that the capacity can only increase. We only need check that $u_\nu \leq 1$. By definition

$$u_\nu(x) = \sum_{k=1}^{2} \int_{S(a_k, r_k)} |x-y|^{2-n} \nu(dy).$$

If $x \in \overline{B(a_1, r_1)}$ since $\mu = \nu$ on $S(a_1, r_1)$ we have

$$\int_{S(a_1,r_1)} |x-y|^{2-n} \nu(dy) = \int_{S(a_1,r_1)} |x-y|^{2-n} \mu(dy).$$

On the other hand, since the distance $|x-y_2| \geq |x-y_1|$ for all $y_2 \in B(a_2, r_2)$ in the support of ν, we have

$$\int_{S(b_2,r_2)} |x-y_2|^{2-n} \nu(dy_2) \leq \int_{S(a_2,r_2)} |x-y_1|^{2-n} \mu(dy_1).$$

A similar calculation holds for $x \in \overline{B(b_2, r_2)}$. Thus, $u_\nu(x) \leq u_\mu(x) \leq 1$.

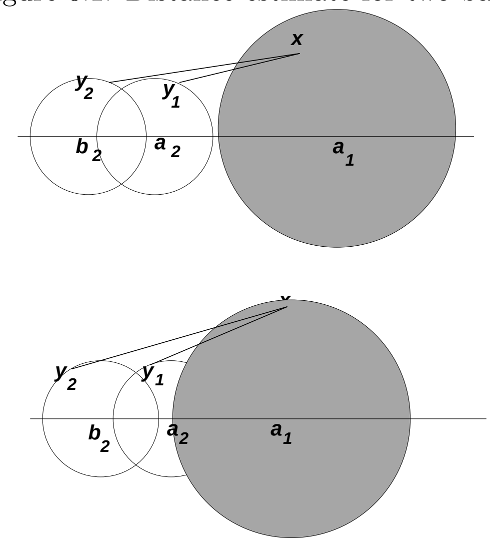
This analysis applies with little change to N balls. If μ is the charge on $S(a_k, r_k)$ we construct ν by translating μ to $S(b_k, r_k)$. As before,

$$u_\nu(x) = \sum_{k=1}^{N} \int_{S(b_k, r_k)} |x-y|^{2-n} \nu(dy).$$

Without loss of generality, suppose $x \in \overline{B(b_j, r_j)}$. Then,

$$\int_{S(b_j,r_j)} |x-y|^{2-n} \nu(dy) = \int_{S(a_j,r_j)} |x-y|^{2-n} \mu(dy),$$

Figure 0.1: Distance estimate for two balls



and on every other ball, $k \neq j$, by the argument for two balls

$$\int_{S(b_k,r_k)} |x-y|^{2-n}\nu(dy) \leq \int_{S(a_k,r_k)} |x-y|^{2-n}\mu(dy).$$

This shows that $u_\nu(x) \leq 1$ on $\overline{B(b_j,r_j)}$. This proves (0.3). $\square$