PDE, HW 2 solutions

1. Heuristically, the fundamental solution provides a prototype of a subharmonic function bounded above. However, this requires a more general definition of subharmonic functions than we have adopted, since the fundamental solution is not continuous. To find subharmonic functions bounded above you could solve $\triangle u = \rho$ where $\rho \geq 0$ is a smooth function with $\int_{\mathbb{R}^n} \rho\, dx = 1$, that is a smoothed version of δ_0. An analytic solution is

$$u(\xi) = \int_{\mathbb{R}^n} K(x,\xi)\rho(x)\, dx.$$

All you need to check is that the integral is well defined. Since $K < 0$ and $\rho \geq 0$, we then have $u \leq 0$. Here is a concrete example. Choose $\rho = (2\pi)^{-n/2} e^{-|x|^2/2}$ to find

$$|u(\xi)| = \frac{1}{(n-2)\omega_n (2\pi)^{n/2}} \int_{\mathbb{R}^n} |x-\xi|^{2-n} e^{-|x|^2/2}\, dx.$$

Split the integral into a piece where $|x-\xi| > 1$ and $|x-\xi| \leq 1$. Then

$$\int_{|x-\xi|\geq 1} |x-\xi|^{2-n} e^{-|x|^2/2}\, dx \leq \int_{\mathbb{R}^n} e^{-|x|^2/2}\, dx = (2\pi)^{n/2}.$$

As for the singular part, we have

$$\int_{|x-\xi|\leq 1} |x-\xi|^{2-n} e^{-|x|^2/2}\, dx \leq \int_{|x-\xi|\leq 1} |x-\xi|^{2-n}\, dx = \omega_n \int_0^1 r\, dr = \frac{\omega_n}{2}.$$

Problem 1, p. 106. (a) The importance of this problem is that it arises in the solution of the wave equation $\triangle v = \partial_{tt} v$. If one looks for standing waves $v = e^{i\lambda t} u$ one is led to the reduced wave equation $\triangle u + \lambda^2 u = 0$, or more generally $\triangle u + cu = 0$. Radially symmetric solutions $u = \psi(r)$ must solve the ODE

$$\psi'' + \frac{n-1}{r}\psi' + c\psi = 0. \tag{0.1}$$

It is hard to solve this equation explicitly (even for $n = 3$) without some knowledge of Bessel functions (this is outlined below). For the purposes of this question, it suffices to take a hint from parts (b) and (c) and *verify* that $\cos\sqrt{c}r/r$ and $\sin\sqrt{c}r/r$ are solutions to (0.1). Once we have two linearly independent solutions, we know that any solution to (0.1) is of the form

$$\psi = c_1 \frac{\cos(\sqrt{c}r)}{r} + c_2 \frac{\sin(\sqrt{c}r)}{r}. \tag{0.2}$$

Note though that the first solution works only on a punctured domain. (b). Use Green's identity in the form

$$\int_{\overline{U}} \left(v\left(\triangle u + cu\right) - u\left(\triangle v + cv\right)\right) dx = \int_{\partial U} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) dS_x. \tag{0.3}$$

Apply this to the domain $U\backslash B(\xi,\varepsilon)$ with $v = K(x,\xi)$, and take $\varepsilon \to 0$. The calculation is essentially the same as on p.96, with minor modifications. (c) Use the hint provided, and choose the Green's function

$$G(x,\xi) = \frac{1}{4\pi r}\left(-\cos(\sqrt{c}r) + \frac{\cos(\sqrt{c}\rho)}{\sin(\sqrt{c}\rho)}\sin(\sqrt{c}r)\right) = \frac{\sin\left(\sqrt{c}(r-\rho)\right)}{4\pi r\sin\left(\sqrt{c}\rho\right)}.$$

On $S(\xi,\rho)$ we have

$$\frac{\partial G}{\partial n_x} = \frac{\sqrt{c}}{4\pi\rho\sin(\sqrt{c}\rho)} = \frac{\sqrt{c}\rho}{\sin\left(\sqrt{c}\rho\right)}\frac{1}{4\pi\rho^2},$$

which gives the modified mean value equality. (d). In case, $c<0$, we have

$$u(\xi) = \frac{\sqrt{c}\rho}{\sinh\left(\sqrt{c}\rho\right)} ⨍_{S(\xi,\rho)} u(y)dS_y \leq ⨍_{S(\xi,\rho)} u(y)dS_y,$$

since $t \leq \sinh t$. If $c = 0$ u is harmonic. In either case, u is subharmonic. If $u = 0$ on $S(\xi,\rho)$ we find $u(\xi) = 0 = \max_{S(\xi,\rho)} u$. Thus, u is constant by the strong maximum principle. In case $c>0$ the solution $u = r^{-1}\sin(\sqrt{c}r)$ is non-constant and is 0 when $\sqrt{c}r = n\pi$. (e). Suppose $u \in C^2(U)$ satisfies $Lu = 0$ in U. Since regularity is a local property, we as well assume that $U = B(\xi_0, r)$ and $u \in C^2(\overline{U})$ (this is needed for the following integral identity). The fundamental solution yields the representation

$$u(\xi) = \int_{\partial U}\left(\frac{\partial K(x,\xi)}{\partial n_x}u(x) - K(x,\xi)\frac{\partial u}{\partial n_x}\right) dS_x.$$

We observe that $K(x,\xi)$ is analytic in any domain such that $r>0$ (use $r^2 = (x-\zeta)^t(x-\zeta)$). Thus, u is complex differentiable, hence analytic. $\square$

Remark 0.1. Bessel's equation with parameter $\nu \in \mathbb{R}$ is the differential equation

$$r^2 f'' + rf' + \left(r^2 - \nu^2\right) f = 0. \tag{0.4}$$

The standardized linearly independent solutions to (0.4) are denoted J_ν and Y_ν and are called Bessel functions of the first and second kind respectively. Y_ν is divergent as $r \to 0$. The change of variables $f = \psi r^\lambda$, yields

$$r\psi'' + (2\nu+1)\,\psi' + \psi = 0,$$

Thus, if we choose $\nu = n/2 - 1$, the solutions to (0.1) (with $c = 1$) are $\psi(r) = r^{\nu}(c_1 J_\nu(r) + c_2 J_\nu(r))$. This connects with the special solution for $n = 3$ through the Bessel function identities

$$J_{1/2}(r) = \sqrt{\frac{2}{\pi r}} \sin r, \quad Y_{1/2}(r) = \sqrt{\frac{2}{\pi r}} \cos r.$$

If $c \neq 1$ one only needs to rescale, say $r' = \sqrt{c} r$.

A somewhat more direct route to the solution is as follows. Suppose ψ_n solves (0.1). If we set $\varphi = \psi_n'/r$ then we find that φ solves (0.1) with n replaced by $n+2$. Therefore, we find that $\varphi = \psi_{n+2}$. This allows us to find solutions for odd n rather easily. When $n = 1$ we have (when $c = 1$) $\psi'' + \psi = 0$ with solution $\psi = c_1 \cos r + c_2 \sin r$. By succcessive differentiation we have

$$\psi_{2n+1}(r) = \left(\frac{1}{r}\frac{d}{dr}\right)^n (c_1 \cos r + c_2 \sin r).$$

If $n = 2$, Bessel functions are unavoidable, and the simplest solution is $\psi_2(r) = c_1 J_0(r) + c_2 Y_0(r)$, and the rest are obtained by the operation above. The gap between odd and even dimensions is interesting, and will be reappear in the absence of a sharp Huygen's principle for the wave equation in even dimensions.

Problem 4, p. 106. Let U denote the open half-plane $x_2 > 0$, and $M = \sup_{\overline{U}} u < \infty$ and $m = \sup_{\partial U} u \leq M$. Fix $\varepsilon > 0$, and let

$$v^\varepsilon(x_1, x_2) = u(x_1, x_2) - \varepsilon \log \sqrt{(x_1^2 + (1 + x_2)^2)}.$$

Observe that $v^\varepsilon < u$ in U. Apply the weak maximum principle to the domain $U_a := x_1^2 + (1 + x_2)^2 < a^2, x_2 > 0$ to deduce that

$$\sup_{\overline{U}_a} v^\varepsilon \leq \sup_{\partial U_a} v^\varepsilon.$$

On the curved part of ∂U_a, $v^\varepsilon = u - \varepsilon \log a \leq M - \varepsilon \log a \leq m$ if a is large enough. On the flat part of the boundary we have $v \leq u \leq m$. In either case, we have the uniform estimate

$$\sup_{\overline{U}_a} v^\varepsilon(x_1, x_2) \leq m.$$

We now take $a \to \infty$ to find $\sup_{\overline{U}} v^\varepsilon \leq m$. But then, for any $(x_1, x_2) \in U$.

$$u(x_1, x_2) \leq m + \varepsilon \log \sqrt{x_1^2 + (1 + x_2)^2}.$$

The left hand side is independent of ε, so we may take $\varepsilon \to 0$ to obtain $u(x_1, x_2) \leq m$. $\square$

Remark 0.2. A similar proof can be used to prove Liouville's theorem for *subharmonic* functions in $\mathbb{R}^2$. Suppose $u : \mathbb{R}^2 \to \mathbb{R}$ is subharmonic and bounded above. Consider the function $v^\varepsilon = u - \varepsilon \log r$ in the domain $r = |x| > 1$. Observe that $v^\varepsilon < u$ here. Arguing as above, we then have

$$\sup_{|x|>1} v^\varepsilon = \max_{S(0,1)} v^\varepsilon = \max_{S(0,1)} u = \max_{\overline{B(0,1)}} u.$$

Thus, for any x with $|x| > 1$ we have

$$u(x) = v^\varepsilon(x) + \varepsilon \log r \leq \max_{\overline{B(0,1)}} u + \varepsilon \log r.$$

Since the left hand side is independent of ε, we have $u(x) \leq \max_{\overline{B(0,1)}} u$, and taking the sup over x we have $\sup_{\mathbb{R}^n} u = \max_{\overline{B(0,1)}} u$. Thus, the maximum is attained in the interior and u is constant.

5, p.110. (a) This is known as the *Schwartz reflection principle*. To show $u : B \to \mathbb{R}$ is harmonic, it is enough to verify the mean value property for sufficiently small balls for every $x \in B$. This is clearly true for x such that $x_n > 0$ since we start with a function that is harmonic in B_+. Reflection implies that it is also true for $x_n < 0$. Finally, when $x_n = 0$ the integral over the upper half ball cancels that over the lower half ball. (b) By reflection we obtain an entire bounded harmonic function, which is constant by Liouville's theorem. □

8, p.110. We begin with the mean value equality

$$\partial_{\xi_i} u(0) = \frac{n}{\omega_n a^{n+1}} \int_{S(0,a)} x_i u(x)\, dS_x.$$

Let $M = \sup_{S(\xi,a)} |u|$. Take absolute values to get

$$|\partial_{\xi_i} u(0)| \leq \frac{Mn}{\omega_n a^{n+1}} \int_{S(0,a)} |x_i|\, dS_x = \frac{Mn}{\omega_n a} \int_{S(0,1)} |x_i|\, dS_x.$$

We have rescaled in the last step. The integral can be computed exactly. By rotational symmetry, we may suppose $x_i = x_n$. All points in $S(0,1)$ at height x_n form an $n-2$ dimensional sphere of radius $\sqrt{1-x_n^2}$. Therefore, the $n-1$-dimensional volume of an infinitesimal slice is

$$\omega_{n-1}(\sqrt{1-x_n^2})^{n-2} \frac{dx_n}{\sqrt{1-x_n^2}} = \omega_{n-1}(\sin\theta)^{n-2} d\theta,$$

where θ is the polar angle from the x_n axis and $x_n = \cos\theta$. Thus,

$$\int_{S(0,1)} |x_n|\, dS_x = \omega_{n-1} \int_0^{\pi} \cos\theta (\sin\theta)^{n-2}\, d\theta = \frac{2\omega_{n-1}}{n-1}.$$

This gives γ_n. The calculation is sharp: if $u = \operatorname{sgn} x_n$ on $S(0,1)$ we find

$$\partial_{\xi_n} u(0) = \frac{n}{\omega_n} \int_{S(0,1)} |x_n|\, dS_x = \gamma_n.$$

Strictly speaking, we have not shown that we can solve the Dirichlet problem with discontinuous data. Observe however that Poisson's integral formula does define a harmonic function for this boundary data, and the estimates on derivatives require only $\sup |u|$. $\square$

9, p. 110. (a) This has been worked out in lecture for the earlier constant. (b). We need to consider convergence of the power series

$$\sum_{\alpha} \frac{\partial^\alpha u(\xi)}{\alpha!} (x-\xi)^\alpha.$$

We take absolute values, and use the estimate on $\partial^\alpha u(\xi)$ to bound the expression above by

$$\begin{aligned}
&\sum_{m=0}^{\infty} \sum_{|\alpha|=m} \left(\frac{m\gamma_n}{a}\right)^m \frac{1}{\alpha!} |x_1 - \xi_1|_1^\alpha \ldots |x_n - \xi_n|_n^\alpha \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{m\gamma_n}{a}\right)^m \sum_{|\alpha|=m} \frac{m!}{\alpha!} |x_1 - \xi_1|_1^\alpha \ldots |x_n - \xi_n|_n^\alpha \\
&= \sum_{m=0}^{\infty} \frac{m^m}{m!} \left(\frac{\gamma_n(|x_1 - \xi_1| + \ldots + |x_n - \xi_n|)}{a}\right)^m.
\end{aligned}$$

We have used the multinomial expansion in the last equation. Use Stirling's approximation $m! \sim \sqrt{2\pi m} m^m e^{-m}$ to see that the above expression converges if and only if

$$\sum_{m=0}^{\infty} \frac{1}{\sqrt{2\pi m}} \left(\frac{e\gamma_n(|x_1 - \xi_1| + \ldots + |x_n - \xi_n|)}{a}\right)^m < \infty.$$

Thus, we need

$$|x_1 - \xi_1| + \ldots |x_n - \xi_n| < \frac{a}{e\gamma_n}.$$

I only get the strict inequality, and can't figure out where the $=$ comes from in John's book. $\square$

3, p. 73. z^p, and thus $g(z) = e^{-z^{-p}}$, is analytic in the slit plane $\mathbb{C}\backslash(-\infty, 0]$. Let $a < x$. Use Cauchy's integral formula on the circle $S(x, a)$ to find

$$g^k(x) = \frac{k!}{2\pi i} \int_{S(x,a)} \frac{g(z)}{(z-x)^{k+1}}\, dz.$$

Therefore, on taking absolute values

$$|g^k(x)| \leq \frac{k!}{a^k} \sup_{S(x,a)} |g|.$$

If we use polar coordinates $z = re^{i\varphi}$, we have

$$g(z) = \exp\left(-r^{-p}e^{-ip\varphi}\right) = \exp\left(-r^{-p}(\cos p\varphi - i \sin p\varphi)\right).$$

Thus, $|g(z)| = \exp(-r^{-p}\cos p\varphi)$. When $z \in S(x, a)$ the polar angle φ is at most $\tan^{-1}(a/x) := \varphi_0$, and r is at least $x - a = x(1 - \tan\varphi_0)$. Thus,

$$\max_{z \in S(x,a)} |g| \leq \exp\left(-x^{-p}(1 - \tan\varphi_0)^{-p}\cos\varphi_0\right).$$

As $\varphi_0 \to 0$, the factor $(1 - \tan\varphi_0)^{-p}\cos\varphi_0 \to 1$ from below. We may therefore, choose $\varphi_0(p)$ such that $\max |g| \leq e^{-x^{-p}/2}$ on $S(x, a)$. Let $\theta(p) := 1 - \tan\varphi_0 < 1$. We then have

$$|g^{(k)}(x)| \leq \frac{k!}{(\theta x)^k} \exp\left(\frac{-x^{-p}}{2}\right).$$

Part (b) is also interesting (you didn't have to turn this in). To find a uniform bound in x, we must maximimize the right hand side. Differentiate to find that the maximum is attained when $x^{-p} = 2k/p$. Consequently,

$$\begin{aligned}|g^{(k)}(x)| \leq \frac{k!}{\theta^k}\left(\frac{2k}{pe}\right)^{k/p} &= (k!)^{1+1/p}\left(\frac{k^k}{k!}\right)^{1/p}\left(\frac{2}{pe\theta^p}\right)^{k/p} \\ &\sim (k!)^{1+1/p}(2\pi k)^{-1/p}\left(2^{1/p}p^{-1/p}\theta\right)^{-k},\end{aligned}$$

by Stirling's approximation. ∎