PDE, HW 1 solutions

For future reference, here is a computation of ω_n (you did not need to turn this in). You can compute ω_n directly by induction. A slick approach uses the Gaussian integral

$$\int_{\mathbb{R}} e^{-|x_1|^2/2}\, dx_1 = \sqrt{2\pi}.$$

Let us compute the Gaussian integral in $\mathbb{R}^n$ two different ways. First, since the integral factors we have

$$\int_{\mathbb{R}^n} e^{-|x|^2/2}\, dx = \int_{\mathbb{R}^n} e^{-(x_1^2+\ldots+x_n^2)/2}\, dx_1 \ldots dx_n = (2\pi)^{n/2}.$$

On the other hand, one may switch to polar coordinates to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-|x|^2/2}\, dx &= \omega_n \int_0^\infty e^{-r^2/2} r^{n-1} dr \\ &= \omega_n \int_0^\infty e^{-t}(2t)^{n/2-1} dt = 2^{n/2-1}\Gamma(\frac{n}{2}),\end{aligned}$$

after the change of variables $r^2 = 2t$. Equate the two calculations to find,

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

The volume of a ball of radius r is given by

$$|B(0,r)| = \omega_n \int_0^r \rho^{n-1}\, d\rho = \frac{\omega_n}{n} r^n.$$

One of the reasons these exact values are of interest is that we can now compute the sharp constant in the isoperimetric inequality. A loose form of the isoperimetric inequality is "of all domains with a fixed surface area, the ball has the most volume". More precisely, for sufficiently regular bounded domains $U \subset \mathbb{R}^n$, there is a constant γ_n such that

$$|U|_n \leq \gamma_n |\partial U|_{n-1}^{n/(n-1)}.$$

Here $|G|_k$ denotes the k-dimensional volume of a set G. The inequality is sharp precisely when U is a ball. In this case, we have

$$|B(x,r)|_n = \frac{\omega_n}{n} r^n = \gamma_n \left(\omega_n r^{n-1}\right)^{n/(n-1)} = \gamma_n \omega_n^{n/n-1} r^n.$$

Observe that r^n cancels (this is *scale-invariance*) and we are left with

$$\gamma_n = \omega_n^{-1/(n-1)}/n.$$

The behaviour of this constant as $n \to \infty$ is surprisingly relevant. You can use Stirling's approximation

$$\Gamma(\alpha) \sim \sqrt{2\pi}\alpha^{\alpha+1/2}e^{-\alpha}$$

to show that

$$\gamma_n \sim \frac{1}{\sqrt{2\pi e n}}, \qquad n \to \infty.$$

1. The mean value inequality can be used to *define* subharmonic functions. Henceforth, we will use the following definition.

Definition 0.1. A function $u \in C(U)$ is subharmonic if for every $x \in U$ there exists $\delta(x) > 0$ such that $S(x, \delta) \subset U$ and

$$u(x) \leq \fint_{S(x,r)} u(y)dS_y, \quad 0 < r \leq \delta.$$

Yet another definition is the following.

Definition 0.2. A function $u \in C(U)$ is subharmonic if for every $B(x, r) \subset\subset U$ and every harmonic function v defined on a domain containing $B(x, r)$ such that $u \leq v$ on $S(x, r)$, we have $u \leq v$ in $B(x, r)$.

Show that these definitions are equivalent; that is, each implies the other. You may assume the existence and uniqueness of solutions for the Dirichlet problem in the ball.

Proof. Assume Defn. 2. Let $\delta(x) = \operatorname{dist}(x, \partial U)/2$, so that $B(x, r) \subset U$ for $0 < r \leq \delta$. Let v be the harmonic function in $B(x, r)$ defined by the boundary values $v = u$ on $S(x, r)$. Then Defn 2 implies at once that

$$u(x) \leq v(x) = \fint_{S(x,r)} v(y)dS_y = \fint_{S(x,r)} u(y)dS_y.$$

Assume Defn. 1. Consider any ball $B(x, r) \subset U$ and a harmonic function v defined on a domain that includes this ball. Then the difference $w = u - v$ is subharmonic in the sense of Defn 1, continuous on $\overline{B(x, r)}$ and satisfies the mean value inequality obtained by integrating over the radial coordinate

$$w(y) \leq \fint_{B(y,\delta(y))} w(z)dS_z, \qquad y \in B(x, r).$$

It then follows that w satisfies the strong, and hence weak maximum principle, so that $w \leq 0$ in $B(x,r)$. (See the proof in the lecture notes, only the continuity of w is needed once one has the mean-value inequality. The C^2 assumption was used only to derive the mean value inequality) $\square$

2. Let $f : \mathbb{R} \to R$ be convex; that is $f(ax + (1-a)y) \leq af(x) + (1-a)f(y)$ for every $a \in [0,1]$. Convex functions are continuous and satisfy Jensen's inequality (look this up, if you haven't seen it before). Let $u : U \to \mathbb{R}$ be harmonic. Show that $f \circ u$ is subharmonic. Deduce that the functions $|u|^p, p \geq 1$ and $|Du|^2$ are subharmonic.

Proof. Jensen's inequality states that if G is a bounded domain, and the average of a function $u : G \to \mathbb{R}$ defined by

$$⨍_G u(y)\, dy = \frac{1}{|G|} \int_U u(y)\, dy,$$

then

$$f(⨍_G u(y)\, dy) \leq ⨍_G f(u(y))\, dy.$$

If u is harmonic, and $B(x,r) \subset U$ the mean value equality states that $u(x) = ⨍_{B(x,r)} u(y)\, dy$. It is then immediate from Jensen's inequality that

$$f(u(x)) \leq ⨍_{B(x,r)} f(u(y))\, dy.$$

The function $f(x) = |x|^p$ is convex for $p \geq 1$, and Du is harmonic if u is (well, strictly speaking we haven't proved this yet, but most of you assumed this). $\square$

3. Use Harnack's inequality to prove Liouville's theorem: a harmonic function on $\mathbb{R}^n$ that is bounded below is constant.

Proof. Let $m = \inf_{\mathbb{R}^n} u$. Replacing u by $u - m$ we may suppose that $m = 0$. In the proof of Harnack's inequality we obtained the following estimate

$$\sup_{B(0,r)} u \leq 3^n \inf_{B(0,r)} u.$$

The factor 3^n is independent of r and we may take $r \to \infty$ to obtain $\sup_{\mathbb{R}^n} u \leq 3^n \inf_{\mathbb{R}^n u} = 0$. Thus, $u \equiv 0$. $\square$

4. Inversion in the unit sphere is the map $f : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n$ defined by

$$f(x) = \frac{x}{|x|^2}.$$

(The origin is mapped to the point at infinity). Show that this map is conformal. Interpret the gradient Df geometrically.

Proof. It may help to do the calculation in coordinates. If $f_i = x_i/|x|^2$, then

$$f_{i,j} := \partial_{x_j} f_i = \frac{\delta_{ij}}{|x|^2} - \frac{2x_i x_j}{|x|^4},$$

where δ_{ij} is 1 if $i = j$ and 0 otherwise (this is called Kronecker's symbol). In coordinate independent notation, this is the matrix

$$Df = |x|^{-2} \left(I - \frac{2xx^t}{|x|^2} \right).$$

x is a column vector, and xx^t is the outer or tensor product of x with itself. Now compute,

$$(Df)^t Df = |x|^{-4} \left(I - 4|x|^{-2} xx^t + 4|x|^{-4} xx^t xx^t \right) = |x|^{-4} I.$$

To interpet Df geometrically, suppose for simplicity that $x = (1, 0, \ldots 0)$. Then in coordinates, Df is the matrix $\operatorname{diag}(-1, 1, \ldots, 1)$. This corresponds to reflection in the plane normal to x. In general, Df is reflection in the plane normal to x composed with rescaling by $|x|^{-2}$. $\square$

5. Suppose $n = 2$. Let $f = (f_1(x), f_2(x))$ be a C^2 conformal map from an open subset $\mathbb{R}^2 \supset U \to \mathbb{R}^2$. Show that f_1 and f_2 are harmonic, that is $\triangle f_1 = \triangle f_2 = 0$. Thus, show that if $u : U \to \mathbb{R}$ is harmonic, so is $v = u \circ f$.

Proof. Conformal matrices in two dimensions are always in one of the following two forms

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \qquad a^2 + b^2 > 0.$$

Confusingly enough, these are sometimes called conformal and anti-conformal matrices respectively (for us, conformal always means $F^t F = \lambda I, \lambda > 0$). The first has positive determinant and the second negative.

Let us first prove this. Suppose F is a conformal matrix. Then we have $F^tF = \lambda I$, and we also have $F^tF = \lambda I$. If

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then the diagonal terms in the matrix equations $F^tF = \lambda I = FF^t$ yield

$$\begin{aligned} a^2 + c^2 &= \lambda & \qquad a^2 + b^2 &= \lambda \\ c^2 + d^2 &= \lambda & \qquad b^2 + d^2 &= \lambda. \end{aligned}$$

Therefore, $c = \pm b$ and $d = \pm a$. Now the vanishing of the off-diagonal terms requires $d = a, c = -b$ or $c = b, d = -a$.

On any connected component of U, $\det(DF)$ is a continuous function, thus $\det(Df)$ is either always positive, or always negative. Consequently, Df is always in one of the two standard forms. If $\det(Df) > 0$ we obtain the Cauchy-Riemann equations

$$f_{1,1} = f_{2,2} \qquad f_{1,2} = -f_{2,1},$$

and if $\det(Df) < 0$, the anti-Cauchy-Riemann equations

$$f_{1,1} = -f_{2,2} \qquad f_{1,2} = f_{2,1}.$$

Since f is C^2 we may equate cross-derivatives. Differentiate the first equation with respect to x_1, the second relative to x_2, and add to obtain $\triangle f_1 = 0$. Similarly, $\triangle f_2 = 0$. Finally, if $v = u \circ f$, the chain rule implies

$$\triangle v = \mathrm{tr}(Df^t\, D^2u\, Df) + Du \cdot \triangle f.$$

Since f is conformal, $Df^t = \lambda Df^{-1}$, therefore,

$$\mathrm{tr}(Df^t\, D^2u\, Df) = \lambda \mathrm{tr}(Df^{-1}\, D^2u\, Df) = \lambda \mathrm{tr}(D^2u) = \lambda \triangle u = 0.$$

In two dimensions, we have also shown that conformality implies $\triangle f = 0$. Thus, $\triangle v = 0$. $\square$

6. Show that inversion is not harmonic for $n \geq 3$ (a vector field is harmonic if each of its components is harmonic). The fix for this is in the following problem.

Proof. We will show that

$$\triangle f = 2(2-n)\frac{x}{|x|^4} \tag{0.1}$$

which vanishes only if $n = 2$. Again, it may help to work with indices. Continuing from problem 4, we have

$$f_{i,jk} = -2|x|^{-4}\left(\delta_{ij}x_k + \delta_{ik}x_j + \delta_{jk}x_i\right) + 8|x|^{-6}x_ix_jx_k.$$

The Laplacian $\triangle f_i$ is obtained by summing $\sum_{j=1}^n f_{i,jj}$. Since this operation is so common, the Einstein summation convention is usually adopted: we sum over any repeated index. For example, $f_{i,jj}$ means $\sum_{j=1}^n f_{i,jj}$. Apply this calculation to the derivative above, and use $\delta_{jj} = n$ to obtain (0.1). $\square$

7. *Kelvin's transformation.* Let u be a harmonic function on $\mathbb{R}^n$. Show that $v(x) = |x|^{2-n}u(x/|x|^2)$ is a harmonic function for $x \neq 0$.

Proof. For future reference, here is a general identity for change of variables. If $v(x) = \lambda(x)u(f(x))$, then

$$\triangle v = u\triangle\lambda + 2Du\,Df\,D\lambda^t + \lambda Du\cdot\triangle f + \lambda\mathrm{Tr}((Df)^t\,D^2u\,Df). \tag{0.2}$$

The convention is that f is a column vector, $D\lambda$ and Du are row vectors. Please derive this formula using the chain rule to be sure you understand it.

Apply this formula with $\lambda = |x|^{2-n}$ and $f = x/|x|^2$. First note, that the first and last terms in (0.2) vanish. Since λ is a multiple of the fundamental solution, $\triangle\lambda = 0$ for $x \neq 0$. The calculations in problem 4 show that f is conformal, therefore

$$\mathrm{Tr}((Df)^t\,D^2u\,Df) = |x|^{-4}\triangle u = 0.$$

Now use the computation of problem 6 to find that

$$\lambda Du\cdot\triangle f = 2(2-n)|x|^{-(n+2)}Du\cdot x.$$

On the other hand,

$$2Du\,Df\,D\lambda^t = 2Du\frac{1}{|x|^2}(I - 2\frac{xx^t}{|x|^2})(2-n)|x|^{-n}x = -2(2-n)|x|^{-(n+2)}Du\cdot x.$$

$\square$

Remark 0.3. The sense in which Kelvin's transformation is a replacement for transformation by conformal maps is the following. Let's suppose the goal is to generate new harmonic functions from old by transformations $v = u \circ f$, that is $\lambda \equiv 1$. Now we find that

$$\triangle v = \mathrm{tr}((Df)^t\, D^2 u\, Df) + Du \cdot \triangle f.$$

Therefore, in order that $\triangle v = 0$ the conditions $Df^t Df = \lambda I$ (f is conformal), and $\triangle f = 0$ (f is harmonic) are clearly sufficient. It turns out that the conditions that f be conformal and f be harmonic are also necessary for every harmonic function u to be taken to a harmonic function v (try to prove this, you will learn something). So it would seem that by choosing maps that are both harmonic and conformal we can generate new harmonic functions. This is fine for $n = 2$ since conformality even implies harmonicity. Things are more interesting when $n \geq 3$: the *only* conformal transformations for $n \geq 3$ are Möbius transformations; that is a finite composition of inversions, rotations, and translations. This is another famous theorem of Liouville. As we have seen, inversion is not harmonic, thus $v = u \circ f$ cannot be harmonic for all but linear conformal maps. In order to obtain a new harmonic map, we must also rescale as in Kelvin's transformation.

8. Nonuniqueness for the exterior problem. Let $U = \{x | |x| > 1\}$ be the exterior of the unit ball in $\mathbb{R}^n, n \geq 3$. Consider the Dirichlet problem

$$\triangle v = 0, \quad x \in U, \qquad v = 1, \quad |x| = 1.$$

Show that there are infinitely many solutions to this problem. Which, if any, is the most appropriate solution?

Proof. Any function of the form $v(x) = a|x|^{2-n} + b$ is harmonic in U. In order to satisfy the boundary condition we only need $a + b = 1$.

Surprisingly, the most natural solution is *not* $v \equiv 1$. In fact, the 'appropriate' solution to the exterior problem is the one determined by Kelvin's transformation because it reflects the conformal invariance of Laplace's equation. If we were solving the interior problem with the same boundary conditions, the unambiguous solution is $u = 1, x \in B(0,1)$. Then Kelvin's transformation yields the solution $v(x) = |x|^{n-2}$ for the exterior problem. □

This solution turns out to be the unique solution to the exterior problem if a decay condition is imposed along with the PDE. The moral is that in order to obtain uniqueness for the exterior problem, we must impose decay conditions at infinity. There are infinitely many solutions in two dimensions also. Simply take $u = 1 + a \log |x|$, for any $a \in \mathbb{R}$. In this case, the 'appropriate' solution is $u = 1$ (Kelvin again).