1 1-D Wave Equation

$$u_{tt} = c^2 u_{xx} = 0 \tag{1.1}$$

for $x \in \mathbb{R}$ and $t > 0$ with $u(x,0) = f(x)$, $u_t(x,0) = g(x)$. D'Alembert's formula:

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} g(y)\,dy\right].$$

Geometric identity:

$$u(A) + u(C) = u(B) + u(D). \tag{1.2}$$

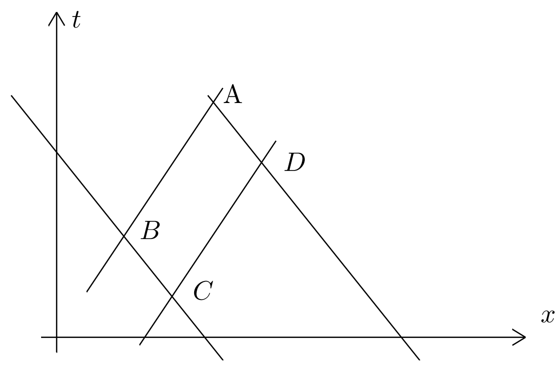


Figure 1.1. Sketch for the geometric identity.

We have: C^2 solution of (1.1) $\Leftrightarrow$ (1.2) for every characteristic parallogram.

1.1 Boundary conditions

Good and bad boundary conditions:

$$0 = u_t + c\,u_x,$$

supposing $c > 0$.

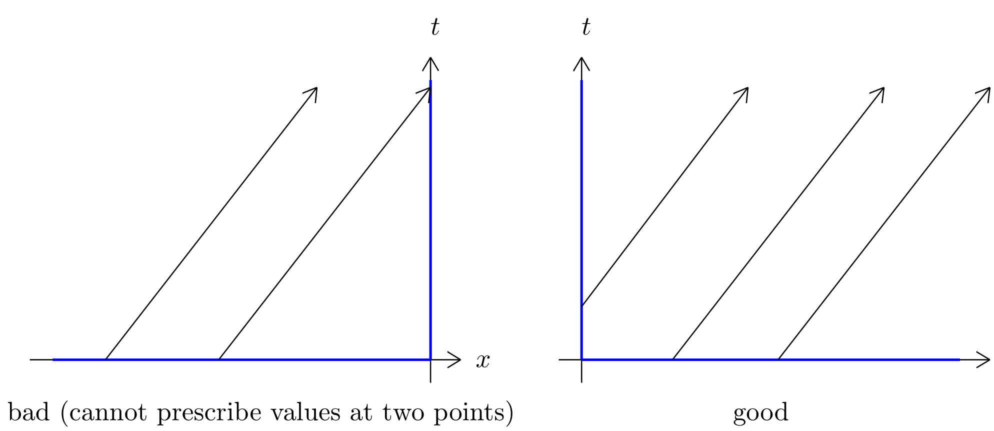


Figure 1.2. Good and bad boundary conditions for the transport equation.

Example:

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in (0,\infty), t > 0$$

$u(x,0)=f(x)$, $u_t(x,0)=g(x)$ for $x\in\mathbb{R}$. $u(0,t)=0$ for $t\geqslant 0$ with the assumption that $f(0)=0$.

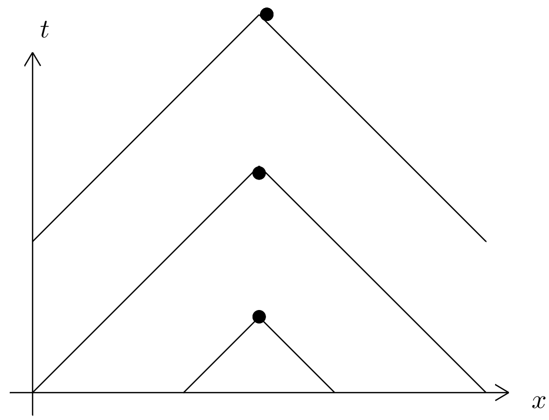


Figure 1.3. Domain of dependence.

The dependency on ICs outside of the domain is solved by the *method of reflection*. Extend u to all of $\mathbb{R}$, say $\tilde{u}$.

$$\tilde{u}(x,t)=\frac{1}{2}\left[\tilde{f}(x+ct)+\tilde{f}(x-ct)+\frac{1}{c}\int_{x-ct}^{x+ct}\tilde{g}(y)dy\right].$$

$$\tilde{u}(0,t)=\frac{1}{2}\left[\tilde{f}(ct)+\tilde{f}(ct)+\frac{1}{c}\int_{ct}^{ct}\tilde{g}(y)dy\right].$$

Choose odd extension:

$$\tilde{u}(x,t)=\begin{cases} u(x,t) & x\geqslant 0,\\ -u(-x,t) & x<0.\end{cases}$$

Similarly for $\tilde{f}$, $\tilde{g}$. Then $\tilde{u}(0,t)=0=u(0,t)$. $u(x,t)=\tilde{u}(x,t)$ for $x>0$.

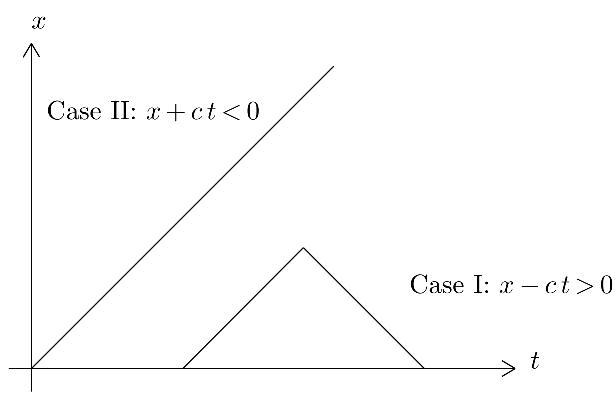


Figure 1.4. Different cases arising for the determination of the domain of dependence.

Case 1: D'Alembert as before.
Case 2:

$$u(x,t)=\frac{1}{2}\left[f(x+ct)+\underbrace{f(ct-x)}_{\text{odd ext.}}+\frac{1}{c}\int_{ct-x}^{x+ct}g(y)dy\right].$$

If $g\equiv 0$, this corresponds to reflection as follows:

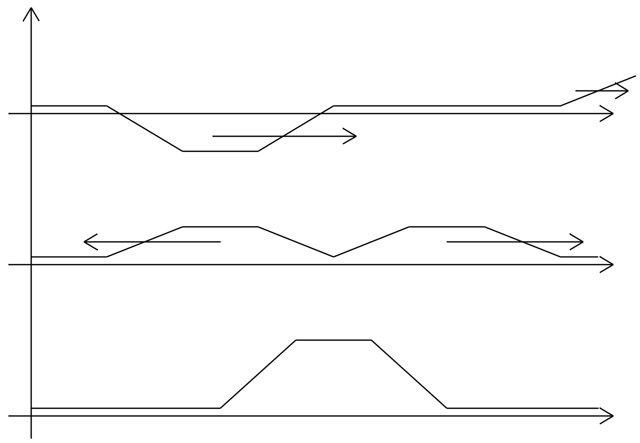


Figure 1.5. Series of snapshots of solutions with $g=0$.

Initial boundary value problem:

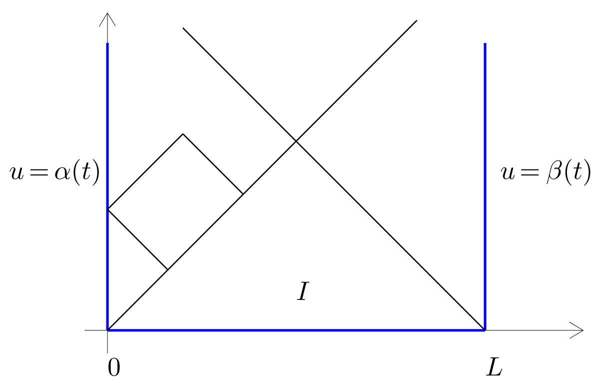


Figure 1.6. Initial boundary value problem. We can satisfy the parallelogram identity using geometry.

For arbitrary α, β the equation need not have a continuous solution:

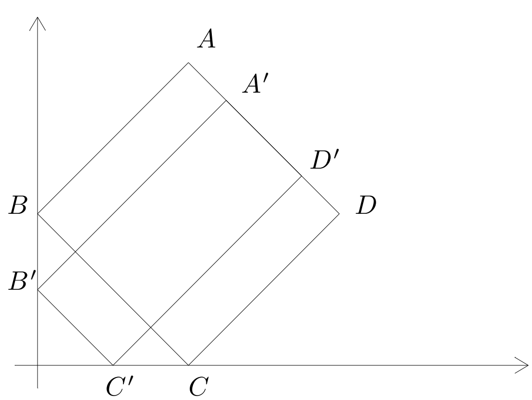


Figure 1.7. Discontinuous solutions in corners.

Assume $u \in C((0, L] \times (0, \infty))$.

$$\begin{aligned} u(B) &= \alpha(B), \\ u(C) &= f(C). \end{aligned}$$

$u(A) + u(C) = u(B) + u(D)$. $A \to D \Rightarrow u(A) \to u(D)$, $u(C) = u(B) \Rightarrow \lim_{t \to 0} \alpha(t) = \lim_{x \to 0} f(x)$. Similarly, if we want $u \in C^1$, this requires $\alpha'(0) = g(0)$, etc.

1.2 Method of Spherical Means

$$\partial_t^2 u - c^2 \Delta u = 0$$

for all $x \in \mathbb{R}^n$ and $t > 0$ with

$$\begin{aligned} u(x, 0) &= f(x), \\ u_t(x, 0) &= g(x). \end{aligned}$$

If $h: \mathbb{R}^n \to \mathbb{R}$, let

$$\begin{aligned} M_h(x, r) &= \frac{1}{\omega_n r^{n-1}} \int_{S(x,r)} h(y) dS_y \\ &= \frac{1}{\omega_n} \int_{|\omega| = 1} h(x + r\omega) dS_\omega. \end{aligned}$$

Assume that h is continuous. Then

1. $\lim_{r \to 0} M_h(x, r) = h(x)$ for every $x \in \mathbb{R}^n$.
2. $M_h(x, r)$ is a continuous and even function.

If $h \in C^2(\mathbb{R}^n)$, then

$$\Delta_x M_h(x, r) = \frac{\partial^2}{\partial r^2} M_h + \frac{n-1}{r} \frac{\partial M_h}{\partial_r}.$$

If you view M_h as a function M_h: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which is spherically symmetric, then the above equation states that the Laplacian in the first n variables equals the Laplacian in the second n. Spherical means of

$$\partial_t^2 u - c^2 \Delta_x u = 0.$$

Then

$$\partial_t^2 M_u - c^2 \Delta_x M_u = 0$$

and

$$\partial_t^2 M_u - \left[\frac{\partial^2}{\partial_r^2 M_u} + \frac{n-1}{r} \frac{\partial M_u}{\partial r} \right] = 0.$$

1.3 Wave equation in $\mathbb{R}^n$

$$\Box u := u_{tt} - c^2 \Delta u = 0 \qquad (*)$$

for $x \in \mathbb{R}^n \times (0, \infty)$ with $u = f$ and $u_t = g$ for $x \in \mathbb{R}^n$ and $t = 0$. Now do Fourier analysis: If h $\in L^1(\mathbb{R}^n)$, consider

$$\hat{h}(\xi) := \int_{\mathbb{R}^n} e^{-ix\xi} h(x) dx.$$

If we take the FT of $(*)$, we get

$$\hat{u}_{tt} + c^2 |\xi|^2 \hat{u} = 0$$

for $\xi \in \mathbb{R}^n$ and $t > 0$, $\hat{u}(\xi, 0) = \hat{f}$, $\hat{u}(\xi, 0) = \hat{g}$. $\hat{u}(\xi, t) = A \cos(c|\xi|t) + B \sin(c|\xi|t)$. Use ICs to find

$$\hat{u}(\xi, t) = \hat{f}(\xi) \cos(c|\xi|t) + \hat{g}(\xi) \frac{\sin(c|\xi|t)}{c|\xi|}.$$

Analogous caclulation for heat equation:

$$u_t - u_{xx} = 0 \Rightarrow \hat{u}_t + |\xi|^2 \hat{u} = 0, \quad \hat{u}(\xi, 0) = \hat{f}$$

yields $\hat{u}(\xi, t) = e^{-|\xi|^2 t} \hat{f}(\xi)$. Then observe that multiplication becomes convolution.

Observe that

$$\cos(c|\xi|t) = \partial_t \left(\frac{\sin(c|\xi|t)}{c|\xi|} \right).$$

If we could find a $k(x, t)$ such that

$$\frac{\sin(c|\xi|t)}{c|\xi|} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} k(x, t) dx,$$

this would lead to a solution formula

$$u(x, t) = \int_{\mathbb{R}^n} k(x - y, t) g(y) dy + \partial_t \int_{\mathbb{R}^n} k(x - y, t) f(y) dy.$$

Suppose $n = 1$, we know that our solution formula must coincide with D'Alembert's formula

$$u(x, t) = \frac{1}{2} \left[f(x + ct) + f(x - ct) + \frac{1}{c} \int_{x-ct}^{x+ct} g(y) dy \right].$$

Here

$$\begin{aligned} k(x, t) &= \frac{1}{2c} \mathbf{1}_{\{|x| \leqslant ct\}}, \\ \partial_t k(x, t) &= \frac{1}{2} \left[\delta_{\{x = ct\}} + \delta_{\{x = -ct\}} \right]. \end{aligned}$$

Solution formula for $n = 3$:

Theorem 1.1. *$u \in C^\infty(\mathbb{R}^3 \times \mathbb{R})$ is a solution to the wave equation with C^∞ initial data f, g if and only if*

$$u(x, t) = ⨍_{S(x, ct)} [t g(y) + f(y) + D f(y)(y - x)] dS_y.$$

Here,

$$k(x, t) = \frac{1}{4\pi c^2 t} \cdot dS_y \Big|_{|x| = ct} = t \cdot \text{uniform measure on } \{|x| = ct\}.$$

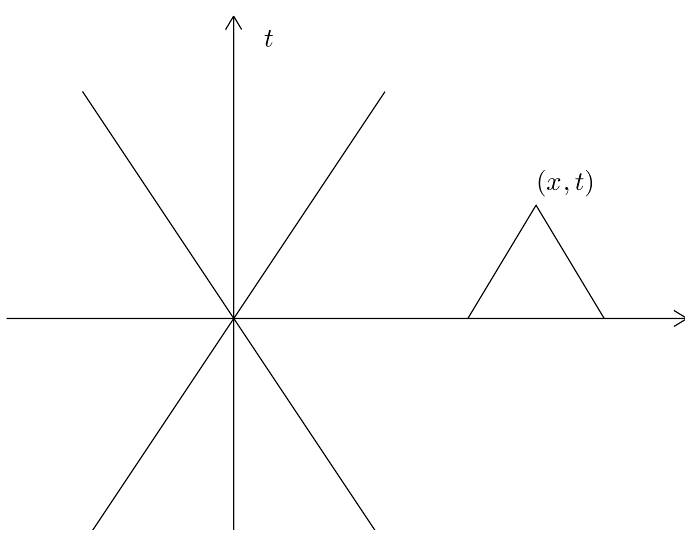


Figure 1.8.

1.4 Method of spherical means

Definition 1.2. *Suppose $h: \mathbb{R}^n \to \mathbb{R}$ is continuous. Define $M_h: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by*

$$M_h(x,r) = ⨍_{S(x,r)} h(y) dS_y = \frac{1}{\omega_n} \int_{|\omega|=1} h(x + r\omega) \cdot d\omega.$$

Notice that

$$\lim_{r \to 0} M_h(x,r) = h(x)$$

if h is continuous.

Darboux's equation: Suppose $h \in C^2(\mathbb{R}^n)$. Then

$$\Delta_x M_h(x) = \frac{\partial^2}{\partial r} M_h + \frac{n-1}{r} \cdot \frac{\partial M_h}{r}.$$

Proof. Similar to the mean value property for Laplace's equation.

$$\begin{aligned}
\int_0^r \Delta_x M_h(x,\rho) \rho^{n-1} \cdot d\rho &= \int_0^r \Delta_x \frac{1}{\omega_n} \int_{|\omega|=1} h(x+\rho\omega) \cdot d\omega \, \rho^{n-1} d\rho \\
&= \int_{B(0,r)} \Delta_x h(x+y) \cdot dy = \frac{1}{\omega_n} \int_{S(0,r)} \frac{\partial h}{\partial n_y}(x+y) dy \\
(y = r\omega, dy = r^{n-1} d\omega) \quad &= \frac{1}{\omega_n} \int_{S(0,r)} Dh(x+y) \cdot n_y \, dy \\
&= \frac{r^{n-1}}{\omega_n} \int_{|\omega|=1} \frac{d}{dr}(h(x+r\omega) \cdot d\omega = r^{n-1} \frac{\partial M_h}{\partial r}.
\end{aligned}$$

Then

$$\int_0^r \Delta_x M_h(x,\rho) \rho^{n-1} \cdot d\rho = r^{n-1} \frac{d}{dr} M_h.$$

Differentiate

$$\begin{aligned}
\Delta_x M_h r^{n-1} &= \frac{d}{dr}\left[r^{n-1} \cdot \frac{dM_h}{dr} \right] \\
&= r^{n-1} \cdot \frac{d^2}{dr^2} + (n-1) r^{n-2} \frac{dM_h}{dr}.
\end{aligned}$$

Altogether

$$\Delta_x M_h = \frac{\partial^2 M_h}{\partial r^2} + \frac{(n-1)}{r} \frac{\partial M_h}{\partial r}.$$

□

Look at spherical means of $(*)$:

$$u_{tt} - c^2 \Delta u = 0$$

Assume $u \in C^2(\mathbb{R}^n \times (0,\infty))$. Take spherical means:

$$M_{u_{tt}} = (M_u)_{tt},$$

which means

$$\begin{aligned}
\partial_t^2 ⨍_{S(x,r)} u(y,t) dS_y &= ⨍_{S(x,r)} \partial_t^2 u(y,t) dy, \\
(M_u)_{tt} &= M_{u_{tt}}.
\end{aligned}$$

And

$$M_h(\Delta_x u) \overset{\text{Darboux}}{=} \frac{\partial^2 M_h}{\partial r^2} + (n-1)\frac{\partial M_h}{\partial r}.$$

Therefore, we have

$$(M_u)_{tt} = c^2\left[\frac{\partial^2 M_h}{\partial r^2} + (n-1)\frac{\partial M_h}{\partial r}\right].$$

If $n=1$, we can solve by D'Alembert. For $n=3$:

$$\frac{\partial^2}{\partial r^2}(r\,M_h) \;=\; \frac{\partial}{\partial r}\left(r\frac{\partial M_h}{\partial r} + M_h\right) = r\frac{\partial^2 M_h}{\partial r^2} + 2\cdot\frac{\partial M_h}{\partial r}.$$

So if $n=3$, we have

$$(r\,M_u)_{tt} = c^2\frac{\partial^2}{\partial r^2}(r\,M_h)$$

This is a 1D wave equation (in r!). Solve for $r\,M_h$ by D'Alembert.

$$M_h(x,r,t) \;=\; \frac{1}{2r}\left[(r+c\,t)M_f(x,r+c\,t) + \underbrace{(r-c\,t)}_{\text{a)}}M_f(x,r-c\,t)\right] + \underbrace{\frac{1}{2c\,r}\int_{r-ct}^{r+ct} r' M_g(x,r')dr'}_{\text{b)}}$$

Pass to limit $r\to 0$ in b)

$$\frac{1}{2c\,r}\int_{r-ct}^{r+ct} r' M_g(x,r')dr' \;=\; \frac{1}{2c\,r}\int_{ct-r}^{ct+r} r' M_g(x,r')dr'$$

M_g is even, $r\,M_g$ is odd. So

$$\lim_{r\to 0}\text{b)} = \frac{1}{c}\cdot c\,t\,M_g(x,c\,t) = t\,M_g(x,c\,t).$$

$$t\,M_g(x,c\,t) = t⨍_{|x-y|=ct} g(y)dS_y.$$

Similarly, a): (M_f even in r)

$$\begin{aligned} &= \frac{1}{2}[M_f(x,r+c\,t) + M_f(x,c\,t-r)] + \frac{1}{2\,r}c\,t\,[M_f(x,c\,t+r) - M_f(x,c\,t-r)] \\ \lim_{r\to 0} * &= M_f(x,c\,t) + c\,t\,\partial_2 M_f(x,c\,t) = \partial_t(t\,M_f(x,c\,t)). \end{aligned}$$

For any $\varphi\in C^\infty(\mathbb{R}^3)$ define

$$(K_t*\varphi)(x) := t\int_{|x-y|=ct}\varphi(y)dS_y.$$

Then if $f,g\in C^\infty$, our solution to $\Box u = 0$ is

$$\boxed{u(x,t) = (K_t*g)(x) + \partial_t(K_t*f)(x).}$$

Aside: Check that

$$\int_{|y|=ct} e^{-i\xi\cdot y}dS_y = \frac{\operatorname{sinc}(c\,t|\xi|)}{c|\xi|}.$$

Remark 1.3. Huygens' principle:

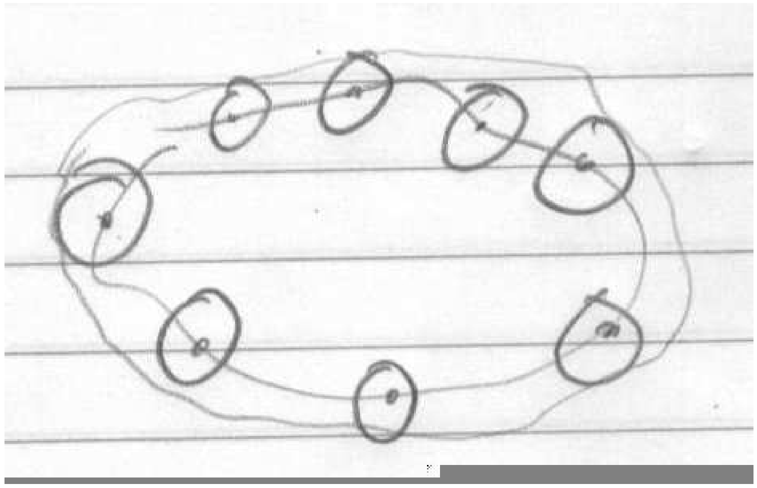


Figure 1.9. Huygens' principle.

We consider data f, g with compact support. Let

$$\Sigma(t) = \operatorname{supp}(u(x,t)) \subset \mathbb{R}^3,$$

where obviously

$$\Sigma(0) = \operatorname{supp}(f) \cup \operatorname{supp}(g).$$

Then Huygens' principle is stated as

$$\Sigma(t) \subset \{x \colon \operatorname{dist}(x, \Sigma(0)) = c\,t\}.$$

Example 1.4. Consider radial data g and $f \equiv 0$.

$$u(x,t) = t \fint_{|x-y|=ct} g(y) dS_y.$$

$$u(x,t) \neq 0 \Leftrightarrow S(x, c\,t) \cap B(0, \rho) \neq \emptyset.$$

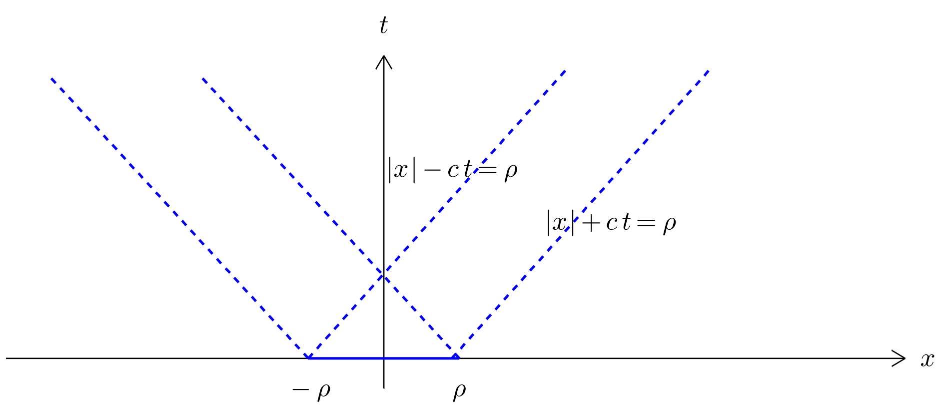
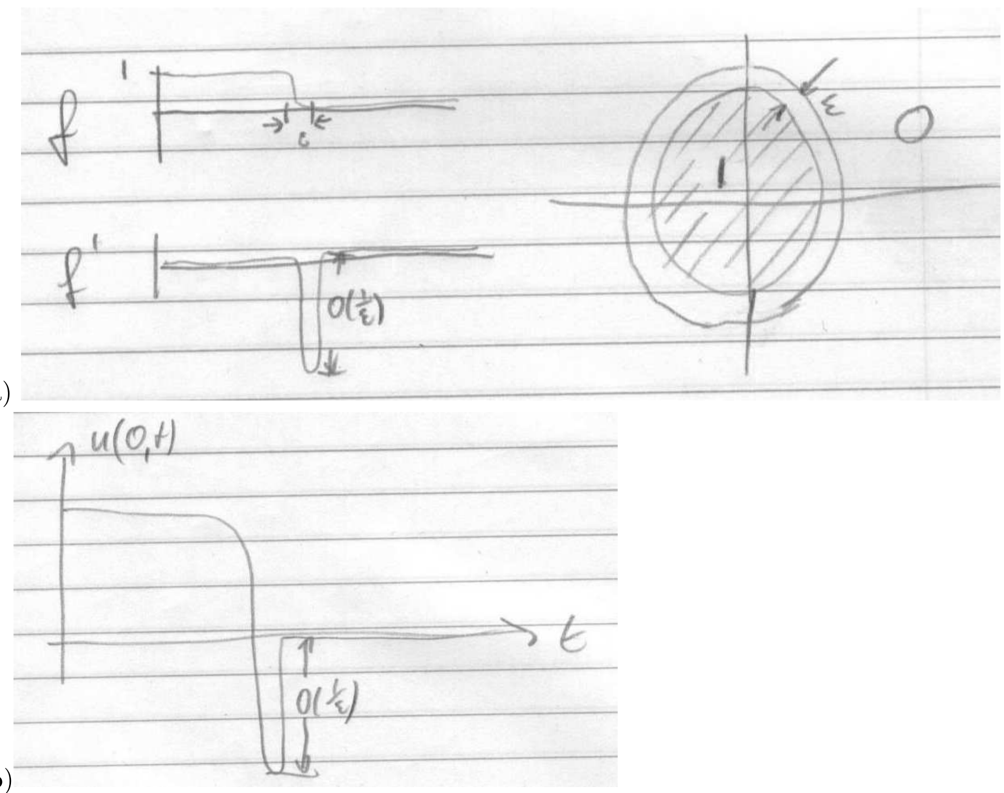


Figure 1.10. How radial data g spreads in time.

Focusing: Assume $g=0$, f radial.

$$\begin{aligned}
u(x,t) &= \partial_t(t\,M_f(x,ct)) = M_f(x,ct) + t\,\partial_t M_f(x,ct)\\
\partial_t M_f(x,ct) &= \partial_t\left(\fint_{|x-y|=ct} f(y)\,dS_y\right)\\
&= \partial_t\left(\fint_{|\omega|=1} f(x+ct\,\omega)\,d\omega\right)\\
&= \fint_{|\omega|=1} Df(x+ct\omega)\cdot(c\omega)\,d\omega\\
&= c\fint_{|\omega|=1}\frac{\partial f}{\partial n_\omega}(x+ct\omega)\,d\omega.
\end{aligned}$$



a) b)

Figure 1.11. a) Spread of data with radial f. b) The sharp dropoff in $u(0,t)$.

$$u(x,t) = \fint_{|x-y|=ct} f(y)\,dS_y + ct\fint_{|x-y|=ct}\frac{\partial f}{\partial n_y}\,dS_y.$$

Thus

$$\|u(x,t)\|_\infty \not\leqslant C\|u(x,0)\|_\infty.$$

More precisely, there exists a sequence $u_0^\varepsilon \in C^\infty(\mathbb{R}^n)$ and t_ε such that

$$\lim_{\varepsilon\downarrow 0}\frac{\sup_x |u^\varepsilon(x,t_\varepsilon)|}{\sup_x |u_0^\varepsilon(x)|} = +\infty.$$

Contrast with solution in $n=1$:

$$\|S(t)u_0\|_{L^p} \leqslant \|u_0\|_{L^p}, \quad 1\leqslant p\leqslant\infty,$$

where $S(t)$ is the shift operator. "=" solution to the wave equation.

Littman's Theorem $S_3(t)=$solution operator for wave equation in $\mathbb{R}^3$.

$$\sup_{f\in L^p(\mathbb{R}^3)}\frac{\|S_3(t)u_0\|_{L^p}}{\|u_0\|_{L^p}}=+\infty.$$

1.5 Hadamard's Method of Descent

Trick: Treat as 3-dimensional wave equation.

Notation: $x\in\mathbb{R}^2$, $\tilde{x}=(x,x_3)\in\mathbb{R}^3$. If $h:\mathbb{R}^2\to\mathbb{R}$, define $\tilde{h}:\mathbb{R}^3\to\mathbb{R}$ by $\tilde{h}(\tilde{x})=\tilde{h}((x,x_3)=h(x)$. Suppose u solves $\partial_t^2u-c^2\Delta_xu=0$ for $x\in\mathbb{R}^2$ and $t>0$ with $u(x,0)=f(x)$ and $u_t(x,t)=g(x)$. Then

$$\begin{aligned}\partial_t^2\tilde{u}-c^2\Delta_{\tilde{x}} &= 0\\ \tilde{u}(\tilde{x},0) &= \tilde{f}(x)\\ \tilde{u}_t(\tilde{x},0) &= \tilde{g}(x)\end{aligned}$$

for $\tilde{x}\in\mathbb{R}^3$, $t>0$.

$$\tilde{u}(\tilde{x},\tilde{t})=\partial_t(\tilde{K}_t*\tilde{f})+\tilde{K}_t*\tilde{g},$$

where

$$=$$

$$\begin{aligned}\tilde{K}_t*\tilde{h} &= t\fint_{|\tilde{x}-\tilde{y}|=ct}\tilde{h}(y)dS_y\\ &= t\fint_{|\tilde{\omega}|=1}\tilde{h}(x+ct\tilde{\omega})d\tilde{\omega}.\end{aligned}$$

with $\tilde{\omega}\in\mathbb{R}^3=(\omega,\omega_3)$ for $\omega\in\mathbb{R}^2$. Then

$$\tilde{h}(\tilde{x}+ct\tilde{\omega})=h(x+ct\omega).$$

$$\fint_{|\tilde{\omega}|=1}h(x+ct\omega\,d\tilde{\omega}.$$

$\tilde{\omega}=(\omega,\omega_3)$. On $|\tilde{\omega}|=1$, we have

$$\omega_3=\pm\sqrt{1-|\omega|^2}=\pm\sqrt{1-(\omega_1^2+\omega_2^2)}.$$

Then

$$\frac{\partial\omega_3}{\partial\omega_i}=\frac{-\omega_i}{\sqrt{1-|\omega|^2}}$$

for $i=1,2$. Thus the Jacobian is

$$\sqrt{1+\left(\frac{\partial\omega_3}{\partial\omega_1}\right)^2+\left(\frac{\partial\omega_3}{\partial\omega_2}\right)^2}=\frac{1}{\sqrt{1-|\omega|^2}}.$$

Thus

$$t\fint_{|\tilde{\omega}|=1}h(x+ct\omega)d\tilde{\omega}=\frac{2t}{4\pi}\int_{|\omega|\leqslant 1}\frac{h(x+ct\omega)}{\sqrt{1-|\omega|^2}}d\omega_1d\omega_2.$$

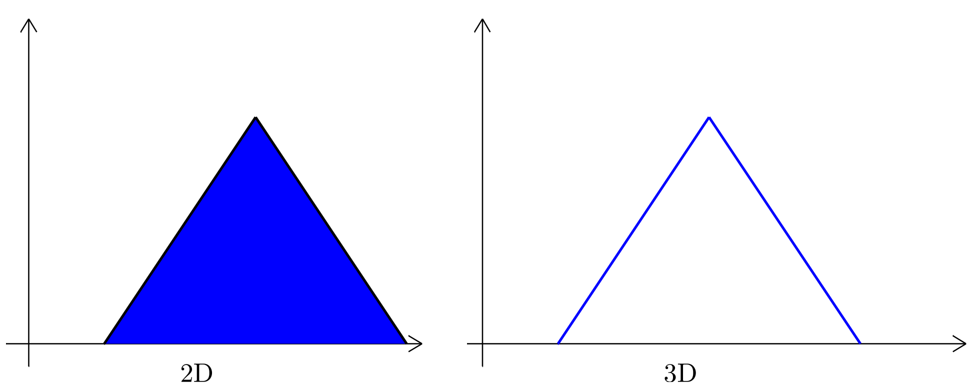


Figure 1.12. Domains of dependence, conceptually, for 2D and 3D.

1.6 Hadamard's Solution for all odd $n \geqslant 3$

[cf. Evans, 4.3?] $n = 2k+1$, $k \geqslant 1$. $k = (n-1)/2$, $c = 1$. The general formula is

$$u(x,t) = \partial_t(K_t * f) + K_t * g$$

where for any $h \in C_c^\infty$ we have

$$(K_t * h)(x) = \frac{\omega_n}{\pi^k 2^{k+1}} \left(\frac{1}{t} \cdot \frac{\partial}{\partial t}\right)^{(n-3)/2} \left[t^{n-2} \fint_{|x-y|=t} h(y) dS_y \right].$$

Check: If $n = 3$, $\omega_n = 4\pi$, so we get our usual formula.

Now, Consider $g \equiv 0$ in $u_{tt} - \Delta u = 0$, $x \in \mathbb{R}^{2k+1}$, $t > 0$, $u(x,0) = f(x)$, $u_t(x,0) = 0$. Extend u to $t < 0$ by $u(x,-t) = u(x,t)$ (which is OK because $\partial_t u = 0$ at $t = 0$)

Consider for $t > 0$

$$\begin{aligned} v(x,t) &:= \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-s^2/4t} u(x,s) ds \\ &= \int_{\mathbb{R}} k(s,t) u(x,s)\, ds \end{aligned}$$

Find solution for the heat equation in 1D. Use that $\partial_t k = \partial_s^2 k$.

$$\begin{aligned} \partial_t v &= \int_{\mathbb{R}} \partial_t k\, u(x,s) ds \\ &= \int_{\mathbb{R}} k(s,t) \partial_s^2 u(x,s) ds \\ &= \int_{\mathbb{R}} k(s,t) \Delta_x u(x,s) ds = \Delta_x \int_{\mathbb{R}} k(s,t) u(x,s) ds. \end{aligned}$$

$\partial_t v = \Delta_x v$, $x \in \mathbb{R}^n$, $t > 0$. Also, as $t \to 0$, $v(x,t) \to f(x)$. Therefore,

$$\begin{aligned} v(x,t) &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|y|^2/4t} f(x-y) dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_0^\infty e^{-r^2/4t} r^{n-1} \fint f(x - r\omega) \cdot d\omega\, dr \\ &= \frac{\omega_n}{(4\pi t)^{n/2}} \int_0^\infty e^{-r^2/4t} r^{n-1} M_f(x,r) dr \end{aligned}$$

Change variables using $\lambda = 1/4t$ and equate $(*)$ and $(\#)$ (what are $*$ and $\#$?)

$$\int_0^\infty e^{-\lambda r^2} u(x,r) dr = \frac{\omega_n}{2} \cdot \frac{1}{\pi^k} \int_0^\infty e^{-\lambda r^2} \lambda^k r^{n-1} M_f(x,r) dr.$$

Then, use the Laplace transform for $h \in L^1(\mathbb{R}_+)$:

$$h^{\#}(\lambda) = \int_0^\infty e^{-\lambda \varphi} h(p) d\varphi.$$

Basic fact: $h^{\#}$ is invertible. Observe that

$$\frac{d}{dr}(e^{-\lambda r^2}) = -\lambda e^{-\lambda r^2}.$$

In particular,

$$\left(-\frac{1}{2r} \cdot \frac{d}{dr}\right)^k e^{-\lambda r^2} = \lambda^k e^{-\lambda r^2}.$$

Therefore

$$\begin{aligned} \int_0^\infty \lambda^k e^{-\lambda r^2} r^{n-1} M_f(x,r) dr &= \frac{(-1)^k}{2^k} \int_0^\infty \left(\frac{1}{r} \cdot \frac{d}{dr}\right)^k e^{-\lambda r^2} (r^{2k} M_f(x,r)) dr \\ &= \frac{1}{2^k} \int_0^\infty e^{-\lambda r^2} \left[r \cdot \left(\frac{1}{r} \cdot \frac{d}{dr}\right)^k (r^{2k-1} M_f(x,r)) \right] dr. \end{aligned}$$

Now have Laplace transforms on both sides, use uniqueness of the Laplace transform to find

$$\begin{aligned} u(x,t) &= \frac{\omega_n}{\pi^k 2^{k+1}} t\left(\frac{1}{t}\cdot\frac{\partial}{\partial t}\right)^k \left[t^{n-2} M_f(x,t)\right] \\ &= \frac{\omega_n}{\pi^k 2^{k+1}} t\left(\frac{1}{t}\cdot\frac{\partial}{\partial t}\right)^{(n-3)/2} \left[t^{n-2} M_f(x,t)\right] \end{aligned}$$

2 Distributions

Let $U \subset \mathbb{R}^n$ be open.

Definition 2.1. *The set of* test functions $D(U)$ *is the set of* $C_c^\infty(U)$ *(C^∞ with compact support). The topology on this set is given by* $\varphi_k \to \varphi$ *in* $D(U)$ *iff*

a) *there is a fixed compact set* $F \subset U$ *such that* $\operatorname{supp} \varphi_k \subset F$ *for every* k

b) $\sup_F |\partial^\alpha \varphi_k - \partial^\alpha \varphi| \to 0$ *for every multi-index* α.

Definition 2.2. *A* distribution *is a continuous linear functional on* $D(U)$*. We write* $L \in D'(U)$ *and* (L, φ)*.*

Definition 2.3. *[Convergence on D'] A sequence* $L_k \xrightarrow{D'} L$ *iff* $(L_k, \varphi) \to (L, \varphi)$ *for every test function* φ*.*

Example 2.4. $L_{\mathrm{loc}}^p(U) := \{f: U \to \mathbb{R}: f \text{ measurable}, \int_{U'} |f|^p dx < \infty \forall U' \subset\subset U\}$.

An example of this is $U = \mathbb{R}$ and $f(x) = e^{x^2}$.

We associate to every $f \in L_{\mathrm{loc}}^p(U)$ a distribution L_f (here: $1 \leqslant p \leqslant \infty$).

$$(L_f, \varphi) := \int_U f(x)\varphi(x) dx.$$

Suppose $\varphi_k \xrightarrow{D} \varphi$. Need to check

$$(L_f, \varphi_k) \to (L_f, \varphi).$$

Since $\operatorname{supp}\varphi_k \subset F \subset\subset U$, we have

$$\begin{aligned} \left|(L_f, \varphi_k) - (L_f, \varphi)\right| &= \left|\int_F f(x)(\varphi_k - \varphi(x)) dx\right| \\ &\leqslant \underbrace{\left(\int_F |f(x)| dx\right)}_{\text{bounded}} \underbrace{\sup_F |\varphi_k - \varphi|}_{\to 0}. \end{aligned}$$

If $q > p$,

$$\int_F |f(x)|^p dx \leqslant \left(\int_F 1 dx\right)^{1-p/q} \left(\int_F |f(x)|^q\right)^{1/q}.$$

Thus, $L_{\mathrm{loc}}^q(U) \subset L_{\mathrm{loc}}^p(U)$ for every $p \leqslant q$. (*Note:* This is *not* true for $L^p(U)$.)

Example 2.5. If μ is a *Radon measure* on U, then we can define

$$(L_\mu, \varphi) = \int_U \varphi(x) \mu(dx).$$

Example 2.6. If $\mu = \delta_y$,

$$(L_\mu, \varphi) = \varphi(y).$$

Definition 2.7. *If L is a distribution, we define $\partial^\alpha L$ for every multi-index α by*

$$(\partial^\alpha L, \varphi) := (-1)^{|\alpha|} (L, \partial^\alpha \varphi).$$

This definition is motivated through integration by parts, noting that the boundary terms do not matter since we are on a bounded domain.

Example 2.8. If L is generated by δ_0,

$$(\partial^\alpha L, \varphi) = (-1)^\alpha \partial^\alpha \varphi(0).$$

Theorem 2.9. *$\partial^\alpha: D' \to D'$ is continuous. That is, if $L_k \xrightarrow{D} L$, then $\partial^\alpha L_k \xrightarrow{D} \partial^\alpha L$.*

Proof. Fix $\varphi \in D(U)$. Consider

$$\begin{array}{rcl} (\partial^\alpha L_k, \varphi) & \to & (\partial^\alpha L, \varphi) \\ \| & & \| \\ (-1)^\alpha (L_k, \partial^\alpha \varphi) & \to & (-1)^\alpha (L, \partial^\alpha \varphi). \end{array}$$

□

Definition 2.10. *Suppose P is a partial differential operator of order N, that is*

$$P = \sum_{|\alpha| \leqslant N} c_\alpha(x) \partial^\alpha$$

with $c_\alpha \in C^\infty(U)$.

Example 2.11. $P = \Delta$ is an operator of order 2. $P = \partial_t - \Delta$. $P = \partial_t^2 - c^2 \Delta$.

Fundamental solution for Δ:

$$\Delta K(x - y) = \delta_y \quad \text{in } D'.$$

All this means is for every $\varphi \in D$

$$\int_U \Delta K(x - y) \varphi(x) dx = \int_U \varphi(x) \delta_y(dx) = \varphi(y).$$

Definition 2.12. *We say that u solves $Pu = 0$ in D' iff*

$$\left(u, P^\dagger \varphi\right) = 0$$

for every test function φ. Here, $P^\dagger$ is the adjoint operator obtained through integration by parts: If $c_\alpha(x) = c_\alpha$ independent of x, then

$$P^\dagger = \sum_{|\alpha| \leqslant N} (-1)^{|\alpha|} c_\alpha \partial^\alpha.$$

Example 2.13. $P = \partial_t - D \Rightarrow P^\dagger = -\partial_t - \Delta$.

Example 2.14. More nontrivial examples of distributions:

1. *Cauchy Principal Value (PV) on $\mathbb{R}$:*

$$(L, v) := \lim_{\varepsilon \to 0} \int_{|x| \geqslant \varepsilon} \frac{\varphi(x)}{(x)} dx.$$

2. $U = (0, 1)$

$$(L, \varphi) = \sum_{k=1}^{\infty} \left(\frac{d^k}{dx^k} \varphi\right) \left(\frac{1}{k}\right),$$

which is well-defined because φ has compact support.

Uniform convergence in topology?

2.1 The Schwartz Class

Definition 2.15. $\mathcal{S}(\mathbb{R}^n)$ *Set* $\varphi \in C^\infty(\mathbb{R}^n)$ *with rapid decay:*

$$\|\varphi\|_{\alpha,\beta} := \sup_x |x^\alpha \partial^\beta(x)| < \infty$$

for all multiindices α, β. *Topology on this class:* $\varphi_k \to \varphi$ *on* $\mathcal{S}(\mathbb{R}^n)$ *iff* $\|\varphi_k - \varphi\|_{\alpha,\beta} \to 0$ *for all* α, β.

Example 2.16. If $\varphi \in D(\mathbb{R}^n)$ then $\varphi \in \mathcal{S}(\mathbb{R}^n)$. If $\varphi_k \to \varphi$ in $D(\mathbb{R}^n) \Rightarrow \varphi_k \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$.

Example 2.17. $\varphi(x) = e^{-|x|^2}$ is in $\mathcal{S}(\mathbb{R}^n)$, but not in $D(\mathbb{R}^n)$.

$$\partial^\beta \varphi(x) = \underbrace{P_\beta(x)}_{\text{Polynomial}} e^{-|x|^2},$$

so $\|x^\alpha \partial^\beta \varphi(x)\|_{L^\infty(\mathbb{R}^n)} < \infty$.

Example 2.18. $e^{-(1+|x|^2)^\varepsilon} \in \mathcal{S}(\mathbb{R}^n)$ for every $\varepsilon > 0$.

Example 2.19.
$$\frac{1}{(1+|x|^2)^N} \in C^\infty,$$

but not in $\mathcal{S}(\mathbb{R}^n)$ for any N. For example,

$$\sup_x \left| \frac{x^\alpha}{(1+|x|^2)^N} \right| = \infty$$

if $\alpha = (3N, 0, \ldots, 0)$.

We can define a *metric* on $\mathcal{S}(\mathbb{R}^n)$:

$$\rho(\varphi, \psi) = \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{|\alpha|+|\beta|=k} \frac{\|\varphi - \psi\|_{\alpha,\beta}}{1 + \|\varphi - \psi\|_{\alpha,\beta}}.$$

Claim: $\varphi_k \to \varphi$ in $\mathcal{S}(\mathbb{R}^n) \Leftrightarrow \rho(\varphi_k, \varphi) \to 0$.

Theorem 2.20. $\mathcal{S}(\mathbb{R}^n)$ *is a complete metric space.*

Proof. Arzelà-Ascoli. □

2.2 Fourier Transform

Motivation: For the wave equation, we find formally that

$$\mathcal{F}K_t = \frac{\sin c|\xi|t}{c|\xi|}.$$

Definition 2.21. *The Fourier transform on* $\mathcal{S}(\mathbb{R}^n)$ *is given by*

$$(\mathcal{F}\varphi)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x) dx.$$

For brevity, also let $\hat{\varphi}(\xi) = (\mathcal{F}\varphi)(\xi)$.

Theorem 2.22. $\mathcal{F}$ *is an isomorphism of* $\mathcal{S}(\mathbb{R}^n)$, *and* $\mathcal{F}\mathcal{F}^* = \mathrm{Id}$, *where*

$$(\mathcal{F}^*\varphi)(\xi) = (\mathcal{F}\varphi)(-\xi).$$

2.2.1 Basic Estimates

$$\begin{aligned} |\hat{\varphi}(\xi)| &\leqslant \frac{1}{(2\pi)^{n/2}} \int |\varphi(x)| \mathrm{d}x \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1+|x|)^{n+1} \frac{|\varphi(x)|}{(1+|x|)^{n+1}} \mathrm{d}x \\ &\leqslant C \|(1+|x|)^{n+1}\|_\infty < \infty. \end{aligned}$$

Also,

$$\begin{aligned}\partial_\xi^\beta \hat{\varphi}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \partial_\xi^\beta e^{-ix\cdot\beta} \varphi(x) \mathrm{d}x \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (-i\,x)^\beta e^{-ix\cdot\beta} \varphi(x) \mathrm{d}x \\ \Rightarrow \|\partial_\xi^\beta \hat{\varphi}(\xi)\|_{L^\infty} &\leqslant C\|(1+|x|)^{n+1} x^\beta \varphi\|_{L^\infty}.\end{aligned}$$

Thus show $\hat{\varphi} \in C^\infty(\mathbb{R}^n)$:

$$\begin{aligned}(-i\xi)^\alpha \hat{\varphi}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (-i\xi)^\alpha e^{-ix\cdot\xi} \varphi(x) \mathrm{d}x \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \partial_x^\alpha (e^{-ix\cdot\xi}) \varphi(x) \mathrm{d}x \\ &= \frac{(-1)^{|\alpha|}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \partial_x^\alpha \varphi(x) \mathrm{d}x \\ \Rightarrow \|\xi^\alpha \hat{\varphi}(\xi)\|_{L^\infty} &\leqslant C\|(1+|x|)^{n+1} \partial_x^\alpha \varphi\|_{L^\infty}.\end{aligned}$$

Combine both estimates to find

$$\|\hat{\varphi}\|_{\alpha,\beta} = \|\xi^\alpha \partial_\xi^\beta \hat{\varphi}\|_{L^\infty} \leqslant C\|(1+|x|)^{n+1} x^\beta \partial_x^\alpha \varphi\|_{L^\infty}.$$

Example 2.23. If $\varphi(x) = e^{-|x|^2/2}$. Then $\hat{\varphi}(\xi) = e^{-|\xi|^2/2}$. $\mathcal{F}\varphi = \varphi$.

2.2.2 Symmetries and the Fourier Transform

1. *Dilation:* $(\sigma_\lambda \varphi)(x) = \varphi(x/\lambda)$.

$$\mathcal{F}(\varphi(x/\lambda))(\xi) = \frac{\lambda^n}{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x/\lambda) \mathrm{d}(x/\lambda) = \lambda^n (\mathcal{F}\varphi)(\xi\lambda).$$

   Thus $\widehat{\sigma_\lambda \varphi} = \lambda^n \sigma_{1/\lambda} \hat{\varphi}$.

2. *Translation* $\tau_h \varphi(x) = \varphi(x-h)$ for $h \in \mathbb{R}^n$. $\mathcal{F}(\tau_h \varphi)(\xi) = e^{-ih\cdot\xi} \hat{\varphi}(\xi)$.

2.2.3 Inversion Formula

For every $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\varphi(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{\varphi}(\xi) \mathrm{d}\xi.$$

$\varphi(x) = \mathcal{F}^* \hat{\varphi} = (\mathcal{R}\mathcal{F})\hat{\varphi}$, where $(\mathcal{R}\varphi)(x) = \varphi(-x)$.

Proof. (of Schwartz's Theorem) Show $\mathcal{F}^*\mathcal{F} e^{-|x|^2/2} = e^{-|x|^2/2}$.

Extend to dilations and translations. Thus find $\mathcal{F}^*\mathcal{F} = \mathrm{Id}$ on $\mathcal{S}$, because it is so on a dense subset. $\mathcal{F}$ is 1-1, $\mathcal{F}^*$ is onto $\Rightarrow$ but $\mathcal{F}^* = \mathcal{R}\mathcal{F}$, so the claim is proven. □

Theorem 2.24. *$\mathcal{F}$ defines a continuous linear operator from $L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$, with*

$$\|\hat{f}\|_{L^\infty} \leqslant \frac{1}{(2\pi)^n} \|f\|_{L^1}.$$

Theorem 2.25. *$\mathcal{F}$ defines an isometry of $L^2(\mathbb{R}^n)$.*

Theorem 2.26. *$\mathcal{F}$ defines a continuous linear operator from $L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ with $1 \leqslant p \leqslant 2$ and*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Ideas:

- Show $\mathcal{S}(\mathbb{R}^n)$ dense in $L^p(\mathbb{R}^n)$ with $1 \leqslant p < \infty$.
- Extend $\mathcal{F}$ from $\mathcal{S}$ to L^p.

Proposition 2.27. *$C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Take a function

$$\eta_N(x) := \begin{cases} 1 & |x| \leqslant N-1, \\ 0 & |x| \geqslant N+1. \end{cases}$$

Given $\varphi \in \mathcal{S}(\mathbb{R}^n)$, consider $\varphi_N := \varphi \eta_N$.

$$\partial^\alpha \varphi_n = \partial^\alpha(\varphi \eta_N) = \sum_{|\alpha'| \leqslant |\alpha|} \partial^{\alpha'} \varphi \partial^{\alpha - \alpha'} \eta_N.$$

So $\|x^\beta \partial^\alpha \varphi_N\|_{L^\infty} < \infty$. □

Theorem 2.28. *$C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \leqslant p < \infty$.*

Proof. *By Mollification.* Choose $\eta \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\eta) \subset B(0,1)$ and

$$\int_{\mathbb{R}^n} \eta(x) \mathrm{d}x = 1.$$

For any n, define $\eta_N(x) = N^n \eta(Nx)$. Then

$$\int_{\mathbb{R}^n} \eta_N(x) \mathrm{d}x = 1.$$

To show:

$$f * \eta_N \overset{L^p}{\rightarrow} f$$

for any $f \in L^p(\mathbb{R}^n)$.

Step 1: Suppose $f(x) = \mathbf{1}_Q(x)$ for a rectangle Q. In this case, we know $\eta_N * f = f$ at any x with $\operatorname{dist}(x, \partial Q) \geqslant 1/N$. Therefore, $\eta_N * f \rightarrow f$ a.e. as $N \rightarrow \infty$.

$$\int_{\mathbb{R}^n} |\eta_N * f(x) - f(x)|^p \mathrm{d}x \rightarrow 0$$

by Dominated Convergence.

(*Aside: Density of C_c^∞ in $\mathcal{S}(\mathbb{R}^n)$*. (Relation to Proposition 2.27?) Given $\varphi \in \mathcal{S}(\mathbb{R}^n)$, consider $\varphi_N := \varphi \eta_N$. We have $\|\varphi_N - \varphi\|_{\alpha,\beta} \rightarrow 0$ for every α, β. In particular, we have

$$\|(|x|^{n+1} + 1)(\varphi_n - \varphi)\|_{L^\infty} \rightarrow 0.$$

$$\int_{\mathbb{R}^n} |\varphi_n - \varphi| \mathrm{d}x = \int_{\mathbb{R}} \frac{1 + |x|^{n+1}}{(1+x)^{n+1}} |\varphi_n - \varphi| \mathrm{d}x \leqslant \left(\int_{\mathbb{R}} \frac{1}{1 + |x|^{n+1}} \mathrm{d}x \right) \ldots ?$$

End aside.)

Step 2: Step functions are dense in $L^p(\mathbb{R}^n)$ for $1 \leqslant p < \infty$.

Step 3: "Maximal inequality", i.e.

$$\|f * \eta_N\|_{L^p} \leqslant C \|f\| L^p,$$

which we obtain by Young's inequality.

$$\begin{aligned} \|f * \eta_N\|_{L^p} &\leqslant C_p \|\eta_N\|_{L^1} \|f\|_{L^p} \\ &= C_p \|\eta\|_{L^1} \|f\|_{L^p}, \end{aligned}$$

where the constant depends on η, but not on N.

Step 4: Suppose $f \in L^p(\mathbb{R}^n)$. Pick g to be a step function such that $\|f - g\|_{L^p} < \varepsilon$ for $1 \leqslant p < \infty$. Then

$$\begin{aligned} \|f * \eta_N - f\|_{L^p} &\leqslant \|f * \eta_N - g * \eta_N\|_{L^p} + \|g * \eta_N - g\|_{L^p} + \|f - g\|_{L^p} \\ &\leqslant (C_p \|\eta\|_{L^1} + 1) \|f - g\|_{L^p} + \|g * \eta_N - g\|_{L^p}. \end{aligned}$$

□

Onwards to prove the L^2 isometry, we define

$$(f,g)_{L^2(\mathbb{R}^n)} := \int_{\mathbb{R}^n} f(x)\overline{g(x)}\mathrm{d}x.$$

Proposition 2.29. (Plancherel) *Suppose $f,g\in\mathcal{S}(\mathbb{R}^n)$. Then*

$$(\mathcal{F}f,\mathcal{F}g)_{L^2(\mathbb{R}^n)} = (f,g)_{L^2(\mathbb{R}^n)}.$$

Proof.

$$\begin{aligned}(\mathcal{F}f,\mathcal{F}g)_{L^2(\mathbb{R}^n)} &\overset{\text{Definition}}{=} \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} \bar{g}(x)\left(\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\bar{f}(\xi)\mathrm{d}\xi\right)\mathrm{d}x \\ &= \int_{\mathbb{R}^n} f(x)\bar{g}(x)dx.\end{aligned}$$

□

Definition 2.30. *$\mathcal{F}: L^1(\mathbb{R}^n)\to\dot{C}(\mathbb{R}^n)$ is the extension of $\mathcal{F}:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, where*

$$\dot{C}(\mathbb{R}^n) := \{h:\mathbb{R}^n\to\mathbb{R} \text{ such that } h(x)\to 0 \text{ as } |x|\to\infty\}.$$

Proposition 2.31. *This extension is well-defined.*

Proof. Suppose

$$\begin{aligned}\varphi_k &\overset{L^1}{\to} f,\\ \psi_k &\overset{L^1}{\to} f.\end{aligned}$$

Then $\|\mathcal{F}\varphi_k - \mathcal{F}\psi_k\|\to 0$:

$$\begin{aligned}|(\hat{\varphi}_k-\hat{\psi}_k)(\xi)| &= \frac{1}{(2\pi)^{n/2}}\left|\int_{\mathbb{R}^n} e^{-ix\cdot\xi}(\varphi_k-\psi_k)\right| \\ &\leqslant \frac{1}{(2\pi)^{n/2}}\|\varphi_k-\psi_k\|_{L^1} \\ &\leqslant \frac{1}{(2\pi)^{n/2}}[\|\varphi_k - f\|_{L^1} + \|f-\psi_k\|_{L^1}]\to 0.\end{aligned}$$

□

Warning: There is something to be proved for $L^2(\mathbb{R}^n)$ because

$$\frac{1}{(2\pi)^{n/2}}\int e^{-ix\cdot\xi}f(x)\mathrm{d}x$$

is *not* defined when $f\in L^2(\mathbb{R}^n)$. However $\mathcal{F}f$ in the sense of L^2-lim $\mathcal{F}\varphi_N$ where $\varphi_N\in\mathcal{S}(\mathbb{R}^n)\to f$ in L^2.

We had proven

$$\begin{aligned}\left\|\hat{f}\right\|_{L^\infty} &\leqslant \frac{1}{(2\pi)^{n/2}}\|f\|_{L^1},\\ \left\|\hat{f}\right\|_{L^2} &= \|f\|_{L^2}.\end{aligned}$$

Definition 2.32. *A linear operator $K:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ is of* type (r,s) *if*

$$\|K\varphi\|_{L^s}\leqslant C(r,s)\|\varphi\|_{L^r}.$$

Example 2.33. $\mathcal{F}$ is of type $(1,\infty)$ and $(2,2)$.

Theorem 2.34. (Riesz-Thorin Convexity Theorem) *Suppose K is of type (r_i, s_i) for $i = 0, 1$. Then K is of type (r, s) where*

$$\begin{aligned}\frac{1}{r} &= \frac{\theta}{r_0} + \frac{1-\theta}{r_1},\\ \frac{1}{s} &= \frac{\theta}{s_0} + \frac{1-\theta}{s_1}\end{aligned}$$

for $0 \leqslant \theta \leqslant 1$. Moreover,

$$C(r, s) \leqslant C_0^{\theta} C_1^{1-\theta}.$$

Proof. Yosida/Hadamard's 3-circle theorem (maximum principle). □

Corollary 2.35. *$\mathcal{F}: \mathcal{S} \to \mathcal{S}$ has a unique extension $\mathcal{F}: L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ where $1 \leqslant p \leqslant 2$ and $1/p' + 1/p = 1$.*

Summary:

- $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ isomorphism
- $\mathcal{F}: L^1 \to \dot{C}$ (either by extension or directly) not an isomorphism
- $\mathcal{F}: L^2 \to L^2$ (by extension) isomorphism
- $\mathcal{F}: L^p \to L^{p'}$ (by interpolation)

Definition 2.36. *$\mathcal{S}'(\mathbb{R}^n)$ is the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$, called the space of* tempered distributions. *Its topology is given by $L_k \to L$ in $\mathcal{S}'$ iff*

$$(L_k, \varphi) \to (L, \varphi)$$

for all $\varphi \in \mathcal{S}$.

Altogether, we have $D \subset \mathcal{S} \subset \mathcal{S}' \subset D'$.

Example 2.37.

1. Suppose $f \in L^1$. Define a tempered distribution

$$(f, \varphi) := \int_{\mathbb{R}^n} f\varphi,$$

which is obviously continuous.

2. (A *non*-example) If $f(x) = e^{|x|^2}$, then $f \in L^1_{\mathrm{loc}}$, so it defines a distribution, but not a *tempered* distribution.
3. $f(x) = e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$, but

$$\int_{\mathbb{R}^n} f\varphi = \infty.$$

4. If f is such that

$$\left\|(1 + |x|^2)^{-M} f\right\|_{L^1} < \infty$$

for some M, then $f \in \mathcal{S}'$.

Proof.
$$|(f, \varphi)| = \left|\int f\varphi\right| \leqslant \left\|(1 + |x|^2)^{-M} f\right\|_{L^1} \left\|(1 + |x|^2)^M \varphi\right\|_{L^\infty}.$$
□

Proposition 2.38. *Suppose $L \in \mathcal{S}'$. Then there exists $C > 0$, $N \in \mathbb{N}$ such that*

$$|(L, \varphi)| \leqslant C \|\varphi\|_N \tag{2.1}$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$, where

$$\|\varphi\|_N = \sum_{|\alpha|, |\beta| \leqslant N} \left\|x^{\alpha} \partial^{\beta} \varphi\right\|_{L^\infty}.$$

Corollary 2.39. *A distribution $L \in D'$ defines a tempered distribution $\Leftrightarrow$ there exist c, N such that (2.1) holds for $\varphi \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. Suppose (2.1) is not true. Then there exist φ_k, N_k such that

$$|(L, \varphi_k)| > k \|\varphi_k\|_{N_k}.$$

Let

$$\psi_k := \frac{\varphi_k}{\|\varphi_k\|_{N_k}} \cdot \frac{1}{k}.$$

Then

$$\|\psi_k\|_{N_k} = \frac{1}{k} \to 0.$$

But $|(L, \psi_k)| > 1$. But $\psi_k \to 0$ in $\mathcal{S}(\mathbb{R}^n) \Rightarrow L$ not continuous. □

Definition 2.40. *If $K: \mathcal{S} \to \mathcal{S}$ is linear, continuous, then the* transpose *of K is the linear operator such that for every $L \in \mathcal{S}'$*

$$(L, K\varphi) = \left(K^t L, \varphi\right).$$

Theorem 2.41. *a) $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{S}'(\mathbb{R}^n)$.*

b) $D(\mathbb{R}^n)$ is dense in $D'(\mathbb{R}^n)$.

Proof. Mollification, but first verify some properties. Fix $\eta \in D(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \eta = 1.$$

Let $\eta_m(x) = m^n \eta(m\,x)$. We want to say $\eta_m * L$ is a C^∞ function for a distribution L.

Definition 2.42. *$L \in D'(\mathbb{R}^n)$, $\eta \in D(\mathbb{R}^n)$, $\eta * L$ is the distribution defined by*

$$(\eta * L, \varphi) = (L, (R\eta) * \varphi),$$

where $R\eta(x) = \eta(-x)$. If L were a function f,

$$\begin{aligned}
(\eta * L, \varphi) &= \int_{\mathbb{R}^n} (\eta * f)(x)\varphi(x)\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \eta(x-y) f(y)\mathrm{d}y\varphi(x)\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \eta(x-y)\varphi(x)\mathrm{d}x \right) f(y) \\
&= \int_{\mathbb{R}^n} (R\eta * \varphi)(y) f(y)\mathrm{d}y.
\end{aligned}$$

Theorem 2.43. *$D(\mathbb{R}^n)$ is dense in $D'(\mathbb{R}^n)$. That is, if f is a distribution, then there exists a sequence of $L_k \in D$ such that $L_k \to L$ in D'.*

Proof. By 1) Mollification and 2) Truncation.

Proposition 2.44. *$L * \eta$ is a C^∞ function. More precisely, $L * \eta$ is equivalent to the distribution defined by the C^∞ function*

$$\gamma(x) = (L, \tau_x(R\eta)),$$

where $\tau_x f(y) = f(y - x)$.

Proof. 1) $\gamma: \mathbb{R}^n \to \mathbb{R}$ is clear.

2) γ is continuous: If $x_k \to x$, then $\gamma(x_k) \to \gamma(x)$. Check

$$\gamma(x_k) = (L, \tau_{x_k}(R\eta)).$$

And $\tau_{x_k}(R\eta) \to \tau_x(R\eta)$ *in* D.

- We can choose F s.t. $\operatorname{supp}(\tau_{x_k}(R\eta)) \subset F$ for all k.
- $R\eta(y - x_k) \to R\eta(y - x)$,
- $\partial^\alpha(R\eta)(y - x_k) \to \partial^\alpha R\eta(y - x)$,

where the last two properties hold uniformly on F.

3) $\gamma \in C^1$: Use finite differences. Consider

$$\frac{\gamma(x + h\, e_j) - \gamma(x)}{h} = \left(L, \frac{\tau_{x + h e_j}(R\eta) - \tau_x(R\eta)}{h} \right).$$

Observe that

$$\frac{1}{h}\left[\tau_{x + h e_j}(R\eta) - \tau_x(R\eta) \right] \to \tau_x(\partial_{x_j} R\eta)$$

in D.

4) $\gamma \in C^\infty$: Induction.

5) Show that $L * \eta \overset{D'}{=} \gamma$. That is

$$(L * \eta, \varphi) \overset{\text{Def}}{=} (L, R\eta * \varphi) \overset{?}{=} \int_{\mathbb{R}^n} \gamma(x)\varphi(x)\mathrm{d}x.$$

$$\begin{aligned}
\int_{\mathbb{R}^n} \gamma(x)\varphi(x)\mathrm{d}x &= \lim_{h \to 0} h^{-n} \sum_{y \in h\mathbb{Z}^n} \gamma(y)\varphi(y) \\
&= \lim_{h \to 0} h^{-n} \sum_{y \in h\mathbb{Z}^n} (L, \tau_y(R\eta))\varphi(y) \\
&= \lim_{h \to 0} \left(L, h^{-n} \sum_{y \in h\mathbb{Z}^n} \tau_y(R\eta)\varphi(y) \right).
\end{aligned}$$

Show that the Riemann sum

$$h^{-n} \sum_{y \in h\mathbb{Z}^n} \tau_y(R\eta)\varphi(y) \to R\eta * \varphi$$

in D. □

Operations with $*$*:*

1. $\eta * L := L * \eta$.
2. $\partial^\alpha(L * \eta) \overset{D'}{=} \partial^\alpha L * \eta \overset{D'}{=} L * \partial^\alpha \eta$.

Proof of Theorem: Fix $\eta \in D(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \eta(x)\mathrm{d}x = 1$. Let $\eta_m(x) = m^n \eta(m\, x)$. Then

$$\int_{\mathbb{R}^n} \eta_m(x)\, \mathrm{d}x = 1.$$

We know from our proposition from that $\eta_m * L$ is C^∞. Consider the cutoff function

$$\chi_m(x) := \begin{cases} 1 & |x| \leqslant m, \\ 0 & |x| > m. \end{cases}$$

Consider $L_m = \chi_m(\eta_m * L)$. $L_m \in D(\mathbb{R}^n)$.

$$\partial^\alpha(\chi_m \gamma_m) = \sum \binom{\alpha}{\beta} \partial^{\alpha - \beta} \chi_m \partial^{\alpha - \beta} \gamma_m$$

Claim: $L_m \to L$ in D'.

$$\begin{aligned}
(L_m, \varphi) &= (\chi_m(\eta_m * L), \varphi) = (\eta_m * L, \chi_m \varphi) \\
&\overset{\text{Def}}{=} (L, (R\eta_m) * (\chi_m \varphi)).
\end{aligned}$$

Finally, show

$$(R\eta_m) * \varphi \quad \overset{m \text{ large}}{=} \quad (R\eta_m) * \chi_m \varphi \quad \to \quad \varphi \quad \text{in } D'.$$

□ □

Definition 2.45. *Suppose $K: \mathcal{S} \to \mathcal{S}$ is linear. We define $K^t: \mathcal{S}' \to \mathcal{S}'$ as the linear operator*

$$\big(K^t L, \varphi\big) := (L, K\varphi).$$

Proposition 2.46. *Suppose $K: \mathcal{S} \to \mathcal{S}$ is linear and continuous. Suppose that $K_t|_S$ is continuous. Then, there exists a unique, continuous extension of K^t to $\mathcal{S}'$.*

Corollary 2.47. *$\mathcal{F}: \mathcal{S}' \to \mathcal{S}'$ is continuous.*

Let's go back to PDE now. Examples:

1. $\mathcal{F}\delta = 1/(2\pi)^{n/2}$.
2. Let $0 < \beta < n$ and $C_\beta = \Gamma((n-\beta)/2)$. Then $\mathcal{F}(C_\beta |x|^{-\beta}) = C_{n-\beta}|x|^{-(n-\beta)}$. Why we care: $\Delta u = \delta_0$. In Fourier space:

$$\begin{aligned} -|\xi|^2 \hat{u} &= \frac{1}{(2\pi)^{n/2}} \\ \Rightarrow \hat{u} &= \frac{-1}{(2\pi)^{n/2}} |\xi|^{-2}. \\ \Rightarrow \mathcal{F}^{-1}\hat{u} &= \frac{-1}{(2\pi)^{n/2}} \frac{C_{n-2}}{C_2} |x|^{2-n}. \end{aligned}$$

Prove (1) and (2) by testing against Gaussians.

2.3 Duhamel's Principle

Consider constant coefficient linear PDE

$$\partial_t^m u + \partial_t^{m-1}\Bigg(\sum_{|\alpha| \overset{?}{=} 1} c_{1,\alpha}\partial^\alpha\Bigg)u + \partial_t^{m-2}\Bigg(\sum_{|\alpha| \overset{?}{=} 2} c_{2,\alpha}\partial^\alpha\Bigg)u + \cdots + \sum_{|\alpha| \leqslant m} c_{m,\alpha}\partial^\alpha u = 0.$$

Here $u: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, m is the order of the equation, $c_{m,\alpha} \in \mathbb{R}$.

Shorthand $P(D,\tau)u = 0$. Herre $D = (\partial_{\alpha_1}, \ldots, \partial_{\alpha_n})$ and $\tau = \partial_t$. Differentiation operators

$$P(D,\tau) = \tau^m + \tau^{m-1}P_1(D) + \cdots + P_m(D).$$

$P_k(D) =$ polynomial in D of order $\leqslant k$.

General Problem:

$$P(D,\tau)u = \omega$$

for $x \in \mathbb{R}^n$, $t > 0$ with

$$\begin{aligned} u &= f_0 \\ \partial_t u = \tau u &= f_1 \\ \vdots \quad & \quad \vdots \\ \partial_t^{m-1} u = \tau^{m-1} u &= f_{m-1} \end{aligned}$$

at $t = 0$.

Standard Problem:

$$P(D,\tau)u = 0$$

with

$$\begin{aligned} u &= 0 \\ \partial_t u = \tau u &= 0 \\ \vdots \quad & \quad \vdots \\ \partial_t^{m-1} u = \tau^{m-1} u &= g \end{aligned}$$

at $t=0$. (Initial conditions). Solution of General Problem from Standard Problem. First, suppose $\omega \neq 0$ and $f_0 = f_1 = \cdots = f_{m-1} = 0$.

Consider the solution to a family of standard problems:

$$\begin{aligned} P(D,\tau)U(x,t,s) &= 0 \quad (s \leqslant t) \\ \tau^{m-1}U(x,t,s) &= \omega(x,s) \quad (t=s) \\ \tau^k U(x,t,,s) &= 0 \quad (t=s, 0 \leqslant k \leqslant m-2) \end{aligned}$$

Consider

$$u(x,t) = \int_0^t U(x,t,s)\mathrm{d}s.$$

This gives us

$$\begin{aligned} P(D,z)u(x,t) &= \int_0^t P(D,\tau)U(x,t,s)\mathrm{d}s + (\tau^{m-1} + \tau^{m-2}P_1(D) + \cdots + P_{n-1}(D))U(x,t,t) \\ &= 0 + \omega(x,t) + 0 \end{aligned}$$

as desired. Similarly, getting rid of non-standard initial conditions involves consideration of

$$\begin{aligned} P(D,\tau) &= 0 \\ u &= f_0 \\ \tau u &= f_1 \\ \vdots & \quad \vdots \\ \tau^{m-1}u &= f_{m-1} \end{aligned}$$

Let u_g dentote the solution to the standard problem. Consider

$u = u_{f_{m-1}} + (\tau + P_1(D))u_{f_{m-2}} + (\tau^2 + P_1(D)\tau + P_2(D))u_{f_{m-3}} + \cdots + (\tau^{m-1} + P_1(D)\tau^{m-2} + \cdots + P_{m-1}(D))u_{f_0}$.

Then

$$\begin{aligned} P(D,\tau)u &= P(D,\tau)u_{f_{m-1}} + (\tau + P_1(D))P(D,\tau)u_{f_{m-2}} + \cdots \\ &= 0 \end{aligned}$$

since $P(D,\tau)u_{f_k} = 0$ for $0 \leqslant k \leqslant m-1$. We need to check the initial conditions: At $t=0$, $\tau^l u_{f_k} = 0$, $0 \leqslant l \leqslant m-2$. Thus, all terms except the last one are 0. The last term is

$$\left[\tau^{m-1} + P_1(D)\tau^{m-2} + \cdots + P_{n-1}(D)\right]u_{f_0} = \tau^{m-1}u_{f_0} + \text{time derivatives of order } \leqslant m-2\ (=0) = f_0.$$

Henceforth, only consider the standard problem

$$\begin{aligned} P(D,\tau) &= 0, \\ \tau^k u(x,0) &= 0 \quad (0 \leqslant k \leqslant m-2), \\ \tau^{m-1}u(x,0) &= g. \end{aligned}$$

Solve by Fourier analysis:

$$\hat{u}(\xi,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x,t)\mathrm{d}x.$$

Fourier transform of the above standard problem yields

$$\begin{aligned} P(i\xi,\tau)\hat{u} &= 0, \\ \tau^k \hat{u}(\xi,\tau) &= 0, \\ \tau^{m-1}\hat{u}(\xi,0) &= \hat{g}(\xi) \end{aligned}$$

Fix ξ and suppose $Z(\xi,t)$ denotes the solution t_0 to the ODE

$$P(i\xi,\tau)Z(\xi,t) = 0$$

with initial conditions

$$\tau^k Z(\xi,0)=0 \quad (0\leqslant k\leqslant m-1), \quad \tau^{m-1}Z(\xi,0)=1.$$

This is a constant coefficients ODE, an analytic solution for it exists for all t. Clearly, by linearity

$$\hat{u}(\xi,t)=Z(\xi,t)\hat{g}(\xi)$$

and

$$u(x,t)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}Z(\xi,t)\hat{g}(\xi)\mathrm{d}\xi.$$

We want $u\in C^m$ ("classical solution"). Problem: Need to show that $Z(\xi,t)$ does not grow too fast (=faster than a polynomial) in ξ. Formally,

$$\partial^\alpha\tau^k u(x,t) \;=\; \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}(i\xi)^\alpha\tau^k Z(\xi,t)\hat{g}(\xi)\mathrm{d}\xi.$$

Key estimate: For any $T>0$, there exists C_T, N such that

$$\max_{0\leqslant k\leqslant m}\sup_{0\leqslant\tau\leqslant T}\sup_{\xi\in\mathbb{R}^n}\left|\tau^k Z(\xi,t)\right|\leqslant C_T(1+|\xi|)^N$$

Definition 2.48. *The above standard problem is called* hyperbolic *if there exists a C^m solution for every $g\in\mathcal{S}(\mathbb{R}^n)$.*

Theorem 2.49. (Gårding's criterion) *The problem is hyperbolic iff $\exists c\in\mathbb{R}$ such that $P(i\xi,\lambda)\neq 0$ for all $\xi\in\mathbb{R}^n$ and λ with $\mathrm{Im}(\lambda)\leqslant -c$.*

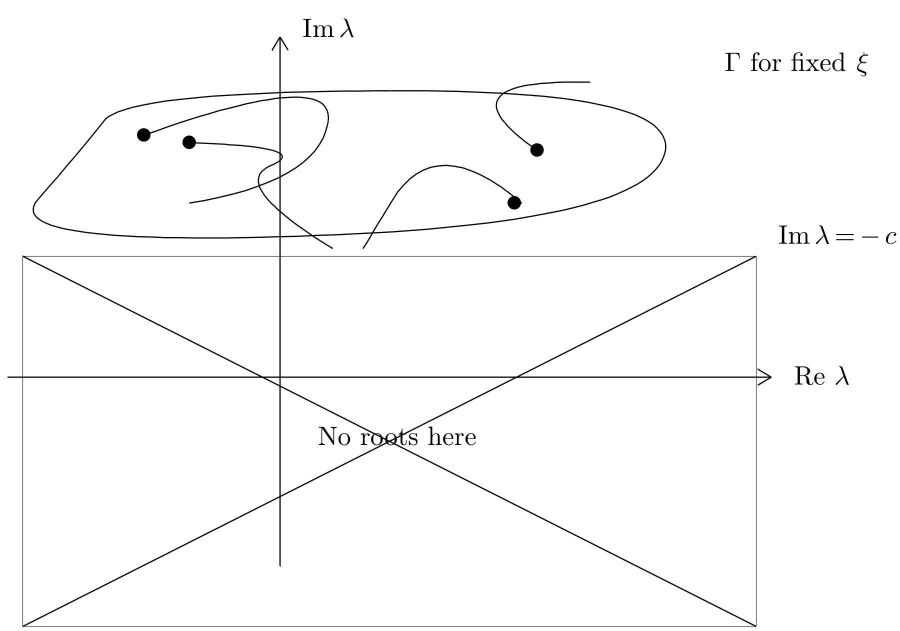


Figure 2.1. Nice cartoon.

Proof. *Cartoon:* Typical solutions to $P(i\xi,\tau)Z=0$ are of the form $Z=e^{i\lambda t}$ with $P(i\xi,i\lambda)=0$. We will only prove "$\Leftarrow$": We'll prove the estimate

$$\max_{0\leqslant k\leqslant m}\sup_{0\leqslant\tau\leqslant T}\sup_{\xi\in\mathbb{R}^n}|Z(\xi,t)|\leqslant C_T(1+|\xi|)^N$$

assuming $P(i\xi, i\lambda) \neq 0$ for $\operatorname{Im}(\lambda) \geqslant -c$. Formula for $Z(\xi, t)$:

$$Z(\xi, t) = \frac{1}{2\pi} \int_\Gamma \frac{e^{i\lambda t}}{P(i\xi, i\lambda)} \mathrm{d}\lambda.$$

Claim: $P(i\xi, \tau) Z = 0$ $(t > 0)$, $\tau^k Z = 0$ $(0 \leqslant k \leqslant m-2,\ t = 0)$, $\tau^{m-1} Z = 1$ $(t = 0)$.

$$\tau^k Z = \frac{1}{2\pi} \int_\Gamma \frac{(i\lambda)^k e^{i\lambda t}}{P(i\xi, i\lambda)} \mathrm{d}\lambda.$$

Therefore

$$\begin{aligned} P(i\xi, \tau) Z &= \frac{1}{2\pi} \int_\Gamma P(i\xi, i\lambda) \frac{e^{i\lambda t}}{P(i\xi, i\lambda)} \mathrm{d}\lambda \\ &= \frac{1}{2\pi} \int_\Gamma e^{i\lambda t} \mathrm{d}\lambda = 0 \end{aligned}$$

by Cauchy's Theorem. Suppose $0 \leqslant k \leqslant m-2$. Let $t = 0 \Rightarrow e^{i\lambda t} = 1$.

$$\tau^k Z = \frac{1}{2\pi} \int_\Gamma \frac{(i\lambda)^k}{(i\lambda)^n \left(1 + o\left(\frac{1}{|\lambda|}\right)\right)} \mathrm{d}\lambda.$$

Suppose that Γ is the circle of radius $R \gg 1$ with center at 0. Then

$$|\tau^k Z| \leqslant \frac{1}{2\pi} \frac{R^k}{R^n \left(1 + o\left(\frac{1}{R}\right)\right)} \cdot 2\pi R = R^{k-(m-1)} \left(1 + o\left(\frac{1}{R}\right)\right) \to 0$$

if $k \leqslant m-2$. Thus, $\boxed{\tau^k Z = 0}$ for any Γ enclosing all roots.

When $k = m-1$, we have

$$\tau^{m-1} Z = \frac{1}{2\pi i} \int_\Gamma \frac{1}{\lambda \left(1 + \underbrace{o\left(\frac{1}{\lambda}\right)}_{\text{analytic}}\right)} \mathrm{d}\lambda = 1.$$

Step 2) Claim: Any root of $P(i\xi, i\lambda)$ staisfies

$$|\lambda(\xi)| \leqslant M(1 + |\xi|).$$

Estimate growth of roots: Suppose λ solves $P(i\xi, i\lambda) = 0$. Then

$$(i\lambda)^n + (i\lambda)^{n-1} P_1(i\xi) + \cdots + P_m(i\xi) = 0.$$

Thus,

$$-(i\lambda)^m = (i\lambda)^{m-1} P_1(i\xi) + (i\lambda)^{m-2} P_2(i\xi) + \cdots + P_m(i\xi).$$

Observe that

$$|P_k(i\xi)| \leqslant C_k (1 + |\xi|)^k \tag{2.2}$$

for every k, $1 \leqslant k \leqslant m$. Therefore,

$$|\lambda|^m \leqslant C \sum_{k=1}^m |\lambda|^{m-k} (1 + |\xi|)^k.$$

Claim: this implies:

$$|\lambda| \leqslant (1 + C)(1 + |\xi|).$$

Let

$$\theta = \frac{|\lambda|}{1 + |\xi|}.$$

Then (2.2) implies

$$\theta^m \leqslant C \sum_{k=1}^m \theta^k \Rightarrow \theta^m \leqslant \frac{\theta^m - 1}{\theta - 1} \quad (\theta \neq 1).$$

Cases:

- $\theta \leqslant 1 \Leftrightarrow |\lambda| \leqslant 1 + |\xi| \Rightarrow$ nothing to prove.
- $\theta > 1 \Rightarrow \theta^m \leqslant C\theta^m/(\theta - 1) \Rightarrow \theta \leqslant 1 + C \Rightarrow |\lambda| \leqslant (1 + C)(1 + |\xi|)$.

Step 3. Claim:

$$|\tau^k Z(\xi, t)| \leqslant M m\, e^{(1+c)t}(1 + |\xi|)^k.$$

Here M=bound from step 2, m=order of $P(D, \tau)$, c=constant in Gårding's criterion.

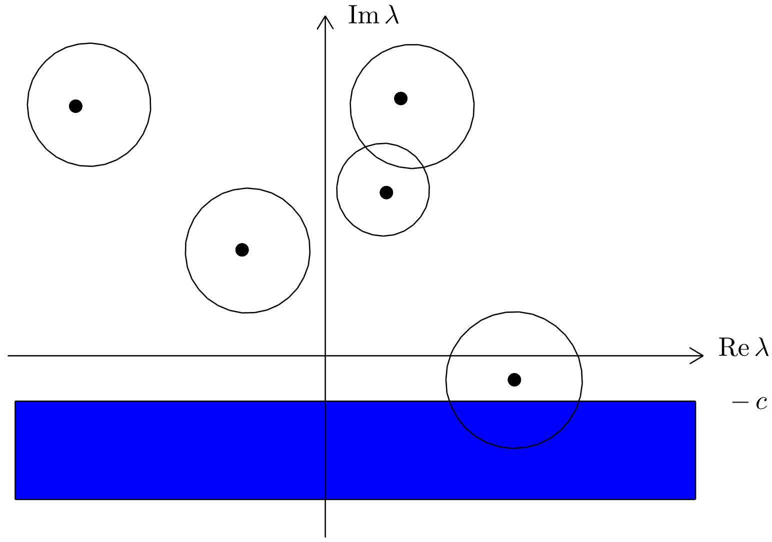


Figure 2.2. Sketch.

Fix $\xi \in \mathbb{R}^n$. Let Γ=union of circles of unit radius abound each λ_k. (wlog, no λ_k on the boundary, else consider circles of radius $1 + \varepsilon$)

$$P(i\xi, i\lambda) = i^m \prod_{k=1}^{m} (\lambda - \lambda_k(\xi)).$$

On Γ we have $|\lambda - \lambda_k(\xi)| \geqslant 1$ for all λ. Therefore $|P(i\xi, i\lambda)| \geqslant 1$ on Γ.

$$\tau^k Z(\xi, t) = \frac{1}{2\pi} \int_\Gamma \frac{(i\lambda)^k e^{i\lambda t}}{P(i\xi, i\lambda)} \mathrm{d}\lambda$$

Bound on $|e^{i\lambda t}|$ on Γ. we have $\mathrm{Im}(\lambda) \geqslant -c - 1$ by Gårding's assumption.

$$|e^{i\lambda t}| = e^{-(\mathrm{Im}\,\lambda)t} \leqslant e^{(1+c)t}.$$

Thus,

$$\begin{aligned}
|\tau^k Z(\xi, t)| &\leqslant \frac{1}{2\pi}\left(\sup_{\lambda \in \Gamma} |\lambda|^k\right) e^{(1+c)t}\ \underbrace{(2\pi m)}_{\text{length of } \Gamma} \\
&\leqslant m\, e^{(1+c)t}\left(\sup_l \left(|\lambda_l(\xi)| + 1\right)\right)^k \\
&\leqslant m\, e^{(1+c)t}(M(1 + |\xi|) + 1)^k
\end{aligned}$$

since each $\lambda(\xi) \leqslant M(1 + |\xi|)$.

$$|\tau^k Z(\xi, t)| \leqslant C\, M^k m\, e^{(1+c)t}(1 + |\xi|)^k.$$

Step 4. This implies that

$$\leqslant$$

$$\begin{aligned} |\partial^\alpha \tau^k u(x,t)| &\leqslant \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |\tau^k Z(\xi,t)|\,|\xi|^\alpha \hat{g}(\xi)\mathrm{d}\xi. \\ &\leqslant \frac{C\,M^k e^{(1+c)t}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1+|\xi|)^k |\xi|^\alpha |\hat{g}|(\xi)\mathrm{d}\xi < \infty \end{aligned}$$

because $\hat{g} \in \mathcal{S}$. □

Theorem 2.50. *Assume $P(D,\tau)$ satisfies Gårding's criterion. Then there exist C^∞ solutions for all $g \in \mathcal{S}(\mathbb{R}^n)$.*

For finite regularity, we only need check for $k + |\alpha| \leqslant m$. We need

$$(1+|\xi|)^m |\hat{g}(\xi)| \in L^1(\mathbb{R}^n).$$

Need for every $\varepsilon > 0$

$$(1+|\xi|)^m |\hat{g}(\xi)| \leqslant \frac{C_\varepsilon}{(1+|\xi|^{n+\varepsilon})}$$

or

$$\Big|\hat{g}(\xi)| \leqslant C_\varepsilon (1+|\xi|)^{-(m+n)-\varepsilon}.$$

m=order of $P(D,\tau)$=regularity of solution, n=space dimension.

Example 2.51. $\partial_t^2 - \Delta u = 0$. $(i\lambda)^2 - (i|\xi|)^2 = 0$, $\lambda = \pm|\xi| \rightarrow$ Growth estimate can't be improved.

Gårding stated wrongly!!!

Question: Is a hyperbolic equation hyperbolic in the sense that it is "wavelike" (meaning if g has compact support, $u(x,t)$ has compact support (in x) for each $t > 0$.

Theorem 2.52. (Paley-Wiener) *Suppose $g \in L^1(\mathbb{R}^n)$ with compact support. Then $\hat{g}: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is* entire.

Proof.

$$\hat{g}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{B(0,R)} e^{-ix\cdot\xi} g(x)\mathrm{d}x.$$

Formally differentiate once, then C^∞ follows.

□

Theorem 2.53. *Assume Gårding's criterion (restriction on roots). Then there is a C^∞ solution to the standard problem for $g \in \mathcal{S}(\mathbb{R}^n)$.*

Example 2.54.

$$\begin{aligned} P(D,\tau)u &= u_{tt} - \Delta u \\ P(i\xi, i\lambda) &= -\lambda^2 + |\xi|^2 \end{aligned}$$

The roots are $\lambda = \pm|\xi|$, which satisfies (GC).

Example 2.55. Suppose $P(i\xi, i\lambda)$ is homogeneous

$$P(i\,s\xi, i\,s\,\lambda) = s^n P(i\xi, i\lambda)$$

for every $s \in \mathbb{R}$. (GC) holds $\Leftrightarrow$ all roots are real–otherwise, we can scale them out as far as we need to.

In general, we can write

$$P(i\xi, i\lambda) = p_{m-1}(i\xi, i\lambda) + \cdots + p_0(i\xi, i\lambda),$$

where p_k is homogeneous of degree k.

Corollary 2.56. *Suppose $P(D, \tau)$ is hyperbolic. Then all roots of $p_m(i\xi, i\lambda)$ are real for every $\xi \in \mathbb{R}^n$.*

Corollary 2.57. *Suppose the roots of p_m are real and distinct for all $\xi \in \mathbb{R}^n$. Then P is hyperbolic. (m=order of P).*

Proof. write $\xi = \rho\eta$, $\lambda = \rho\mu$. where $|\eta| = 1$, $\rho = |\xi|$.

$$P(i\xi, i\lambda) = 0 \Leftrightarrow p_m(i\eta, i\mu) + \frac{1}{\rho} p_{m-1}(i\eta, i\mu) + \cdots + \frac{1}{\rho^n} p_0(i\eta, i\mu) = 0.$$

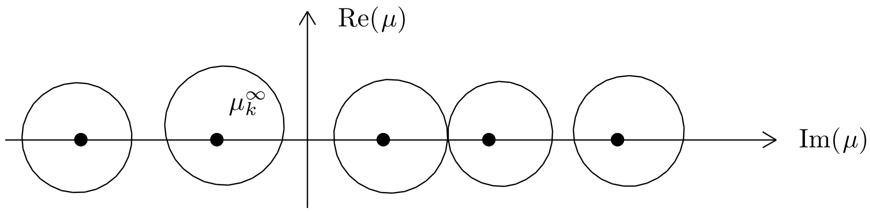


Figure 2.3. Illustrative Sketch. :-)

Use the Implicit Function Theorem to deduce that there exists $\delta > 0$ such that each μ_k^∞ perturbs $\mu_k(p)$ for $1/\rho \leqslant \delta_0$.

$$|\mu_k^\infty - \mu_k(p)| \leqslant \frac{C}{\rho}.$$

We want $f(x(\varepsilon), \varepsilon) = 0$. We know $f(x_0, 0) = 0$. The distinctness is guaranteed by the derivative condition. □

Definition 2.58. *$P(D, \tau)$ is called* strictly hyperbolic *if all $\lambda(\xi)$ are real and distinct. Also say that $p_m(D, \tau)$ is* strictly hyperbolic *if roots are real and distinct.*

Example 2.59. $u_{tt} - \Delta u = 0$ is strictly hyperbolic.

Example 2.60. (*Telegraph equation*) $u_{tt} - \Delta u + k\, u = 0$ with $k \in \mathbb{R}$. By Corollary 2.57, this equation is hyperbolic.

Theorem 2.61. *Suppose $p_m(D, \tau)$ is strictly hyperbolic. Suppose $g \in \mathcal{S}(\mathbb{R}^n)$ and* supp $g \subset B(0, a)$*. Then there exists a c_* such that*

$$\operatorname{supp} u \subset B(0, a + c_* t).$$

c_* is the largest wave speed.

Proof. (Main ingredients)

- Paley-Wiener Theorem: Suppose $g \in L^1(\mathbb{R}^n)$ and supp $g \subset B(0, a)$. Then $\hat{g}$ extends to an entire function $\mathbb{C}^n \to \mathbb{C}^n$ and

  $$|\hat{g}(\xi + i\zeta)| \leqslant \frac{\|g\|_{L^1}}{(2\pi)^{n/2}} e^{a|\zeta|}.$$

  (Proof see below)

- Heuristic:

  - Decay in $f \Rightarrow$ regularity of $\hat{f}$.
  - Regularity of $f \Rightarrow$ decay of $\hat{f}$.

- Estimates of Im(λ) for complex $\xi + i\zeta$. Use strict hyperbolicity to show

  $$\operatorname{Im}(\lambda_k) \leqslant c_*(1 + |\zeta|)$$

  for all $\zeta \in \mathbb{R}^n$.

- Plug into

$$Z(i\xi,t)=\frac{1}{2\pi}\int_{\Gamma}\frac{e^{i\lambda t}}{P(i\xi-\zeta,\lambda)}\mathrm{d}\lambda.$$

- Use

$$u(x,t)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{-ix\cdot(\xi+i\zeta)}Z(\xi+i\zeta,\tau)g(\xi)\mathrm{d}\xi.$$

□

Proof. (of Paley-Wiener)

$$\begin{aligned}|\hat{g}(\xi+i\zeta)| &= \left|\frac{1}{(2\pi)^{n/2}}\int_{B(0,a)}e^{-ix\cdot(\xi+i\zeta)}g(x)\mathrm{d}x\right|\\ &\leqslant \frac{1}{(2\pi)^{n/2}}\int_{B(0,a)}|e^{-ix\cdot(\xi+i\zeta)}|\,|g(x)|\mathrm{d}x\\ &\leqslant \frac{1}{(2\pi)^{n/2}}e^{a|\zeta|}\int_{B(0,a)}|g(x)|\mathrm{d}x.\end{aligned}$$

□