PDE, HW 6, due Monday 11/28/05

Notation: u denotes a solution to the wave equation $\Box u = 0$, with initial data $u(0) = f$, $u_t(0) = g$. The first four problems are from John's book, the next two from Rauch's.

1. Problem 1, p. 45.

2. Problem 1, p. 132.

3. Problem 4, p. 133.

4. Problem, p. 139.

5. Littman's theorem for $n = 3$. Suppose $f = 0$ and $g \in \mathcal{S}(\mathbb{R}^3)$ (this is the class of C^∞ functions all of whose derivatives decay to zero as $|x| \to \infty$). Prove that for $p \neq 2$ and $t \neq 0$,

$$\sup_{g \in \mathcal{S} \setminus \{0\}} \frac{\|u_t(\cdot, t)\|_{L^p}}{\|u_t(\cdot, 0)\|_{L^p}} = \infty.$$

Hint: consider radially symmetric data and the explicit solution formula. Solutions with small support initially spread over an annulus of much larger volume at time t.

6. The Radon transform of $f \in \mathcal{S}(\mathbb{R}^n)$ is a function in $C^\infty(\mathbb{R} \times S^{n-1})$ defined by integrals over hyperplanes

$$h(s, \omega) = \int_{y \cdot \omega = s} f(y)\, dS_y, \quad s \in \mathbb{R}, \;\; |\omega| = 1.$$

Suppose $g = 0$. Show that

$$\lim_{t \to \infty} tu(t, x + ct\omega) = \frac{1}{4\pi} h(x \cdot \omega, \omega)$$

the limit being uniform for $\omega \in S^{n-1}$ and x in compact subsets of $\mathbb{R}^n$. Thus, the solution to the wave equation can be decomposed into plane waves.