PDE, HW 5, due Monday 11/14/05

1. Here and in all that follows, $k(x,t) = (4\pi t)^{-n/2}\exp(-|x|^2/4t)$ is the fundamental solution of the heat equation on $\mathbb{R}^n$. Verify that the fundamental solutions of the heat equation and Laplace's equation are related by

$$\int_0^\infty k(x,t)\,dt = \frac{1}{\omega_n(n-2)}|x|^{2-n}, \quad n \geq 3.$$

2. Prove the following 'semigroup' property of the fundamental solution of the heat equation: if $\star$ denotes convolution

$$k(x,t) = \left(k(\cdot,\frac{t}{m}) \star k(\cdot,\frac{t}{m}) \star \ldots \star k(\cdot,\frac{t}{m})\right)(x), \quad m \in \mathbb{Z}_+.$$

3. The heat equation and convexity:

(a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex and satsifies the growth assumption $f \leq Me^{ax^2}$, $M, a > 0$. Show that the solution to the heat equation $u(x,t) = \int_{\mathbb{R}} k(x-y,t)f(y)\,dy$, $t > 0$ with $u(x,0) = f$ is a convex function in x for every $t > 0$.

(b) Show that $u(x,t_2) \geq u(x,t_1)$ for every $t_2 \geq t_1 \geq 0$ for which $u(x,t)$ is defined.

4. Show that there is a domain $V \subset \mathbb{R}^n \times (0,\infty)$ for which Dirichlet problem for the heat equation does not have a solution.

5. Appell's transformation: Let $u(x,t)$ be a solution to the heat equation for $x \in \mathbb{R}, t < 0$. Let $v(x,t) = k(x,t)u(x/t,-1/t)$ for $x \in \mathbb{R}$, $t > 0$. Show that v solves the heat equation for $x \in \mathbb{R}$, $t > 0$. This is the analogue of inversion for harmonic functions.

6. Let $E[\mu] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{2-n}\mu(dx)\mu(dy)$ denote the Coulomb energy of a finite signed measure. Prove that the Coulomb energy has the properties mentioned in Lecture notes (Thm. 1.56, p. 39, in the current version posted online) by the following method.

(a) Assume part (1), and prove parts (2) and (3) of Thm 1.56.

(b) Use problems (1) and (2) to obtain the identity

$$|x-y|^{2-n} = \omega_n(n-2)\int_0^\infty \int_{\mathbb{R}^n} k(z-x,\frac{t}{2})k(z-y,\frac{t}{2})\,dz\,dt.$$

(c) Substitute this identity in the definition of $E[\mu]$ to show that $E[\mu] \geq 0$.

(d) Show that $E[\mu] = 0$ if and only if $\mu \equiv 0$.